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THE UNIFORM LOCAL ASYMPTOTIC NORMALITY: AN EMPIRICAL PROCESS THEORY APPROACH

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ABSTRACT. We investigate a uniform local asymptotic normality for likelihood ratio processes based on an independent and identically distributed local asymptotic problem. Our tool is an empirical process theory.

1. Introduction

It is known that a local asymptotic normality(LAN) property for the family of distributions, originated by Le Cam [5], often arises in a finite dimensional parametric statistical inference.

Sufficient conditions for LAN have been studied by many authors. See, for example, Fabian and Hannan [2] and Fabian and Hannan [3].

In this paper we investigate a uniform local asymptotic normality (ULAN) for an independent and identically distributed(IID) asymptotic problem. Our tool to develop ULAN is an empirical process theory.

In introducing ULAN, we use the setting of an asymptotic problem of Fabian and Hannan [4] where LAN and a strong local asymptotic normality(SLAN) are dealt with.

In this paper, for studying ULAN, we turn our view points of the likelihood ratios into the stochastic processes, not into the random variables.

In Section 2, we illustrate the concepts of the IID asymptotic problem and Cramer-regular conditions. We introduce *LAN process* and *ULAN process* based on the likelihood ratio processes indexed by a real line. We state the main result for the paper.

In Section 3, we provide the proof of the main result in Section 2.

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2. Preliminaries and the main result

We begin by illustrating the concepts of asymptotic problems.

DEFINITION 2.1. [4] $\mathcal{E} := (\Theta, P_\tau^n)$ is an asymptotic problem if for every n, $\Theta \subseteq \mathbb{R}$ and P_τ^n is a probability measure for every $\tau \in \Theta$. We say that $\mathcal{E} := (\theta, \Theta, P_\tau^n)$ is a local asymptotic problem at θ if $\theta \in \Theta$ is fixed.

DEFINITION 2.2. [4] An asymptotic problem $\mathcal{E} = (\Theta, P_\tau^n)$ is an IID asymptotic problem if there are probability spaces $(\Omega, \mathcal{T}, P_\tau)$ and $X := \{X_i : i \geq 1\}$ such that, for each $\tau \in \Theta$, P_τ is a probability measure on $(\Omega, \mathcal{T})$ and X is a sequence of IID random variables under P_τ and P_τ^n is the restriction of P_τ to the smallest σ-algebra generated by $\{X_1, \ldots, X_n\}$. We denote by $\mathcal{E} = (\Theta, P_\tau, X)$ the IID asymptotic problem generated by X. Similarly, we denote by $\mathcal{E} = (\theta, \Theta, P_\tau, X)$ the IID local asymptotic problem generated by X.

Consider the IID asymptotic problem $\mathcal{E} = (\Theta, P_\tau, X)$ generated by X. Throughout the paper we assume that the asymptotic problem satisfies the following Cramér regular conditions.

Let $\Theta \subseteq \mathbb{R}$. For each $\theta \in \Theta$, there exists a measure μ defined on $(\mathbb{R}, \mathcal{B})$ and P_θ has common density function $p(x;\theta)$ with respect to μ.

C1. The set of all x which satisfies $p(x;\theta) > 0$ does not depend on θ.

C2. The functions $p(x;\theta)$ are three times continuously differentiable with respect to θ and $\frac{\partial^3}{\partial\theta^3} l$ is uniformly bounded where $l(x;\cdot) := \ln p(x;\cdot)$.

C3. The integrals $\int a(x)p(x;\theta)\mu(dx)$ are possible to differentiate with respect to θ under the sign of the integral.

C4. For each $\theta \in \Theta$, $0 < I(\theta) < \infty$ where

$$I(\theta) := \int \frac{\left[\frac{\partial}{\partial\theta} p(x;\theta)\right]^2}{p(x;\theta)} \mu(dx) \text{ for each } \theta \in \Theta$$

is the Fisher's information.

We consider a sequence of real valued functions $\{\varphi_n(u,t) : (u,t) \in \Theta^2\}$ defined by

$$\varphi_n(u) := t + u/\sqrt{nI(t)}.$$

We write

$$\varphi_n := \varphi_n(u,t) \text{ if both } u \text{ and } t \text{ are fixed;}$$
$$\varphi_n(u) := \varphi_n(u,t) \text{ if } t \text{ is fixed.}$$

Throughout the paper we assume that $P^n_{\varphi_n(u,t)} << P^n_t$ for all t, $u \in \Theta$ and for all $n \in \mathbb{N}$.

We introduce the concepts of *the LAN process* and *the ULAN process* for the likelihood ratios.

DEFINITION 2.3. Given an IID asymptotic problem $\mathcal{E} = (\Theta, P_\tau, X)$, we consider a sequence of the random fields $\{Z_n(u,t) : (u,t) \in \Theta^2\}$ defined by

$$(2.1) \qquad Z_n(u,t) := \frac{dP^n_{\varphi_n(u,t)}}{dP^n_t}(X_1, \ldots, X_n) = \prod_{j=1}^{n} \frac{p(X_j; \varphi_n(u,t))}{p(X_j; t)}$$

based on the likelihood ratios. When we fix both u and t, we obtain a sequence $\{Z_n := Z_n(u,t)\}$ of random variables. We call it the LAN process. When we fix only t, we obtain a sequence $\{Z_n(u) := Z_n(u,t)\}$ of continuous time stochastic processes with respect to u. We call it the ULAN process.

REMARK 1. When we discuss a local asymptotic problem at θ, the sequence $\{Z_n(u,t)\}$ of random fields given in (2.1) boils down to a sequence of processes indexed by u.

Throughout the paper we consider a given IID local asymptotic problem $\mathcal{E} = (\theta, \Theta, P_\theta, X)$. Therefore, we fix $t = \theta$ from now on. We first consider the LAN process $\{Z_n\}$ in Definition 2.3.

We illustrate the notion of LAN based on the LAN process.

Define ρ_u on $\mathbb{R}$ by $\rho_u(x) = \exp\{ux - u^2/2\}$ and consider a standard normal random variable χ. Then the process $Z = \{Z(u) : u \in \Theta\}$ defined by

$$(2.2) \qquad Z(u) := \rho_u(\chi) = \exp\{u\chi - u^2/2\}$$

is a Gaussian. In particular, $Z(u)$ is a log normal random variable if u is fixed.

DEFINITION 2.4. Let $\{Z_n\}$ be the LAN process in Definition 2.3. An IID local asymptotic problem $\mathcal{E} = (\theta, \Theta, P_\tau, X)$ is said to obey LAN if for some sequence of numbers φ_n, φ_n is in Θ eventually,

$$\left\{ Z_n = \frac{dP^n_{\varphi_n}}{dP^n_\theta}(X_1, \ldots, X_n) \right\}$$

converges in distribution to a log normal random variable $Z(u)$ given by (2.2).

The Definition 2.4 is equivalent to say that, for some sequence of numbers φ_n, φ_n is in Θ eventually, the identity

$$Z_n = \frac{dP^n_{\varphi_n}}{dP^n_\theta}(X_1, \dots, X_n) = \exp\{\gamma_n(u) + S_n(u)\}\exp\{o_P(1)\}$$

is valid, where the sequence $\{\gamma_n(u)\}$ of random variables converges in distribution to a normal random variable, the sequence $\{S_n(u)\}$ of random variables converges to $-u^2/2$ in probability.

The following LAN for an IID local asymptotic problem is well known.

PROPOSITION 2.5. *Consider the LAN process $\{Z_n\}$. Then, under Cramer regular conditions, the IID local asymptotic problem $\mathcal{E} = (\theta, \Theta, P_\tau, X)$ obeys LAN.*

More explicitly, for each fixed u, and for $\varphi_n = \theta + u/\sqrt{nI(\theta)}$,

$$Z_n = \frac{dP^n_{\varphi_n}}{dP^n_\theta}(X_1, \dots, X_n) = \exp\{\Gamma_n u - u^2/2\}\exp\{o_P(1)\}$$

is valid, where

$$\Gamma_n := \frac{1}{\sqrt{nI(\theta)}}\sum_{j=1}^n \frac{\frac{\partial}{\partial\theta}p(X_j;\theta)}{p(X_j;\theta)}$$

converges in distribution to a standard normal variable.

The proof of Proposition 1 is based on a central limit theorem and a law of the large numbers.

In this paper we develop a generalization of Proposition 1 to a uniform convergent result. Our tool to develop the uniform convergent result is an empirical process theory.

For a $\psi : \Theta \to \mathbb{R}$, we let $||\psi|| := \sup_{u\in\Theta}|\psi(u)|$. Given a subset $\Theta \subseteq \mathbb{R}$, let $D(\Theta)$ be the space of cadlag functions defined on Θ. We endow the space $D(\Theta)$ with the Skorohod topology. We use the following weak convergence.

DEFINITION 2.6. [8] A sequence of $D(\Theta)$-valued random elements $\{Y_n : n \geq 1\}$ converges in distribution to a $D(\Theta)$-valued Borel measurable random element Y, denoted by $Y_n \Rightarrow Y$, if $Eg(Y) = \lim_{n\to\infty} Eg(Y_n)$ for all $g \in C(D(\Theta), ||\cdot||)$, where $C(D(\Theta)), ||\cdot||)$ is the set of real bounded, continuous functions.

We introduce the notion of ULAN based on the ULAN process.

DEFINITION 2.7. Let $\{Z_n(u)\}$ be the ULAN process in Definition 2.3. An IID local asymptotic problem $\mathcal{E} = (\theta, \Theta, P_\tau, X)$ is said to obey

ULAN if for some sequence of real valued functions $\varphi_n(u)$, $\varphi_n(u)$ is in Θ eventually

$$\{Z_n(u) = \frac{dP^n_{\varphi_n(u)}}{dP^n_\theta}(X_1, \dots, X_n) : u \in \Theta\} \Rightarrow Z = \{Z(u) : u \in \Theta\},$$

as random elements of $D(\Theta)$, where Z is the Gaussian process given by (2.2).

Our goal of the paper is to establish ULAN for an IID local asymptotic problem by developing the weak convergence of the ULAN process $\{Z_n(u)\}$.

Given an IID local asymptotic problem $\mathcal{E} = (\theta, \Theta, P_\theta, X)$, we next consider the ULAN process $\{Z_n(u)\}$ in Definition 2.3.

Then, by using Taylor expansion,

$$Z_n(u) \quad = \quad \exp\left\{\gamma_n(u) + S_n(u) + R_n(u)\right\}, \text{ where}$$

$$\gamma_n(u) \quad := \quad \frac{u}{\sqrt{nI(\theta)}} \sum_{j=1}^{n} \frac{\frac{\partial}{\partial\theta} p(X_j;\theta)}{p(X_j;\theta)}, \tag{2.3}$$

$$S_n(u) \quad := \quad \frac{u^2}{2nI(\theta)} \sum_{j=1}^{n} \frac{\partial^2}{\partial\theta^2} l(X_j;\theta), \text{ and} \tag{2.4}$$

$$R_n(u) := \frac{1}{2} \sum_{j=1}^{n} \int_\theta^{\varphi_n(u)} (\varphi_n(u) - \eta_\theta)^2 \frac{\partial^3}{\partial\theta^3} l(X_j;\eta_\theta)\mu(d\eta_\theta). \tag{2.5}$$

We are ready to state the main result.

THEOREM 2.8. *Consider the ULAN process $\{Z_n(u)\}$. Then, under Cramer regular conditions, the IID local asymptotic problem $\mathcal{E} = (\theta, \mathbb{R}, P_\tau, X)$ obeys ULAN.*

More explicitly, for a sequence $\varphi_n(u) = \theta + u/\sqrt{nI(\theta)}$ of real valued functions,

$$Z_n(u) \quad = \quad \frac{dP^n_{\varphi_n}}{dP^n_\theta}(X_1, \dots, X_n) = \exp\{\gamma_n(u) + S_n(u) + R_n(u)\}$$

is valid, where

$$\gamma_n \Rightarrow W = \{u\chi : u \in \mathbb{R}\} \text{ as random elements of } D(\mathbb{R}),$$

with $\{\gamma_n(u)\}$ given by (2.3),

$$\sup_{u\in\mathbb{R}} \left|S_n(u) + u^2/2\right| \to 0 \text{ almost surely,}$$

with $\{S_n(u)\}$ *given by* (2.4), *and*

$$\sup_{u\in\mathbb{R}} |R_n(u)| \to 0 \text{ almost surely,}$$

with $\{S_n(u)\}$ *given by* (2.5).

The proof will be given in the next section.

REMARK 2. Our approach can readily be applied to get a result for a general $\Theta \subseteq \mathbb{R}$ with some additional topological assumptions. But we do not pursue it here in a more concrete fashion.

REMARK 3. Compare the main result with Theorem 9.2.4 in Fabian and Hanann [4] where SLAN property for the IID asymptotic problem is discussed.

3. Proof of the main result

We first consider the case that Θ is a compact interval $\Theta = [0, 1]$. We begin by working the processes as the random elements of $D([0, 1])$, the space of the cadlag functions on a compact interval $[0, 1]$.

Consider a Gaussian process $W = \{W(u) : u \in [0, 1]\}$ defined by

$$W(u) := u\chi,$$

where χ is a standard normal random variable. Then the Gaussian process W has mean zero and covariance structure $Cov(W(u), W(v)) = uv$.

LEMMA 3.1. *Let* $\{\gamma_n(u) : u \in [0, 1]\}$ *be a sequence of processes*

$$\gamma_n(u) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{u}{\sqrt{I(\theta)}} \frac{\frac{\partial}{\partial\theta} p(X_j; \theta)}{p(X_j; \theta)},$$

as in (2.3). *Then*

$$\gamma_n \Rightarrow W \text{ as random elements of } D([0, 1]).$$

Proof. The proof consists of establishing the convergence of the finite dimensional distribution and the tightness of γ_n

CLAIM 3.2. *The finite dimensional distributions of* γ_n *converge to those of* W.

Write

$$\Gamma_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{I(\theta)}} \frac{\frac{\partial}{\partial\theta} p(X_j;\theta)}{p(X_j;\theta)},$$

and

$$\xi_j := \frac{1}{\sqrt{I(\theta)}} \frac{\frac{\partial}{\partial\theta} p(X_j;\theta)}{p(X_j;\theta)}.$$

Then the mean is given by

$$\begin{aligned} E\xi_j &= \int \frac{1}{\sqrt{I(\theta)}} \frac{\frac{\partial}{\partial\theta} p(x;\theta)}{p(x;\theta)} p(x;\theta)\mu(dx) \\ &= \frac{1}{\sqrt{I(\theta)}} \frac{\partial}{\partial\theta} \int p(x;\theta)\mu(dx) = 0, \end{aligned}$$

and the variance is given by

$$\begin{aligned} Var(\xi_j) &= E\xi_j^2 \\ &= \int \left[\frac{1}{\sqrt{I(\theta)}} \frac{\frac{\partial}{\partial\theta} p(x;\theta)}{p(x;\theta)} \right]^2 p(x;\theta)\mu(dx) \\ &= \frac{1}{I(\theta)} \int \frac{\left[\frac{\partial}{\partial\theta} p(x,y;\theta)\right]^2}{p(x,y;\theta)} \mu(dx) = 1. \end{aligned}$$

Then by the central limit theorem, γ_n converges in distribution to χ. Observe that for each $u \in [0,1]$

$$\gamma_n(u) = u\Gamma_n.$$

By the Cramer-wold argument, the finite dimensional distributions of γ_n converge to those of W.

CLAIM 3.3. *The process $\gamma_n = \{\gamma_n(u) : u \in [0,1]\}$ is tight in the sense that given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\limsup_{n\to\infty} P\left\{ \sup_{|u-v|<\delta} |\gamma_n(u) - \gamma_n(v)| > \epsilon \right\} < \epsilon.$$

Let $\epsilon > 0$. Since $\{\Gamma_n\}$ is convergent in distribution, $\{\Gamma_n\}$ is uniformly tight [6]. That is, there exists a number K such that

$$\limsup_{n\to\infty} P\{|\Gamma_n| > K\} < \epsilon. \tag{3.1}$$

Choose $\delta > 0$ so that $K < \epsilon/\delta$. Then

$$\begin{aligned}
P\left\{\sup_{|u-v|<\delta} |\gamma_n(u) - \gamma_n(v)| > \epsilon\right\} &= P\left\{|\Gamma_n| \sup_{|u-v|<\delta} |u-v| > \epsilon\right\} \\
&\leq P\left\{|\Gamma_n|\delta > \epsilon\right\} \\
&= P\left\{|\Gamma_n| > \epsilon/\delta\right\} \\
&\leq P\left\{|\Gamma_n| > K\right\}.
\end{aligned}$$

Therefore, by (3.1), we see that

$$\limsup_{n\to\infty} P\left\{\sup_{|u-v|<\delta} |\gamma_n(u) - \gamma_n(v)| > \epsilon\right\} \leq \limsup_{n\to\infty} P\left\{|\Gamma_n| > K\right\} < \epsilon.$$

The proof of the Claim 3.3 is completed. Tightness of $\{\gamma_n\}$ together with the finite dimensional distributions convergence implies, see Pollard [7], that $\gamma_n \Rightarrow W$ as random elements of $D([0,1])$. The proof of Lemma 3.1 is completed. □

LEMMA 3.4. *Let $\{S_n(u) : u \in [0,1]\}$ be a sequence of processes*

$$S_n(u) = \frac{1}{n}\sum_{j=1}^{n} \frac{u^2}{2I(\theta)} \frac{\partial^2}{\partial\theta^2} l(X_j;\theta)$$

as in (2.4). Then

$$\sup_{u\in[0,1]} \left|S_n(u) + u^2/2\right| \to 0 \text{ almost surely.}$$

Proof. Write

$$T_n := \frac{1}{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial\theta^2} l(X_j;\theta) - E\left[\frac{\partial^2}{\partial\theta^2} l(X_1,;\theta)\right].$$

Then, by the strong law of the large numbers, we get

$$T_n \to 0 \text{ almost surely.} \tag{3.2}$$

Notice that

$$\frac{\partial^2}{\partial\theta^2} l(X_1;\theta) = -\frac{(\frac{\partial}{\partial\theta} p(X_1;\theta))^2}{(p(X_1;\theta))^2} + \frac{\frac{\partial^2}{\partial\theta^2} p(X_1;\theta)}{p(X_1;\theta)}$$

and

$$
\begin{aligned}
E\frac{\frac{\partial^2}{\partial\theta^2}p(X_1;\theta)}{p(X_1;\theta)} &= \int \frac{\partial^2}{\partial\theta^2}p(x;\theta)\mu(dx) \\
&= \frac{\partial^2}{\partial\theta^2}\int p(x;\theta)\mu(dx) \\
&= 0.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
& E\left[\frac{\partial^2}{\partial\theta^2}l(X_1;\theta)\right] \\
&= -\int \left[\frac{\frac{\partial}{\partial\theta}p(x;\theta)}{p(x;\theta)}\right]^2 p(x;\theta)\mu(dx) \\
&= -\int \frac{\left[\frac{\partial}{\partial\theta}p(x;\theta)\right]^2}{p(x;\theta)}\mu(dx) \\
&= -I(\theta).
\end{aligned}
$$

Now, observe that for $u \in [0,1]$

$$
\begin{aligned}
\left|S_n(u) + u^2/2\right| &= \left|\frac{1}{n}\sum_{j=1}^n \frac{u^2}{2I(\theta)}\frac{\partial^2}{\partial\theta^2}l(X_j;\theta) + u^2/2\right| \\
&= \frac{1}{2I(\theta)}\left|\frac{1}{n}\sum_{j=1}^n \frac{\partial^2}{\partial\theta^2}l(X_j;\theta) + I(\theta)\right||u^2| \\
&= \frac{1}{2I(\theta)}|T_n||u^2|.
\end{aligned}
$$

Therefore, from (3.2), we get

$$
\sup_{u\in[0,1]}\left|S_n(u) + u^2/2\right| = \frac{|T_n|}{2I(\theta)}\sup_{u\in[0,1]}|u^2| \le \frac{|T_n|}{2I(\theta)} \to 0
$$

almost surely. The proof of Lemma 3.4 is completed. □

LEMMA 3.5. *Let $\{R_n(u) : u \in [0,1]\}$ be a sequence of processes*

$$
R_n(u) = \frac{1}{2}\sum_{j=1}^n \int_\theta^{\varphi_n(u)} (\varphi_n(u) - \eta_\theta)^2 \frac{\partial^3}{\partial\theta^3}l(X_j;\eta_\theta)\mu(d\eta_\theta),
$$

as in (2.5). *Then*

$$
||R_n|| \to 0
$$

almost surely.

Proof. Notice that $\frac{\partial^3}{\partial\theta^3} l$ is uniformly bounded by $2C$, say, and $u \le 1$. Then we see that

$$\begin{aligned}
|R_n(u)| &\le C\sum_{j=1}^{n}\int_{\theta}^{\varphi_n(u)}(\varphi_n(u)-\eta_\theta)^2\mu(d\eta_\theta)\\
&\le C\sum_{j=1}^{n}\int_{\theta}^{\varphi_n(u)}(\varphi_n(u)-\theta)^2\mu(d\eta_\theta)\\
&\le Cn\left(\frac{u}{\sqrt{nI(\theta)}}\right)^3\\
&\le C\frac{1}{n^{1/2}I(\theta)^{3/2}}.
\end{aligned}$$

Now, since $0 < I(\theta) < \infty$, we conclude

$$||R_n|| \le \frac{C}{n^{1/2}}\frac{1}{I(\theta)^{3/2}} \to 0.$$

The lemma is proved. □

The following Lemma is well known.

LEMMA 3.6. (See Billingsley [1] of Theorem 4.4) *Let $X_n \Rightarrow X$ as random elements of $D([0,1])$ and let $||Y_n - c||_1 \to 0$ almost surely. Then*

$$X_nY_n \Rightarrow cX \text{ as random elements of } D([0,1]).$$

Now, in order to work out the problem for the infinite interval $\Theta = \mathbb{R}$, we need the following Lemma 3.7.

LEMMA 3.7. (See Pollard [6] of Theorem V.23) *Let $X, X_1, X_2, \ldots$ be random elements of $D(\mathbb{R})$ with $P(X \in C) = 1$ for some separable set C. Then,*

$$X_n \Rightarrow X \text{ as random elements of } D(\mathbb{R}),$$

if and only if

$$X_n \Rightarrow X \text{ as random elements of } D([0,k])$$

for each fixed k.

Proof of Theorem 1. Applying last lemmas we get the Theorem 1. □

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