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B.-Y. CHEN INEQUALITIES FOR SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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ABSTRACT. Some B.-Y. Chen inequalities for different kind of submanifolds of generalized complex space forms are established.

1. Introduction

According to Nash's immersion theorem every n-dimensional Riemannian manifold admits an isometric immersion into Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$. Thus, one becomes able to consider any Riemannian manifold as a submanifold of Euclidean space; and this provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold is one of the basic interests of study in the submanifold theory. Gauss-Bonnet Theorem, Isoperimetric inequality and Chern-Lashof Theorem provide relations between extrinsic and extrinsic invariants for a submanifold in a Euclidean space.

In [2], B.-Y. Chen established a sharp inequality for a submanifold in a real space form involving intrinsic invariants, namely the sectional curvatures and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature.

On the other hand, A. Gray introduced the notion of constant type for a nearly Kähler manifold ([6]), which led to definitions of RK-manifolds $\tilde{M}\,(c,\alpha)$ of constant holomorphic sectional curvature c and constant type α ([10]) and generalized complex space forms $\tilde{M}\,(f_1, f_2)$ ([7]). We have

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the inclusion relation $\tilde{M}(c) \subset \tilde{M}(c,\alpha) \subset \tilde{M}(f_1, f_2)$, where $\tilde{M}(c)$ is the complex space form of constant holomorphic sectional curvature c.

Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a generalized space form. In this paper, we establish several such relationships for slant, totally real and invariant submanifolds in generalized complex space forms, complex space forms and RK-manifolds. The paper is organized as follows. Section 2 is preliminary in nature. It contains necessary details about generalized complex space form and its submanifolds. In section 3, we obtain a basic inequality for a submanifold in a generalized complex space form involving intrinsic invariants, namely the scalar curvature and the sectional curvatures of the submanifold on left hand side and the main extrinsic invariant, namely the squared mean curvature on the right hand side. Then, we apply this result to get a B.-Y. Chen inequality between Chen's δ-invariant and squared mean curvature for θ-slant submanifolds in a generalized complex space form. Particular cases are put in a table in concise form. Next, we establish another general inequality for submanifolds of generalized complex space forms and then using it we obtain a B.-Y. Chen's inequality between Chen's $\delta(n_1, \cdots, n_k)$-invariant and squared mean curvature for slant submanifolds. In last, we again list particular cases in a table.

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J, \langle\,,\rangle)$. An almost Hermitian manifold becomes a *nearly Kähler manifold* ([6]) if $\left(\tilde{\nabla}_X J\right) X = 0$, and becomes a *Kähler manifold* if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle\,,\rangle$. An almost Hermitian manifold with J-invariant Riemannian curvature tensor $\tilde{R}$, that is,

$$\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \qquad X, Y, Z, W \in T\tilde{M},$$

is called an *RK-manifold* ([10]). All nearly Kähler manifolds belong to the class of RK-manifolds.

The notion of constant type was first introduced by A. Gray for a nearly Kähler manifold ([6]). An almost Hermitian manifold $\tilde{M}$ is said to have (*pointwise*) *constant type* if for each $p \in \tilde{M}$ and for all $X, Y, Z \in T_p\tilde{M}$ such that

$$\langle X, Y\rangle = \langle X, Z\rangle = \langle X, JY\rangle = \langle X, JZ\rangle = 0, \ \ \langle Y, Y\rangle = 1 = \langle Z, Z\rangle$$

we have

$$\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \tilde{R}(X,Z,X,Z) - \tilde{R}(X,Z,JX,JZ).$$

An RK-manifold $\tilde{M}$ has (pointwise) constant type if and only if there is a differentiable function α on $\tilde{M}$ satisfying ([10])

$$\tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \alpha\{\langle X,X\rangle \langle Y,Y\rangle - \langle X,Y\rangle^2 - \langle X,JY\rangle^2\}$$

for all $X, Y \in T\tilde{M}$. Furthermore, $\tilde{M}$ has *global constant type* if α is constant. The function α is called the constant type of $\tilde{M}$. An RK-manifold of constant holomorphic sectional curvature c and constant type α is denoted by $\tilde{M}(c,\alpha)$. For $\tilde{M}(c,\alpha)$ it is known that ([10])

$$\begin{aligned} 4\tilde{R}(X,Y)Z &= (c+3\alpha)\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\ &\quad + (c-\alpha)\{\langle X,JZ\rangle JY - \langle Y,JZ\rangle JX + 2\langle X,JY\rangle JZ\} \end{aligned}$$

for all $X, Y, Z \in T\tilde{M}$. If $c = \alpha$ then $\tilde{M}(c,\alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (a Kähler manifold of constant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifolds $\tilde{M}(c,\alpha)$ (with the constant type zero).

An almost Hermitian manifold $\tilde{M}$ is called a *generalized complex space form* $\tilde{M}(f_1, f_2)$ ([7]) if its Riemannian curvature tensor $\tilde{R}$ satisfies

$$\begin{aligned} \tilde{R}(X,Y)Z &= f_1\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\ &\quad + f_2\{\langle X,JZ\rangle JY - \langle Y,JZ\rangle JX + 2\langle X,JY\rangle JZ\} \end{aligned} \tag{1}$$

for all $X, Y, Z \in T\tilde{M}$, where f_1 and f_2 are smooth functions on $\tilde{M}$.

The Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. Here, we recall a number of Riemannian invariants ([4]) in a Riemannian manifold. Let M be a Riemannian manifold and L be a r-plane section of T_pM. Choose an orthonormal basis $\{e_1, \cdots, e_r\}$ for L. Let K_{ij} denote the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$. The scalar curvature τ of the r-plane section L is given by

$$\tau(L) = \sum_{1\le i<j\le r} K_{ij}. \tag{2}$$

Given an orthonormal basis $\{e_1, \cdots, e_n\}$ for T_pM, $\tau_{1\cdots r}$ will denote the scalar curvature of the r-plane section spanned by $e_1, \cdots, e_r$. If L is a 2-plane section then $\tau(L)$ reduces to the sectional curvature K of the plane section L. We denote by $K(\pi)$ the sectional curvature of M for a plane section π in T_pM, $p \in M$. The scalar curvature $\tau(p)$ of M at p

is the scalar curvature of the tangent space of M at p. Thus, the scalar curvature τ at p is given by

$$\tau(p) = \sum_{i<j} K_{ij}, \tag{3}$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis for T_pM and K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$. Chen's *δ-invariant* is defined by the following identity

$$\delta_M(p) = \tau(p) - \inf\{K(\pi) \mid \pi \text{ is a plane section } \subset T_pM\}, \tag{4}$$

which is certainly an intrinsic character of M.

For an integer $k \geq 0$, we denote by $\mathcal{S}(n,k)$ the finite set which consists of k-tuples $(n_1, \cdots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of all (unordered) k-tuples with $k \geq 0$ for a fixed positive integer n. For each k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$, we B.-Y. Chen introduced a Riemannian invariant $\delta(n_1, \cdots, n_k)$ defined by

$$\delta(n_1, \cdots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{5}$$

where $L_1, \cdots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \cdots, k$. For each $(n_1, \cdots, n_k) \in \mathcal{S}(n)$ we put

$$a(n_1, \cdots, n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1), \tag{6}$$

$$b(n_1, \cdots, n_k) = \frac{n^2\left(n + k - 1 - \sum_{j=1}^{k} n_j\right)}{2\left(n + k - \sum_{j=1}^{k} n_j\right)}. \tag{7}$$

For more details we refer to [4] and corresponding references therein.

Let M be an n-dimensional submanifold in a manifold $\tilde{M}$ equipped with a Riemannian metric $\langle\,,\rangle$. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$ for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle$. The mean curvature vector H is expressed by $nH = \operatorname{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\tilde{M}$ if $\sigma = 0$.

In a submanifold M of an almost Hermitian manifold, for a vector $0 \neq X_p \in T_pM$, the angle $\theta(X_p)$ between JX_p and the tangent space T_pM is called the *Wirtinger angle* of X_p. If the Wirtinger angle is independent of $p \in M$ and $X_p \in T_pM$, then M is called a *slant submanifold* ([1]). We put $JX = PX + FX$ for $X \in TM$, where PX (resp. FX) is the projection of JX on TM (resp. $T^{\perp}M$). Slant submanifolds of almost Hermitian manifolds are characterized by the condition $P^2 + \lambda^2 I = 0$ for some real number $\lambda \in [0,1]$. Invariant and anti-invariant ([11]) (or totally real) submanifolds are slant submanifolds with $\theta = 0$ ($F = 0$) and $\theta = \pi/2$ ($P = 0$) respectively. For more details about slant submanifolds we refer to [1].

3. B.-Y. Chen inequalities

First we state the following algebraic lemma from [2] for later uses.

LEMMA 3.1. *If $a_1, \cdots, a_n, a_{n+1}$ are $n+1$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + a_{n+1}\right),$$

then $2a_1a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Let M be a submanifold in an almost Hermitian manifold $\tilde{M}$. Let $\pi \subset T_pM$ be a plane section at $p \in M$. Then

$$\Theta(\pi) = \langle Pe_1, e_2\rangle^2 \tag{8}$$

is a real number in $[0,1]$, which is independent of the choice of orthonormal basis $\{e_1, e_2\}$ of π. Moreover, if $\tilde{M}$ is a generalized complex space form, then Gauss equation becomes ([8])

$$\begin{aligned} R(X,Y,Z,W) &= f_1\{\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\} \\ &+ f_2\{\langle X,PZ\rangle\langle PY,W\rangle - \langle Y,PZ\rangle\langle PX,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\} \\ &+ \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle \end{aligned} \tag{9}$$

for all $X,Y,Z,W \in TM$, where R is the curvature tensors of M. Thus, we are able to state the following Lemma.

LEMMA 3.2. *In an n-dimensional submanifold in a generalized complex space form $\tilde{M}(f_1, f_2)$, the scalar curvature and the squared mean curvature satisfy*

$$2\tau = n(n-1) f_1 + 3f_2 \|P\|^2 + n^2 \|H\|^2 - \|\sigma\|^2, \tag{10}$$

where

$$\|P\|^2 = \sum_{i,j=1}^{n} \langle e_i, Pe_j \rangle^2 \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle .$$

For a submanifold M in a real space form $R^m(c)$, B.-Y. Chen ([2]) gave the following

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)c.$$

He ([3]) also established the basic inequality for submanifold M in a complex space form $CP^m(4c)$ (respectively, $CH^m(4c)$, the complex hyperbolic space) of constant holomorphic sectional curvature $4c$ as follows:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n^2+2n-2)c$$

$$(\text{respectively,} \quad \delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)c).$$

Now, we prove the following basic inequality for later uses.

THEOREM 3.3. *Let M be an n-dimensional submanifold isometrically immersed in a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\tau - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2) f_1 + \frac{3}{2} f_2 \|P\|^2 - 3f_2 \Theta(\pi). \tag{11}$$

Equality in (11) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \cdots, e_{2m}\}$ of $T_p^{\perp}M$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) the shape operators $A_r \equiv A_{e_r}$, $r = n+1, \cdots, 2m$, take the following forms:

$$A_{n+1} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda+\mu & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda+\mu \end{pmatrix}, \tag{12}$$

$$(13)\qquad A_r=\begin{pmatrix} c_r & d_r & 0 & \cdots & 0\\ d_r & -c_r & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix},\qquad r=n+2,\cdots,2m.$$

Proof. Let $\pi\subset T_pM$ be a plane section. Choose an orthonormal basis $\{e_1,e_2,\cdots,e_n\}$ for T_pM and $\{e_{n+1},\cdots,e_{2m}\}$ for the normal space $T_p^{\perp}M$ at p such that $\pi=\mathrm{Span}\{e_1,e_2\}$ and the normal vector e_{n+1} is in the direction of the mean curvature vector H. We rewrite (10) as

(14)
$$\frac{1}{n-1}\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2=\sum_{i=1}^{n}\left(\sigma_{ii}^{n+1}\right)^2+\sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2+\rho,$$

where

$$(15)\qquad \rho=2\tau-\frac{n^2(n-2)}{n-1}\|H\|^2-n(n-1)f_1-3f_2\|P\|^2$$

and $\sigma_{ij}^r=\langle\sigma(e_i,e_j),e_r\rangle$, $i,j\in\{1,\cdots,n\}$; $r\in\{n+1,\cdots,2m\}$. Now, applying Lemma 3.1 to (14), we obtain

$$(16)\qquad 2\sigma_{11}^{n+1}\sigma_{22}^{n+1}\geq\sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2+\rho.$$

From (9) it also follows that

(17)
$$K(\pi)=\sigma_{11}^{n+1}\sigma_{22}^{n+1}-\left(\sigma_{12}^{n+1}\right)^2+\sum_{r=n+2}^{2m}\left(\sigma_{11}^{r}\sigma_{22}^{r}-\left(\sigma_{12}^{r}\right)^2\right)+f_1+3f_2\Theta(\pi).$$

Thus, from (16) and (17) we have

$$\begin{aligned} K(\pi)\ &\geq\ f_1+3f_2\Theta(\pi)+\frac{\rho}{2}+\sum_{r=n+1}^{2m}\sum_{j>2}\{(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2\}+\frac{1}{2}\sum_{i\neq j>2}(\sigma_{ij}^{n+1})^2\\ (18)\qquad &\quad+\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(\sigma_{ij}^r)^2+\frac{1}{2}\sum_{r=n+2}^{2m}(\sigma_{11}^r+\sigma_{22}^r)^2.\end{aligned}$$

In view of (15) and (18), we get (11).

If the equality in (11) holds, then the inequalities given by (16) and (18) become equalities. In this case, we have

$$\begin{aligned} &\sigma_{1j}^{n+1} = 0,\ \sigma_{2j}^{n+1} = 0,\ \sigma_{ij}^{n+1} = 0, \quad i \neq j > 2; \\ &\sigma_{1j}^{r} = \sigma_{2j}^{r} = \sigma_{ij}^{r} = 0,\ r = n+2, \cdots, 2m; \quad i,j = 3, \cdots, n; \\ (19) \qquad &\sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \cdots = \sigma_{11}^{2m} + \sigma_{22}^{2m} = 0. \end{aligned}$$

Now, we choose e_1 and e_2 so that $\sigma_{12}^{n+1} = 0$. Applying Lemma 3.1 we also have

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}. \tag{20}$$

Thus, choosing a suitable orthonormal basis $\{e_1, \cdots, e_{2m}\}$, the shape operators of M become of the forms given by (12) and (13). The converse is simple to observe. □

As an application, we prove a B.-Y. Chen inequality for θ-slant submanifolds in a generalized complex space form.

THEOREM 3.4. *For an n-dimensional $(n > 2)$ θ-slant submanifold M isometrically immersed in a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$, at every point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)f_1 + 3f_2\cos^2\theta\right\}. \tag{21}$$

Equality in (21) holds at $p \in M$ if and only if the shape operators of M in $\tilde{M}(f_1, f_2)$ at p take the forms given by (12) and (13).

Proof. Let M be an n-dimensional θ-slant submanifold M in an almost Hermitian manifold with $n \geq 3$, $n = 2l$. Let $p \in M$ and $\{e_i, \sec\theta Pe_i\}$, $i = 1, \cdots, l$, be an orthonormal basis of T_pM. Thus, we have $\|P\|^2 = n\cos^2\theta$. Choosing an orthonormal basis $\{e, \sec\theta Pe\}$ for any plane section $\pi \subset T_pM$, we have $\Theta(\pi) = \cos^2\theta$. Putting these values of $\|P\|^2$ and $\Theta(\pi)$ in (11), we get (21). □

In particular, the above Theorem provides the following Corollary.

COROLLARY 3.5. *We have the following table*:

Manifold	Submanifold	Inequality
$\tilde{M}(f_1, f_2)$	*totally real*	$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1) f_1\right\}$
$\tilde{M}(f_1, f_2)$	*invariant*	$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1) f_1 + 3f_2\right\}$
$\tilde{M}(c, \alpha)$	*θ-slant*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + (n+1)(c+3\alpha) + 3(c-\alpha)\cos^2\theta\right\}$
$\tilde{M}(c, \alpha)$	*totally real*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + (n+1)(c+3\alpha)\right\}$
$\tilde{M}(c, \alpha)$	*invariant*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + (n+4)c + 3n\alpha\right\}$
$\tilde{M}(c)$	*θ-slant*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + \left(n+1+3\cos^2\theta\right)c\right\}$
$\tilde{M}(c)$	*totally real*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + (n+1)c\right\}$
$\tilde{M}(c)$	*invariant*	$\delta_M \leq \frac{n-2}{8}\left\{\frac{4n^2}{n-1}\|H\|^2 + (n+4)c\right\}$
$R(c)$		$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right\}$

Let M be a submanifold of an almost Hermitian manifold. For an r-plane section $L \subset T_pM$ and for any orthonormal basis $\{e_1, \cdots, e_r\}$ of L, we put

$$\Psi(L_j) = \sum_{1 \leq i < j \leq r} \langle Pe_i, e_j \rangle^2. \tag{22}$$

LEMMA 3.6. *Let M be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Let $n_1, \cdots, n_k$ be integers ≥ 2 satisfying $n_1 < n$, $n_1 + \cdots + n_k \leq n$. For $p \in M$, let L_j be an n_j-plane section of T_pM, $j = 1, \cdots, k$. Then we have*

$$\tau - \sum_{j=1}^{k} \tau(L_j) \leq b(n_1, \cdots, n_k)\|H\|^2 + a(n_1, \cdots, n_k) f_1$$
$$+\frac{3}{2}\left\{\|P\|^2 - 2\sum_{j=1}^{k}\Psi(L_j)\right\} f_2. \tag{23}$$

Proof. Choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for T_pM and $\{e_{n+1}, \cdots, e_{2m}\}$ for the normal space $T_p^{\perp}M$ such that the mean curvature vector H is in the direction of the normal vector to e_{n+1}. We put

$$\begin{aligned}
&a_i = \sigma_{ii}^{n+1} = \langle \sigma(e_i, e_i), e_{n+1}\rangle, \qquad i = 1, \cdots, n,\\
&b_1 = a_1,\ b_2 = a_2 + \cdots + a_{n_1},\ b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \cdots,\\
&b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k},\\
&b_{k+2} = a_{n_1+\cdots+n_k+1}, \cdots,\ b_{\gamma+1} = a_n,
\end{aligned}$$

and denote by D_j, $j = 1, \cdots, k$ the sets

$$\begin{aligned}
&D_1 = \{1, \cdots, n_1\},\ D_2 = \{n_1 + 1, \cdots, n_1 + n_2\}, \cdots,\\
&D_k = \{(n_1 + \cdots + n_{k-1}) + 1, \cdots, (n_1 + \cdots + n_{k-1}) + n_k\}.
\end{aligned}$$

Let $L_1, \cdots, L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j$ n_j, defined by

$$\begin{aligned}
&L_1 = \mathrm{span}\{e_1, \cdots, e_{n_1}\},\ L_2 = \mathrm{span}\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}, \cdots,\\
&L_k = \mathrm{span}\{e_{n_1+\cdots+n_{k-1}+1}, \cdots, e_{n_1+\cdots+n_{k-1}+n_k}\}.
\end{aligned}$$

From (9) it follows that

$$\begin{aligned}
\tau(L_j) &= \frac{1}{2}\{n_j(n_j-1)f_1 + 6f_2\Psi(L_j)\}\\
&\quad + \sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}\left[\sigma_{\alpha_j\alpha_j}^r\sigma_{\beta_j\beta_j}^r - (\sigma_{\alpha_j\beta_j}^r)^2\right]. \qquad (24)
\end{aligned}$$

We rewrite (10) as

$$n^2\|H\|^2 = \left(\|\sigma\|^2 + \eta\right)\gamma, \qquad (25)$$

or equivalently,

(26)

$$\left(\sum_{i=1}^n \sigma_{ii}^{n+1}\right)^2 = \gamma\left(\sum_{i=1}^n(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^n(\sigma_{ij}^r)^2 + \eta\right)$$

where

$$\eta = 2\tau - 2b(n_1, \cdots, n_k)\|H\|^2 - n(n-1)f_1 - 3f_2\|P\|^2, \qquad (27)$$

$$\gamma = n + k - \sum_{j=1}^k n_j. \qquad (28)$$

The relation (26) implies that

$$\left(\sum_{i=1}^{\gamma+1} b_i\right)^2 = \gamma\left[\eta + \sum_{i=1}^{\gamma+1} {b_i}^2 + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^r\right)^2 -2\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} - \cdots - 2\sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k},\right],$$

with $\alpha_j, \beta_j \in D_j$, for all $j = 1, \cdots, k$. Applying Lemma 3.1 to the above relation, we have

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \geq \frac{1}{2}\left[\eta + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^r\right)^2\right],$$

which implies that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}\left[\sigma_{\alpha_j\alpha_j}^r\sigma_{\beta_j\beta_j}^r - \left(\sigma_{\alpha_j\beta_j}^r\right)^2\right] \geq \frac{\eta}{2} + \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{(\alpha,\beta)\notin D^2}\left(\sigma_{\alpha\beta}^r\right)^2 + \sum_{r=n+2}^{2m}\sum_{\alpha_j\in D_j}\left(\sigma_{\alpha_j\alpha_j}^r\right)^2,$$

where we denote by $D^2 = (D_1 \times D_1) \cup \cdots \cup (D_k \times D_k)$. Thus, we have

$$\frac{\eta}{2} \leq \sum_{j=1}^{k}\sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}\left[\sigma_{\alpha_j\alpha_j}^r\sigma_{\beta_j\beta_j}^r - (\sigma_{\alpha_j\beta_j}^r)^2\right]. \tag{29}$$

From (6), (24), (27) and (29), we obtain (23). □

B.-Y. Chen gave a general inequality for submanifolds in real space forms as follows ([5]).

THEOREM 3.7. *For any n-dimensional submanifold M in a real space form $R(c)$ and for any k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$, regardless of dimension and codimension, we have*

$$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k)\|H\|^2 + a(n_1, \cdots, n_k)\,c. \tag{30}$$

We extend the above result for submanifolds in a generalized complex space forms.

THEOREM 3.8. *Given an n-dimensional $(n \geq 3)$ θ-slant submanifold M, of a $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$, we* *have*

$$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) f_1 \\ + \frac{3}{2} \left(n - 2 \sum_{j=1}^{k} \left[\frac{n_j}{2} \right] \right) f_2 \cos^2 \theta. \tag{31}$$

Proof. Let M be an n-dimensional θ-slant submanifold M in an almost Hermitian manifold with $n \geq 3$, $n = 2l$. Then, we have $\|P\|^2 = n \cos^2 \theta$. Let $L_1, \cdots, L_k$ be k mutually orthogonal subspaces of T_pM with $\dim L_j = n_j$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM, such that

$$L_1 = \operatorname{span}\{e_1, \cdots, e_{n_1}\},\ L_2 = \operatorname{span}\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}, \cdots, \\ L_k = \operatorname{span}\{e_{n_1+\cdots+n_{k-1}+1}, \cdots, e_{n_1+\cdots+n_{k-1}+n_k}\}.$$

Choosing $e_2 = \sec\theta P e_1, \cdots, e_{2k} = \sec\theta P e_{2l-1}$, for $i = 1, 3, \cdots, 2l-1$, we get $\langle Pe_i, e_{i+1} \rangle = \cos\theta$. Thus, it follows that $\Psi(L_j) = \left[\frac{n_j}{2}\right] \cos^2\theta$ for all $j = 1, \cdots, k$. Putting these values of $\|P\|^2$ and $\Psi(L_j)$ in (23), we get (31). □

COROLLARY 3.9. *For a submanifold M in a manifold $\tilde{M}$, we have the following table*

$\tilde{M}$	M	**Inequality**
$\tilde{M}(f_1, f_2)$	TR	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) f_1$
$\tilde{M}(f_1, f_2)$	I	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) f_1$
		$+\frac{3}{2}\left(n - 2\sum_{j=1}^{k} m_j\right) f_2 \cos^2\theta,$
$\tilde{M}(c, \alpha)$	S	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{(c+3\alpha)}{4}$
		$+\frac{3}{8}\left(n - 2\sum_{j=1}^{k} m_j\right)(c-\alpha)\cos^2\theta$
$\tilde{M}(c, \alpha)$	TR	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{(c+3\alpha)}{4}$
$\tilde{M}(c, \alpha)$	I	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{(c+3\alpha)}{4}$
		$+\frac{3}{8}\left(n-2\sum_{j=1}^{k} m_j\right)(c-\alpha)$
$\tilde{M}(c)$	S	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{c}{4}$
		$+\frac{3}{8}\left(n - 2\sum_{j=1}^{k} m_j\right) c\cos^2\theta$
$\tilde{M}(c)$	TR	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{c}{4}$
$\tilde{M}(c)$	I	$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) \frac{c}{4}$
		$+\frac{3}{8}\left(n - 2\sum_{j=1}^{k} m_j\right) c$
$R(c)$		$\delta(n_1, \cdots, n_k) \leq b(n_1, \cdots, n_k) \|H\|^2 + a(n_1, \cdots, n_k) c,$

where "TR", "I" and "S" stand for totally real, invariant and θ-slant respectively.

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