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A FUBINI THEOREM FOR GENERALIZED ANALYTIC FEYNMAN INTEGRALS AND FOURIER-FEYNMAN TRANSFORMS ON FUNCTION SPACE

SEUNG JUN CHANG AND IL YONG LEE

ABSTRACT. In this paper we use a generalized Brownian motion process to define a generalized analytic Feynman integral. We then establish a Fubini theorem for the function space integral and generalized analytic Feynman integral of a functional F belonging to Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ and we proceed to obtain several integration formulas. Finally, we use this Fubini theorem to obtain several Feynman integration formulas involving analytic generalized Fourier-Feynman transforms. These results subsume similar known results obtained by Huffman, Skoug and Storvick for the standard Wiener process.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. The concept of L_1 analytic Fourier-Feynman transform(FFT) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced an L_2 analytic FFT. In [14], Johnson and Skoug developed an L_p analytic FFT theory for $1 \le p \le 2$ which extended the results in [1, 2] and gave various relationships between the L_1 and the L_2 theories. In [11, 12], Huffman, Skoug and Storvick established a Fubini theorem for various analytic Wiener and Feynman integrals.

In [3], Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of functionals on Wiener space which are a kind of stochastic Fourier transform

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of complex Borel measures on $L_2[0,T]$. In [6], Chang and Chung use a generalized Brownian motion process to define a function space integral. In [9], Chang and Skoug studied the analytic generalized FFT(GFFT) on function space.

In this paper we extend the results of [11, 12] to a very general function space $C_{a,b}[0,T]$ and Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$. Recall that the Wiener process is free of drift and is stationary in time, while the stochastic processes considered in this paper are subject to a drift $a(t)$ and are nonstationary in time.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega,\mathcal{B},P)$ be a probability measure space. A real valued stochastic process Y on $(\Omega,\mathcal{B},P)$ and D is called a generalized Brownian motion process if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega),\cdots,Y(t_n,\omega))$ is normally distributed with density function

$$\begin{aligned} K(\vec{t},\vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j)-b(t_{j-1}))\Big)^{-1/2} \\ \cdot \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((\eta_j-a(t_j))-(\eta_{j-1}-a(t_{j-1})))^2}{b(t_j)-b(t_{j-1})}\bigg\} \end{aligned} \tag{2.1}$$

where $\vec{\eta}=(\eta_1,\cdots,\eta_n)$, $\eta_0=0$, $\vec{t}=(t_1,\cdots,t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$.

As explained in [16, pp.18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D,\mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t), t\in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x)=x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D,\mathcal{B}^D,\mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. By theorem 14.2 [16, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set to hold scale-invariant almost everywhere(s-a.e.) [4, 10, 15].

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

(2.2)
$$L^2_{a,b}[0,T]=\left\{v:\int_0^T v^2(s)db(s)<\infty \text{ and } \int_0^T v^2(s)d|a|(s)<\infty\right\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u,v\in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b}=\int_0^T u(t)v(t)d[b(t)+|a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b}=\sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b}=0$ if and only if $u(t)=0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T],\|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b}=\begin{cases}0, & j\neq k\\ 1, & j=k\end{cases}$$

and for each $v\in L^2_{a,b}[0,T]$, let

$$v_n(t)=\sum_{j=1}^n (v,\phi_j)_{a,b}\phi_j(t) \tag{2.4}$$

for $n = 1, 2, \cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty}\int_0^T v_n(t)dx(t) \tag{2.5}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$\int_{C_{a,b}[0,T]} F(x)d\mu(x) \tag{2.6}$$

whenever the integral exists.

We are now ready to state the definition of the generalized analytic Feynman integral.

DEFINITION 2.1. Let $\mathbb{C}$ denote the complex numbers and let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$. Let $F : C_{a,b}[0,T] \longrightarrow \mathbb{C}$ be such that the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)d\mu(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_{a,b}[0,T]}^{an_\lambda} F(x)d\mu(x) = J^*(\lambda). \tag{2.7}$$

Let $q \neq 0$ be a real number and let F be a functional such that $\int_{C_{a,b}[0,T]}^{an_\lambda} F(x)d\mu(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{anf_q} F(x)d\mu(x) = \lim_{\lambda\to -iq}\int_{C_{a,b}[0,T]}^{an_\lambda} F(x)d\mu(x) \tag{2.8}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Now, we give the definition of the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$.

DEFINITION 2.2. Let $M(L^2_{a,b}[0,T])$ be the space of complex-valued, countably additive Borel measures on $L^2_{a,b}[0,T]$. The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\} df(v) \tag{2.9}$$

for s-a.e. $x \in C_{a,b}[0,T]$ where the associated measure f is an element of $M(L^2_{a,b}[0,T])$.

REMARK 2.3. (i) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{S}(L^2_{a,b}[0,T])$ reduces to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [3]. For further work on $\mathcal{S}$, see the references referred to in Section 20.1 of [13].

(ii) $M(L^2_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(iii) One can show that the correspondence $f \to F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L^2_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L^2_{a,b}[0,T]} |df(v)|.$$

In [3], Cameron and Storvick carry out these arguments in detail for the Banach algebra $\mathcal{S}$.

The following function space integral and generalized analytic Feynman integral formulas are used several times in this paper [5, 9].

$$\int_{C_{a,b}[0,T]} \exp\{i\alpha\langle v, x\rangle\} d\mu(x) = \exp\left\{-\frac{\alpha^2(v^2, b')}{2} + i\alpha(v, a')\right\} \tag{2.10}$$

for all $\alpha > 0$, and

$$\int^{anf_q}_{C_{a,b}[0,T]} \exp\{i\langle v, x\rangle\} d\mu(x) = \exp\left\{-\frac{i(v^2, b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v, a')\right\} \tag{2.11}$$

for all real $q \neq 0$, $(i/q)^{\frac{1}{2}}$ is always chosen to have positive real part and $v \in L^2_{a,b}[0,T]$ where

$$(v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t) \tag{2.12}$$

and

$$(v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t). \tag{2.13}$$

REMARK 2.4. If $a(t) \equiv 0$ on $[0,T]$, then for all $F \in \mathcal{S}(L^2_{a,b}[0,T])$ with associated measure f, the generalized analytic Feynman integral of F will always exist for all real $q \neq 0$ and be given by the formula

$$\int^{anf_q}_{C_{a,b}[0,T]} F(x)d\mu(x) = \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i(v^2,b')}{2q}\right\}df(v). \tag{2.14}$$

However for $a(t)$ as in this section, and proceeding formally using equations (2.9) and (2.11), we see that $\int^{anf_q}_{C_{a,b}[0,T]} F(x)d\mu(x)$ will be given by the formula

(2.15)

$$\int^{anf_q}_{C_{a,b}[0,T]} F(x)d\mu(x) = \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i(v^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v,a')\right\}df(v)$$

if it exists. But the integral on the right hand-side of (2.15) might not exist if the real part of

$$\exp\left\{-\frac{i(v^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v,a')\right\} \tag{2.16}$$

is positive. However

(2.17)

$$\left|\exp\left\{-\frac{i(v^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v,a')\right\}\right| = \begin{cases} \exp\{-(2q)^{-1/2}(v,a')\}, & q > 0 \\ \exp\{(-2q)^{-1/2}(v,a')\}, & q < 0 \end{cases}$$

and so the generalized analytic Feynman integral of F will certainly exist provided the associated measure f satisfies the condition

$$\int_{L^2_{a,b}[0,T]} \exp\left\{|2q|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty. \tag{2.18}$$

3. Generalized Feynman integrals

In this section we establish a Fubini theorem for the function space integral and the generalized analytic Feynman integral for a functional F in a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$. We also use this Fubini theorem to establish several generalized analytic Feynman integration formulas.

In our first Lemma we obtain a Fubini theorem for function space integrals of a functional $F \in \mathcal{S}(L^2_{a.b}[0,T])$.

LEMMA 3.1. *Let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by* (2.9). *Then for all $\alpha, \beta > 0$,*

$$\int_{C_{a,b}[0,T]}\left[\int_{C_{a,b}[0,T]} F(\alpha y+\beta z)\,d\mu(y)\right]d\mu(z) = \int_{C_{a,b}[0,T]}\left[\int_{C_{a,b}[0,T]} F(\alpha y+\beta z)\,d\mu(z)\right]d\mu(y). \tag{3.1}$$

In addition, both expressions in (3.1) *are given by the expression*

$$\int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{1}{2}(\alpha^2+\beta^2)(v^2,b')+i(\alpha+\beta)(v,a')\right\}df(v). \tag{3.2}$$

Proof. Since F is an element of $\mathcal{S}(L^2_{a,b}[0,T])$, we have

$$\int_{C_{a,b}[0,T]} |F(\rho x)|\,d\mu(x) < \infty \tag{3.3}$$

for each $\rho > 0$. Hence by the usual Fubini theorem, we have the equation (3.1) above. Further, by using (2.10), we have for all $\alpha, \beta > 0$,

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\left[\int_{C_{a,b}[0,T]} F(\alpha y+\beta z)\,d\mu(y)\right]d\mu(z)\\
&= \int_{C_{a,b}[0,T]}\left[\int_{C_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\{i\langle v,\alpha y\rangle + i\langle v,\beta z\rangle\}df(v)\,d\mu(y)\right]d\mu(z)\\
&= \int_{L^2_{a,b}[0,T]}\int_{C_{a,b}[0,T]} \exp\{i\langle v,\beta z\rangle\}\\
&\qquad\cdot\left[\int_{C_{a,b}[0,T]} \exp\{i\langle v,\alpha y\rangle\}d\mu(y)\right]d\mu(z)\,df(v)\\
&= \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{\alpha^2}{2}(v^2,b')+i\alpha(v,a')\right\}\\
&\qquad\cdot\left[\int_{C_{a,b}[0,T]} \exp\{i\langle v,\beta z\rangle\}d\mu(z)\right]df(v)\\
&= \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{1}{2}(\alpha^2+\beta^2)(v^2,b')+i(\alpha+\beta)(v,a')\right\}df(v).
\end{aligned} \tag{3.4}$$

□

THEOREM 3.2. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure f satisfies the condition*

$$\int_{L^2_{a,b}[0,T]} \exp\left\{4|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty. \tag{3.5}$$

Then for all nonzero real numbers q_1 and q_2 with $|q_1| \geq |q_0|$, $|q_2| \geq |q_0|$ and $q_1 + q_2 \neq 0$,

$$\begin{aligned}
&\int^{anf_{q_2}}_{C_{a,b}[0,T]}\left[\int^{anf_{q_1}}_{C_{a,b}[0,T]} F(y+z)d\mu(y)\right]d\mu(z)\\
&= \int^{anf_{\frac{q_1q_2}{q_1+q_2}}}_{C_{a,b}[0,T]} F_{q_1,q_2}(x)d\mu(x)\\
&= \int^{anf_{q_1}}_{C_{a,b}[0,T]}\left[\int^{anf_{q_2}}_{C_{a,b}[0,T]} F(y+z)d\mu(z)\right]d\mu(y)
\end{aligned} \tag{3.6}$$

where F_{q_1,q_2} is given by (3.11) below.

Also, all expressions in (3.6) are given by the expression

$$\int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}\right)(v^2,b')+i\left(\left(\frac{i}{q_1}\right)^{\frac{1}{2}}+\left(\frac{i}{q_2}\right)^{\frac{1}{2}}\right)(v,a')\right\}df(v). \tag{3.7}$$

Proof. Using the usual Fubini theorem, (2.15), and (2.10), we have that for all $\lambda_2 > 0$,

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}\left[\int^{anf_{q_1}}_{C_{a,b}[0,T]} F(y+\lambda_2^{-1/2}z)d\mu(y)\right]d\mu(z)\\
&= \int_{L^2_{a,b}[0,T]}\left[\int_{C_{a,b}[0,T]}\left[\int^{anf_{q_1}}_{C_{a,b}[0,T]} \exp\{i\langle v,y\rangle\}d\mu(y)\right]\right.\\
&\qquad\qquad \left.\cdot \exp\{i\lambda_2^{-1/2}\langle v,z\rangle\}d\mu(z)\right]df(v)\\
&= \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i(v^2,b')}{2q_1}+i\left(\frac{i}{q_1}\right)^{\frac{1}{2}}(v,a')\right\}
\end{aligned} \tag{3.8}$$

$$\cdot \left[\int_{C_{a,b}[0,T]} \exp\{i\lambda_2^{-1/2}\langle v,z\rangle\} d\mu(z)\right] df(v)$$

$$= \int_{L^2_{a,b}[0,T]} \exp\bigg\{-\frac{i(v^2,b')}{2q_1} + i\Big(\frac{i}{q_1}\Big)^{\frac{1}{2}}(v,a') - \frac{(v^2,b')}{2\lambda_2} + i\lambda_2^{-1/2}(v,a')\bigg\} df(v).$$

But the last expression above is an analytic function of $\mathbb{C}_+$ and is a continuous function of λ_2 in $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}\lambda \geq 0\}$, and so setting $\lambda_2 = -iq_2$ yields (3.7).

Also, using (2.15) with q replaced with q_2, we obtain that for all $\lambda_1 > 0$

$$\int_{C_{a,b}[0,T]} \left[\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(\lambda_1^{-1/2}y + z) d\mu(z)\right] d\mu(y)$$

$$(3.9) \quad = \int_{L^2_{a,b}[0,T]} \exp\bigg\{-\frac{(v^2,b')}{2\lambda_1} + i\lambda_1^{-1/2}(v,a') - \frac{i(v^2,b')}{2q_2} + i\Big(\frac{i}{q_2}\Big)^{\frac{1}{2}}(v,a')\bigg\} df(v).$$

By the same argument with $\lambda_1 = -iq_1$, we have the expression (3.7) above. Moreover, the expression (3.7) is equal to

(3.10)

$$\int_{L^2_{a,b}[0,T]} \exp\bigg\{-\frac{i}{2}\Big(\frac{1}{q_1} + \frac{1}{q_2}\Big)(v^2,b') + i\Big(\frac{i}{q_1} + \frac{i}{q_2}\Big)^{\frac{1}{2}}(v,a')\bigg\} df_{q_1,q_2}(v)$$

$$= \int_{L^2_{a,b}[0,T]} \exp\bigg\{-\frac{i}{2(\frac{q_1q_2}{q_1+q_2})}(v^2,b') + i\Big(\frac{i}{\frac{q_1q_2}{q_1+q_2}}\Big)^{\frac{1}{2}}(v,a')\bigg\} df_{q_1,q_2}(v)$$

$$= \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}} F_{q_1,q_2}(x) d\mu(x)$$

where

$$(3.11) \qquad F_{q_1,q_2}(x) = \int_{L^2_{a.b}[0,T]} \exp\{i\langle v,x\rangle\} df_{q_1,q_2}(v)$$

and
(3.12)

$$f_{q_1,q_2}(E) = \int_E \exp\Big\{i\Big(\Big(\frac{i}{q_1}\Big)^{\frac12}+\Big(\frac{i}{q_2}\Big)^{\frac12}\Big)(v,a') - i\Big(\frac{i}{q_1}+\frac{i}{q_2}\Big)^{\frac12}(v,a')\Big\}df(v)$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$. Finally, we have that

$$\begin{aligned}
&||f_{q_1,q_2}|| \\
&= \int_{L^2_{a,b}[0,T]} |df_{q_1,q_2}(v)| \\
&\le \int_{L^2_{a,b}[0,T]} \exp\Big\{|2q_1|^{-1/2}\int_0^T |v(s)|d|a|(s)\Big\} \\
&\qquad \cdot \exp\Big\{|2q_2|^{-1/2}\int_0^T |v(s)|d|a|(s)\Big\} \\
&\qquad \cdot \exp\Big\{\Big|\frac{2q_1q_2}{q_1+q_2}\Big|^{-1/2}\int_0^T |v(s)|d|a|(s)\Big\}|df(v)| \\
&\le \int_{L^2_{a,b}[0,T]} \exp\Big\{4|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\Big\}|df(v)| < \infty.
\end{aligned} \tag{3.13}$$

Hence f_{q_1,q_2} is an element of $M(L^2_{a,b}[0,T])$ and so F_{q_1,q_2} is in $\mathcal{S}(L^2_{a,b}[0,T])$. Thus we have the desired results. □

COROLLARY 3.3. *Let q_0 and F be as in Theorem 3.2. Then for all real $q \neq 0$ with $|q| \ge |q_0|$,*
(3.14)

$$\int_{C_{a,b}[0,T]}^{anf_q}\Big[\int_{C_{a,b}[0,T]}^{anf_q} F(y+z)d\mu(y)\Big]d\mu(z) = \int_{C_{a,b}[0,T]}^{anf_{q/2}} F_{q,q}(x)d\mu(x)$$

where

$$F_{q,q}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\}df_{q,q}(v) \tag{3.15}$$

and

$$f_{q,q}(E) = \int_E \exp\Big\{2i\Big(\frac{i}{q}\Big)^{\frac12}(v,a') - i\Big(\frac{2i}{q}\Big)^{\frac12}(v,a')\Big\}df(v) \tag{3.16}$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

THEOREM 3.4. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by (2.9). Let $q_1,\cdots,q_{n-1}$ and q_n be nonzero real numbers satisfying the followings;*

i) $|q_j| \geq |q_0|$ *for all* $j=1,\cdots,n$;

ii) *for all* $j,l=1,\cdots,n$, $q_j+q_l \neq 0$

iii) *for all* $k=2,\cdots,n$, $\sum\limits_{j=1}^{k} \frac{q_1\cdots q_k}{q_j} \neq 0$.

Suppose that the associated measure f of F satisfies the condition

$$(3.17) \qquad \int_{L^2_{a,b}[0,T]} \exp\left\{2n|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty$$

for $n=1,2,\cdots$, *then*

$$(3.18) \qquad \begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_{q_n}}\cdots\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(y_1+\cdots+y_n)d\mu(y_1)\cdots d\mu(y_n) \\ &= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(x)d\mu(x) \end{aligned}$$

where $\alpha_n = \frac{q_1\cdots q_n}{\sum_{j=1}^n \frac{q_1\cdots q_n}{q_j}}$ *and* $F_{q_1,\cdots,q_n}$ *is given by equation* (3.21) *below.*

In addition, both expressions in (3.18) *are given by the expression*

$$(3.19) \qquad \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i}{2}\sum_{j=1}^n \frac{1}{q_j}(v^2,b') + i\sum_{j=1}^n\left(\frac{i}{q_j}\right)^{\frac{1}{2}}(v,a')\right\}df(v).$$

Proof. Using equation (3.6) repeatedly, we obtain that

$$(3.20) \qquad \begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_{q_n}}\cdots\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(y_1+\cdots+y_n)d\mu(y_1)\cdots d\mu(y_n) \\ &= \int_{C_{a,b}[0,T]}^{anf_{q_n}}\cdots\int_{C_{a,b}[0,T]}^{anf_{q_3}} \\ &\cdot \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}} F_{q_1,q_2}(z_1+y_3+\cdots+y_n)d\mu(z_1)d\mu(y_3)\cdots d\mu(y_n) \end{aligned}$$

$$
\begin{aligned}
&= \int_{C_{a,b}[0,T]}^{anf_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{anf_{q_4}} \\
&\quad \cdot \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2q_3}{q_1q_2+q_1q_3+q_2q_3}}} F_{q_1,q_2,q_3}(z_2+y_4+\cdots+y_n) \\
&\quad \cdot d\mu(z_2)d\mu(y_4)\cdots d\mu(y_n) \\
&= \cdots \\
&= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(x)d\mu(x)
\end{aligned}
$$

where

$$
F_{q_1,\cdots,q_n}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\}df_{q_1,\cdots,q_n}(v) \tag{3.21}
$$

and

(3.22)

$$
f_{q_1,\cdots,q_n}(E) = \int_E \exp\left\{i\sum_{j=1}^n\left(\frac{i}{q_j}\right)^{\frac{1}{2}}(v,a') - i\left(\sum_{j=1}^n\frac{i}{q_j}\right)^{\frac{1}{2}}(v,a')\right\}df(v)
$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$. Finally, we have that

(3.23)

$$
\begin{aligned}
&||f_{q_1,\cdots,q_n}|| = \int_{L^2_{a,b}[0,T]} |df_{q_1,\cdots,q_n}(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{\sum_{j=1}^n |2q_j|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\} \\
&\qquad\qquad \cdot \exp\left\{|2\alpha_n|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{2\sum_{j=1}^n |2q_j|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{2n|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty.
\end{aligned}
$$

Hence $f_{q_1,\cdots,q_n}$ is an element of $M(L^2_{a,b}[0,T])$ and so $F_{q_1,\cdots,q_n}$ is in $\mathcal{S}(L^2_{a,b}[0,T])$. Thus we have the desired results. □

Choosing $q_j = q$ for $j = 1,\cdots,n$, we obtain the following corollary to Theorem 3.4.

COROLLARY 3.5. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure f satisfies the condition*

$$\int_{L^2_{a,b}[0,T]} \exp\left\{2n|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty \tag{3.24}$$

for $n = 1, 2, \cdots$. Then for all real q with $|q| \geq |q_0|$,

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_q} \cdots \int_{C_{a,b}[0,T]}^{anf_q} F(y_1 + \cdots + y_n)d\mu(y_1)\cdots d\mu(y_n) \\ &= \int_{C_{a,b}[0,T]}^{anf_{q/n}} F_{q,\cdots,q}(x)d\mu(x) \end{aligned} \tag{3.25}$$

where

$$F_{q,\cdots,q}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\}df_{q,\cdots,q}(v) \tag{3.26}$$

and

$$f_{q,\cdots,q}(E) = \int_E \exp\left\{in\left(\frac{i}{q}\right)^{\frac{1}{2}}(v, a') - i\left(\frac{in}{q}\right)^{\frac{1}{2}}(v, a')\right\}df(v) \tag{3.27}$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

REMARK 3.6. Note that each of the iterated integrals in equation (3.18) above can also be expressed in $(n! - 1)$ other similar ways; for example, all of the expressions in (3.18), also equal the expression

$$\int_{C_{a,b}[0,T]}^{anf_{\frac{q_2\cdots q_n}{\sum_{j=2}^n \frac{q_2\cdots q_n}{q_j}}}} \int_{C_{a,b}[0,T]}^{anf_{q_1}} F_{q_2,\cdots,q_n}(y_1 + x)d\mu(y_1)d\mu(x) \tag{3.28}$$

where

$$F_{q_2,\cdots,q_n}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\}df_{q_2,\cdots,q_n}(v) \tag{3.29}$$

and
(3.30)

$$f_{q_2,\cdots,q_n}(E) = \int_E \exp\left\{i\sum_{j=2}^n\left(\frac{i}{q_j}\right)^{\frac{1}{2}}(v, a') - i\left(\sum_{j=2}^n \frac{i}{q_j}\right)^{\frac{1}{2}}(v, a')\right\}df(v)$$

for every $E \in \mathcal{B}(L^2_{a.b}[0,T])$.

LEMMA 3.7. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure f satisfies the condition (2.18) with q replaced with q_0. Then for all nonzero real number q and for all $\alpha > 0$ with $|\alpha q| \geq |q_0|$,*

$$\int_{C_{a,b}[0,T]}^{anf_{\alpha q}} F(x)d\mu(x) = \int_{C_{a,b}[0,T]}^{anf_q} F(\frac{x}{\sqrt{\alpha}})d\mu(x). \tag{3.31}$$

Proof. By using (2.15), we see that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{\alpha q}} F(x)d\mu(x) \\
&= \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i(v^2,b')}{2\alpha q} + i\left(\frac{i}{\alpha q}\right)^{\frac{1}{2}}(v,a')\right\}df(v) \\
&= \int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i}{2q}(v^2,\frac{b'}{\alpha}) + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v,\frac{a'}{\sqrt{\alpha}})\right\}df(v) \\
&= \int_{C_{a,b}[0,T]}^{anf_q} F(\frac{x}{\sqrt{\alpha}})d\mu(x).
\end{aligned} \tag{3.32}$$

The generalized analytic Feynman integral in equation (3.32) exists because

$$\begin{aligned}
&\int_{L^2_{a,b}[0,T]} \left|\exp\left\{-\frac{i}{2\alpha q}(v^2,b') + i\left(\frac{i}{\alpha q}\right)^{\frac{1}{2}}(v,a')\right\}\right| |df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{|2\alpha q|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| \\
&\leq \int_{L^2_{a,b}[0,T]} \exp\left\{|2q_0|^{-1/2}\int_0^T |v(s)|d|a|(s)\right\}|df(v)| < \infty.
\end{aligned} \tag{3.33}$$

Hence we have the desired result. □

THEOREM 3.8. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ given by (2.9) whose associated measure f satisfies the condition (3.5). Let $\alpha, \beta > 0$ and let q_1 and q_2 be nonzero real* *numbers with* $|q_1|/\alpha^2 \geq |q_0|$, $|q_2|/\beta^2 \geq |q_0|$ *and* $\beta^2 q_1 + \alpha^2 q_2 \neq 0$. *Then*

$$\begin{aligned}(3.34)\qquad &\int_{C_{a,b}[0,T]}^{anf_{q_2}}\left[\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(\alpha y + \beta z)d\mu(y)\right]d\mu(z)\\ &= \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{\beta^2 q_1+\alpha^2 q_2}}} F_{q_1/\alpha^2,q_2/\beta^2}(x)d\mu(x)\end{aligned}$$

where $F_{q_1/\alpha^2,q_2/\beta^2}$ *is given by* (3.37) *below.*

Also, both expressions in (3.34) *are given by the expression*

(3.35)

$$\int_{L^2_{a,b}[0,T]} \exp\left\{-\frac{i}{2}\left(\frac{\alpha^2}{q_1}+\frac{\beta^2}{q_2}\right)(v^2,b') + i\left(\alpha\left(\frac{i}{q_1}\right)^{\frac12} + \beta\left(\frac{i}{q_2}\right)^{\frac12}\right)(v,a')\right\}df(v).$$

Proof. By using (3.31) and (3.6), we see that

$$\begin{aligned}(3.36)\qquad &\int_{C_{a,b}[0,T]}^{anf_{q_2}}\left[\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(\alpha y + \beta z)d\mu(y)\right]d\mu(z)\\ &= \int_{C_{a,b}[0,T]}^{anf_{q_2}}\left[\int_{C_{a,b}[0,T]}^{anf_{q_1/\alpha^2}} F(y + \beta z)d\mu(y)\right]d\mu(z)\\ &= \int_{C_{a,b}[0,T]}^{anf_{q_1/\alpha^2}}\left[\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(y + \beta z)d\mu(z)\right]d\mu(y)\\ &= \int_{C_{a,b}[0,T]}^{anf_{q_1/\alpha^2}}\left[\int_{C_{a,b}[0,T]}^{anf_{q_2/\beta^2}} F(y + z)d\mu(z)\right]d\mu(y)\\ &= \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{\beta^2 q_1+\alpha^2 q_2}}} F_{q_1/\alpha^2,q_2/\beta^2}(x)d\mu(x)\end{aligned}$$

where

$$(3.37)\qquad F_{q_1/\alpha^2,q_2/\beta^2}(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\}df_{q_1/\alpha^2,q_2/\beta^2}(v)$$

and

(3.38)

$$\begin{aligned}f_{q_1/\alpha^2,q_2/\beta^2}(E) = \int_E \exp\Bigg\{&i\left(\alpha\left(\frac{i}{q_1}\right)^{\frac12}+\beta\left(\frac{i}{q_2}\right)^{\frac12}\right)(v,a')\\ &- i\left(\frac{i(\beta^2 q_1+\alpha^2 q_2)}{q_1 q_2}\right)^{\frac12}(v,a')\Bigg\}df(v)\end{aligned}$$

for every $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

Moreover, we have that

$$
\begin{aligned}
||f_{q_1/\alpha^2,q_2/\beta^2}|| &= \int_{L^2_{a,b}[0,T]} |df_{q_1/\alpha^2,q_2/\beta^2}(v)| \\
&\le \int_{L^2_{a,b}[0,T]} \exp\left\{ \left| \frac{2q_1q_2}{\beta^2 q_1 + \alpha^2 q_2} \right|^{-1/2} \int_0^T |v(s)|d|a|(s) \right\} \\
&\qquad \cdot \exp\left\{ |2q_1/\alpha^2|^{-1/2} \int_0^T |v(s)|d|a|(s) \right\} \\
&\qquad \cdot \exp\left\{ |2q_2/\beta^2|^{-1/2} \int_0^T |v(s)|d|a|(s) \right\} |df(v)| \\
&\le \int_{L^2_{a,b}[0,T]} \exp\left\{ 4|2q_0|^{-1/2} \int_0^T |v(s)|d|a|(s) \right\} |df(v)| < \infty.
\end{aligned} \tag{3.39}
$$

Hence $f_{q_1/\alpha^2,q_2/\beta^2}$ is an element of $M(L^2_{a,b}[0,T])$ and so $F_{q_1/\alpha^2,q_2/\beta^2}$ is in $\mathcal{S}(L^2_{a,b}[0,T])$. Thus we have the desired results. □

4. Generalized Fourier-Feynman transforms

In this section, we will establish a Fubini theorem for analytic GFFT for functional $F \in \mathcal{S}(L^2_{a,b}[0,T])$. Then, as corollaries we will obtain several Feynman integration formulas involving analytic GFFT. For simplicity, we restrict our discussion to the case $p = 1$; however most of our results hold for all $p \in [1, 2]$.

We state the definition of the analytic GFFT [7, 9].

DEFINITION 4.1. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$
T_\lambda(F)(y) = \int_{C_{a,b}[0,T]}^{\text{an}_\lambda} F(y+x)\,d\mu(x). \tag{4.1}
$$

Then for $q \in \mathbb{R} - \{0\}$, the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, is defined by the formula ($\lambda \in \mathbb{C}_+$)

$$
T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{4.2}
$$

for s-a.e. $y \in C_{a,b}[0,T]$ whenever the limit exists. That is to say,

$$T_q^{(1)}(F)(y) = \int_{C_{a,b}[0,T]}^{anf_q} F(y+x)\,d\mu(x) \tag{4.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

We note that if $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(F) \approx T_q^{(1)}(G)$.

THEOREM 4.2. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L_{a,b}^2[0,T])$ given by* (2.9) *whose associated measure f satisfies the condition* (3.5). *Let $r > 0$ and let q_1 and q_2 be nonzero real numbers with $|q_1| > |q_0|$, $|q_2| > |q_0|$ and $q_1 + q_2 \neq 0$. Then*

$$\begin{aligned} \int_{C_{a,b}[0,T]}^{anf_{rq_2}} T_{q_1}^{(1)}(F)(\sqrt{r}z)\,d\mu(z) &= \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}} F_{q_1,q_2}(x)\,d\mu(x) \\ &= \int_{C_{a,b}[0,T]}^{anf_{rq_1}} T_{q_2}^{(1)}(F)(\sqrt{r}y)\,d\mu(y) \end{aligned} \tag{4.4}$$

where F_{q_1,q_2} is given by (3.11).

Proof. Using equations (4.3) and (3.34) with $\alpha = 1$, $\beta = \sqrt{r}$, we obtain that

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_{rq_2}} T_{q_1}^{(1)}(F)(\sqrt{r}z)\,d\mu(z) \\ &= \int_{C_{a,b}[0,T]}^{anf_{rq_2}} \left[\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(\sqrt{r}z+y)\,d\mu(y)\right] d\mu(z) \\ &= \int_{C_{a,b}[0,T]}^{anf_{\frac{rq_1q_2}{rq_1+rq_2}}} F_{q_1,rq_2/r}(x)\,d\mu(x) \\ &= \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}} F_{q_1,q_2}(x)\,d\mu(x). \end{aligned} \tag{4.5}$$

By the same argument in equation (4.5) with $\alpha = \sqrt{r}$, $\beta = 1$, we have that

$$\int_{C_{a,b}[0,T]}^{anf_{rq_1}} T_{q_2}^{(1)}(F)(\sqrt{r}y)\,d\mu(y) = \int_{C_{a,b}[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}} F_{q_1,q_2}(x)\,d\mu(x). \tag{4.6}$$

Now equation (4.4) follows from equations (4.5) and (4.6). □

COROLLARY 4.3. *Let q_0 and F be as in Theorem 4.2. Then for all nonzero real numbers q_1 and q_2 with $|q_1| > |q_0|$, $|q_2| > |q_0|$ and $q_1 + q_2 \neq 0$,*

$$\int_{C_{a,b}[0,T]}^{anf_{q_2}} T_{q_1}^{(1)}(F)(z)d\mu(z) = \int_{C_{a,b}[0,T]}^{anf_{q_1}} T_{q_2}^{(1)}(F)(y)d\mu(y). \tag{4.7}$$

COROLLARY 4.4. *Let q_0 and let F be as in Theorem 4.2. Then for all nonzero real number q with $|q| > |q_0|$,*

$$\begin{aligned} \int_{C_{a,b}[0,T]}^{anf_q} T_q^{(1)}(F)(y)d\mu(y) &= \int_{C_{a,b}[0,T]}^{anf_{q/2}} F_{q,q}(x)d\mu(x) \\ &= \int_{C_{a,b}[0,T]}^{anf_q} F_{q,q}(\sqrt{2}x)d\mu(x) \end{aligned} \tag{4.8}$$

where $F_{q,q}$ is given by (3.15).

Proof. The first equality in (4.8) follows by letting $r = 1$ and $q_1 = q_2 = q$ in equation (4.4). The second equality follows from Lemma 3.7.□

THEOREM 4.5. *Let $q_0, q_1, \cdots, q_n$, and let F be as in Theorem 3.4. Then for s-a.e. $z \in C_{a,b}[0,T]$,*

$$\begin{aligned} &T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))(z) \\ &= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(z+x)d\mu(x) \\ &= T_{\alpha_n}^{(1)}(F_{q_1,\cdots,q_n})(z) \end{aligned} \tag{4.9}$$

where $F_{q_1,\cdots,q_n}$ is given by equation (3.21) and α_n is as in Theorem 3.4.

Proof. Using equations (4.3) and (3.18), we obtain that

$$\begin{aligned} &T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\cdots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\cdots))(z) \\ &= \int_{C_{a,b}[0,T]}^{anf_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{anf_{q_1}} F(z+y_1+\cdots+y_n)d\mu(y_1)\cdots d\mu(y_n) \\ &= \int_{C_{a,b}[0,T]}^{anf_{\alpha_n}} F_{q_1,\cdots,q_n}(z+x)d\mu(x) \\ &= T_{\alpha_n}^{(1)}(F_{q_1,\cdots,q_n})(z) \end{aligned} \tag{4.10}$$

for s-a.e. $z \in C_{a,b}[0,T]$. □

Choosing $q_j = q$ for $j = 1, \cdots, n$, we obtain the following corollary to Theorem 4.5.

COROLLARY 4.6. *Let q_0 and F be as in Theorem 4.5 and let q be a nonzero real number with $|q| \geq |q_0|$. Then for s-a.e. $z \in C_{a,b}[0,T]$,*
(4.11)

$$T_q^{(1)}(T_q^{(1)}(F))(z) = T_{q/2}^{(1)}(F_{q,q})(z) = \int_{C_{a,b}[0,T]}^{anf_q} F_{q,q}(z + \sqrt{2}x)d\mu(x),$$

$$\begin{aligned} &T_q^{(1)}(T_q^{(1)}(T_q^{(1)}(F)))(z) \\ &= T_{q/3}^{(1)}(F_{q,q,q})(z) = \int_{C_{a,b}[0,T]}^{anf_q} F_{q,q,q}(z + \sqrt{3}x)d\mu(x), \end{aligned} \tag{4.12}$$

and in general,

$$\begin{aligned} &T_q^{(1)}(T_q^{(1)}(\cdots(T_q^{(1)}(F))\cdots))(z) \\ &= T_{q/n}^{(1)}(F_{q,\cdots,q})(z) = \int_{C_{a,b}[0,T]}^{anf_q} F_{q,\cdots,q}(z + \sqrt{n}x)d\mu(x). \end{aligned} \tag{4.13}$$

COROLLARY 4.7. *Let q_0 and F be as in Theorem 4.5 and let q_1 and q_2 be nonzero real numbers with $|q_1| \geq |q_0|, |q_2| \geq |q_0|$, and $q_1 + q_2 \neq 0$. Then for s-a.e. $z \in C_{a,b}[0,T]$,*

$$T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))(z) = T_{\frac{q_1 q_2}{q_1+q_2}}^{(1)}(F_{q_1,q_2})(z) = T_{q_1}^{(1)}(T_{q_2}^{(1)}(F))(z) \tag{4.14}$$

where F_{q_1,q_2} is given by (3.11).

References

[1] M. D. Brue, *A functional transform for Feynman integrals similar to the Fourier transforms*, Thesis, University of Minnesota, 1972.

[2] R. H. Cameron and D. A. Storvick, *An L_2 analytic Fourier-Feynman transform*, Michigan Math. J. **23** (1976), 1–30.

[3] ———, *Some Banach algebras of analytic Feynman integrable functionals, Analytic Functions* (*Kozubnik, 1979*), Lecture Notes in Math. **798**, Springer, Berlin (1980), 18–67.

[4] K. S. Chang, *Scale-invariant measurability in Yeh-Wiener space*, J. Korean Math. Soc. **19** (1982), 61–67.

[5] S. J. Chang and J. G. Choi, *Relationships of convolution products, generalized transforms and the first variation on function space*, Int. J. Math. Math. Sci. **29** (2002), 591–608.

[6] S. J. Chang and D. M. Chung, *Conditional function space integrals with applications*, Rocky Mountain J. Math. **26** (1996), 37–62.
[7] S. J. Chang, S. J. Kang and D. Skoug, *Conditional generalized analytic Feynman integrals and a generalized integral equation*, Int. J. Math. Math. Sci. **23** (2000), 759–776.
[8] S. J. Chang, C. Park and D. Skoug, *Translation theorems for Fourier-Feynman transforms and conditional Fourier-Feynman transforms*, Rocky Mountain J. Math. **30** (2000), 477–496.
[9] S. J. Chang and D. Skoug, *Generalized Fourier-Feynman transforms and a first variation on function space*, to appear in the Integral transforms and special functions.
[10] D. M. Chung, *Scale-invariant measurability in abstract Wiener spaces*, Pacific J. Math. **130** (1987), 27–40.
[11] T. Huffman, D. Skoug and D. Storvick, *A Fubini theorem for analytic Feynman integrals with applications*, J. Korean Math. Soc. **38** (2001), 409–420.
[12] ———, *Integration formulas involving Fourier-Feynman transforms via a Fubini theorem*, J. Korean Math. Soc. **38** (2001), 421–435.
[13] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University press, New York, 2000.
[14] G. W. Johnson and D. L. Skoug, *An L_p analytic Fourier-Feynman transform*, Michigan Math. J. **26** (1979), 103–127.
[15] ———, *Scale-invariant measurability in Wiener space*, Pacific J. Math **83** (1979), 157–176.
[16] J. Yeh, *Stochastic processes and the Wiener integral*, Marcel Dekker, Inc., New York, 1973.

DEPARTMENT OF MATHEMATICS, DANKOOK UNIVERSITY, CHEONAN 330-714, KOREA
E-mail: sejchang@dankook.ac.kr
iylee@dankook.ac.kr