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SOME EXAMPLES OF HYPERBOLIC HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACE

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ABSTRACT. In the previous paper [6], the author constructed hyperbolic hypersurfaces of degree 2^n in the n-dimensional complex projective space for every $n \geq 3$. The purpose of this paper is to give some improvement of this result and to show some general methods of constructions of hyperbolic hypersurfaces of higher degree, which enable us to construct hyperbolic hypersurfaces of degree d in the n-dimensional complex projective space for every $d \geq 2 \times 6^n$.

§1. Introduction

Since S. Kobayashi conjectured that generic hypersurfaces of high degrees in n-dimensional projective space $P^n(\mathbb{C})$ are hyperbolic ([8]), many reseachers constructed various examples of hyperbolic hypersurfaces in $P^n(\mathbb{C})$. Here, we mean by a hyperbolic hypersurface a hypersurface which is hyperbolic in the sense of S. Kobayashi. For the case where $n = 3$, the first example was given by R. Brody and M. Green, who gave a hyperbolic hypersurface in $P^3(\mathbb{C})$ of even degree ≥ 50 ([2]). Afterwards, new types of hyperbolic hypersurfaces of degree d in $P^3(\mathbb{C})$ were given by A. Nadel for $d = 6p + 3 \geq 21$ ([10]), by J. El Goul for $d \geq 14$ ([7]), by J. P. Demailly ([3]) and by Y. T. Siu–S. K. Yeung ([13]) for $d \geq 11$ respectively. Moreover, J. P. Demailly-J. El Goul proved that a very generic hypersurface of degree ≥ 21 in $P^3(\mathbb{C})$ is hyperbolic in [4] and M. Shirosaki constructed a hyperbolic hypersurface of degree 10 in [12].

On the other hand, for the case where $n \geq 4$, in [9] K. Masuda–J. Noguchi proved that there exists a hyperbolic hypersurface of every degree $d \geq d(n)$ for some positive integer $d(n)$ depending only on n and

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gave some concrete examples of hyperbolic hypersurfaces in $P^n(\mathbb{C})$ for $n \leq 5$. Moreover, Siu–Yeung gave examples of hypersurfaces in $P^n(\mathbb{C})$ of degree $16(n-1)^2$ in [13].

By improving Shirosaki's argument in [12], the author succeeds in constructing concrete examples of hyperbolic hypersurfaces of degree 8 in $P^3(\mathbb{C})$, which is one of the lowest degrees among known hyperbolic hypersurfaces in $P^3(\mathbb{C})$ ([6]). More generally, he showed the following:

THEOREM 1.1. *There exists a family of hyperbolic hypersurfaces of degrees 2^n in $P^n(\mathbb{C})$.*

In this paper, we give some slightly improved version of Theorem 1.1 and, by using this, prove the following improvement of the above-mentioned result of Masuda–Noguchi:

THEOREM 1.2. *For every $d \geq 2 \times 6^n$ $(n \geq 3)$, there exists a family of hyperbolic hypersurfaces of degree d in $P^n(\mathbb{C})$.*

Recently, in their paper [11] B. Shiffman and M. Zaidenberg gave the following improvement of the above-mentioned result of Siu–Yeung:

THEOREM 1.3. *Let $m \geq 2n-1$. For every $d \geq (m-1)^2$ and generic linear functions $h_1, \ldots, h_m$ on $\mathbb{C}^{n+1}$, the hypersurface*

$$X_{n-1} := \left\{ z \in P^n(\mathbb{C}) : \sum_{j=1}^{m} h_j(z)^d = 0 \right\}$$

is hyperbolic. In particular, there exist algebraic families of hyperbolic hypersurfaces of degree d in $P^n(\mathbb{C})$ for every $d \geq 4(n-1)^2$.

We note that B. Shiffman and M. Zaidenberg only show the existence of hyperbolic hypersurfaces of the above type, but do not construct concrete examples.

The motivation for Theorem 1.2 was suggested by J. Noguchi. The author would like to thank him.

§2. Hyperbolic hypersurfaces of low degrees

We call a complex space M Brody hyperbolic if there is no nonconstant holomorphic map of $\mathbb{C}$ into M. As was shown by R. Brody ([1]), a compact complex manifold is Brody hyperbolic if and only if it is hyperbolic in the sense of S. Kobayashi. In the following, a compact hyperbolic space means a compact Brody hyperbolic space.

We first give the following:

DEFINITION 2.1. We call a homogeneous polynomial $Q(w)$ of degree d in $w = (w_0, w_1, \dots, w_n)$ an *H-polynomial* if it satisfies the following two conditions:

(H1) for an arbitrary holomorphic map f of $\mathbb{C}$ into $P^n(\mathbb{C})$ with a reduced representation $f := (f_0 : f_1 : \cdots : f_n)$, namely, a representation in terms of homogeneous coordinates on $P^n(\mathbb{C})$ with holomorphic functions f_i's without common zeros, if it satisfies

$$Q(f_0, f_1, \dots, f_n) = cf_0^d$$

for some $c \in \mathbb{C}$, then f is a constant.

(H2) if a holomorphic map f of $\mathbb{C}$ into $P^{n-1}(\mathbb{C})$ with a reduced representation $f := (f_1 : \cdots : f_n)$ satisfies the identity

$$Q(0, f_1, \dots, f_n) = cf_{n+1}^d$$

for some $c \in \mathbb{C}$ and some entire function f_{n+1}, then f is a constant.

THEOREM 2.2. *Let $Q(w_0, w_1, \dots, w_n)$ be an H-polynomial. Then,*

(i) $V := \{(w_0 : \cdots : w_n) : Q(w_0, \dots, w_n) = 0\}$ *is hyperbolic and*

(ii) *for* $W := \{(w_1 : \cdots : w_n) : Q(0, w_1, \dots, w_n) = 0\} \subset P^{n-1}(\mathbb{C})$, $P^{n-1}(\mathbb{C}) \setminus W$ *is Brody hyperbolic.*

Proof. Take an arbitrary holomorphic map $f : \mathbb{C} \to P^n(\mathbb{C})$ with a reduced representation $f := (f_0 : f_1 : \cdots : f_n)$ such that $f(\mathbb{C}) \subseteq V$ and so $Q(f_0, f_1, \dots, f_n) = 0$. By the assumption (H1) for the particular case $c = 0$, f is a constant, which gives the assertion (i). Next, to see (ii), take a holomorphic map f of $\mathbb{C}$ into $P^{n-1}(\mathbb{C})$ with a reduced representation $f = (f_1 : \cdots : f_n)$ such that $f(\mathbb{C}) \cap W = \emptyset$. Then, the entire function $Q(0, f_1, \dots, f_n)$ has no zeros. Therefore, we can find an entire function f_{n+1} such that $Q(0, f_1, \dots, f_n) = f_{n+1}^d$. We can apply (H2) to see the assertion (ii). □

For the case where $n = 2$ we can show the following:

THEOREM 2.3 ([6, §4]). *Let $Q(u_0, u_1, u_2)$ be a homogeneous polynomial of degree $d \geq 4$ and consider the associated inhomogeneous polynomial $\tilde{Q}(v, w) := Q(1, v, w)$. Assume that*

(C1) *the simultaneous equations*

$$\tilde{Q}_v(v, w) = \tilde{Q}_w(v, w) = 0$$

have only finitely many solutions, say $P_k := (v_k, w_k)$ $(1 \leq k \leq N)$,

(C2) $\tilde{Q}(P_k) \neq \tilde{Q}(P_\ell)$ *for* $1 \leq k < \ell \leq N$,

(C3) $\{(u_1, u_2) : Q_{u_i}(0, u_1, u_2) = 0, i = 0, 1, 2\} = \{(0, 0)\}$.

(C4) *the Hessian* $\varphi := \tilde{Q}_{vv}\tilde{Q}_{ww} - \tilde{Q}_{vw}^2 \neq 0$ *at* (v_k, w_k) $(1 \leq k \leq N)$.

Then, Q is an H-polynomial.

For the readers' convenience, we give here an outline of the proof.

To prove Theorem 2.3, we consider an algebraic curve

$$V_c : Q(u_0, u_1, u_2) = cu_0^d \ (c \in \mathbf{C})$$

in $P^2(\mathbb{C})$ for arbitrary $c \in \mathbb{C}$. By the assumption (C3), V_c has no singularities on the infinite line $H_\infty := \{u_0 = 0\}$ and, by (C1), the singularities of V_c are contained in the set $\{P_1, \ldots, P_N\}$. On the other hand, by (C2), there is at most one point P_k contained in V_c. Moreover, we can use the assumption (C4) to conclude that a possible singularity of V_c is at worst an ordinary singularity. We recall here Plücker's genus formula, by which the genus $g(V_c)$ is given by

$$g(V_c) = \frac{(d-1)(d-2)}{2} - \text{ the number of ordinary singularities } \geq 2.$$

We now use the following Picard theorem.

THEOREM 2.4. *Let V be a compact Riemann surface of genus greater than one. Then, there exists no nonconstant holomorphic map of $\mathbb{C}$ into V.*

This concludes that the holomorphic map $f : \mathbb{C} \to V_c$ is a constant and Q satisfies the condition (H1).

To see the condition (H2), we consider the following algebraic curve

$$\tilde{V}_c : Q(u_1, u_2) = cu_3^d \ (c \in \mathbb{C})$$

in $P^2(\mathbb{C})$. By the same argument as the above, we can prove the genus of $\tilde{V}_c$ is greater than one. Therefore, Q satisfies the condition (H1) and so it is an H-polynomial.

By a generic homogeneous polynomial of degree d in $n+1$ variables we mean an arbitrary polynomial in a Zariski dense set in the space of all nonzero homogeneous polynomials of degree d in $n+1$ variables which is canonically identified with the space $\mathbb{C}^{N(d,n)}$, where $N(d,n) = \binom{n+d}{d}$.

PROPOSITION 2.5. *A generic homogeneous polynomial of degree $d(\geq 4)$ satisfies the condition in Theorem 2.3 and so is an H-polynomial.*

For the proof, refer to the original paper [5].

For the case $n \geq 3$, we can prove the following:

THEOREM 2.6. *Let $Q(u_0, u_1, \ldots, u_n)$ be an H-polynomial of degree d_0 and $P(u_0, u_{n+1})$ a homogeneous polynomial of degree $d_1(\geq 3)$ such that $P(u_0, u_{n+1})$ and $\tilde{P}(w) := P(1, w)$ satisfy the conditions;*

(P1) $P(0, u_{n+1}) \not\equiv 0$,
(P2) $\tilde{P}'(w)$ *has only simple zeros* $\alpha_1, \alpha_2, \dots, \alpha_{d_1-1}$,
(P3) $\tilde{P}(\alpha_k) \neq \tilde{P}(\alpha_\ell)$ *for* $1 \leq k < \ell \leq d_1 - 1$.
For $m \geq 2$, *if* $d_1 := md_0$ *and*

$$\frac{2}{d_1 - 2} + \frac{1}{m} < 1,$$

then

$$R(u_0, u_1, \dots, u_n, u_{n+1}) := P(u_0, u_{n+1}) - Q(u_0, u_1, \dots, u_n)^m$$

is an H-polynomial.

REMARK. We can easily show that a generic homogeneous polynomial $P(w_0, w_1)$ of degree d_1 satisfies the conditions (P1), (P2) and (P3).

For the proof of Theorem 2.5, we recall the following result from the second main theorem of holomorphic curves in $P^n(\mathbb{C})$:

THEOREM 2.7. *Let f be a holomorphic map of $\mathbb{C}$ into $P^n(\mathbb{C})$ which is nondegenerate, namely, whose image is not included in any hyperplane in $P^n(\mathbb{C})$, and let $H_1, \dots, H_q$ be hyperplanes in $P^n(\mathbb{C})$ which are located in general position. Assume that there are positive integers $m_1, \dots, m_q$ such that each pull-back $f^*(H_j)$ of H_j considered as a divisor does not have positive multiplicity smaller than m_j. Then,*

$$\sum_{j=1}^{q} \left(1 - \frac{n}{m_j}\right) \leq n + 1.$$

For the proof, refer to p.112 in [5].

As a special case $n = 1$, we have the following theorem for meromorphic functions on $\mathbb{C}$.

COROLLARY 2.8. *Let φ be a nonconstant meromorphic function on $\mathbb{C}$. Assume that there are mutually distinct values $\alpha_1, \dots, \alpha_q$ and positive integers $m_1, \dots, m_q$ such that $f(z) - \alpha_j$ does not have zeros of multiplicity smaller than m_j respectively. Then,*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) \leq 2.$$

Proof of Theorem 2.6. Consider a holomorphic map $f : \mathbb{C} \to P^{n+1}(\mathbb{C})$ with a reduced representation $f := (f_0 : \cdots : f_{n+1})$ satisfying the identity

$$R(f_0, \ldots, f_{n+1}) = cf_0^{d_1}.$$

If $f_0 \equiv 0$, we have

$$Q(0, f_1, \ldots, f_n)^m = ef_{n+1}^{d_1}$$

and so

$$Q(0, f_1, \ldots, f_n) = e' f_{n+1}^{d_0}$$

for some constant e, e'. Hence, f is a constant by the assumption (H2) for Q. Otherwise, setting $\varphi := f_{n+1}/f_0$, we have

$$\tilde{P}(\varphi) - c = Q\left(1, \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right)^m. \tag{1}$$

On the other hand, if $\tilde{P}(w) - c$ has a multiple zero w_0, then w_0 is equal to some α_j and its multiplicity is not larger than two by the assumption (P2). Moreover, by (P3), there is at most one α_j with $c = P(\alpha_j)$. These imply that $\tilde{P}(w) - c$ has at least $d_1 - 2$ simple zeros $\beta_1, \beta_2, \ldots, \beta_M$, where $\beta_j \neq \alpha_k$ for any j, k. Take a zero of $\varphi - \beta_j$. Since the right hand side of (1) has no zero of multiplicity smaller than m, $\varphi(z) - \beta_j$ has also no zero of multiplicity smaller than m. Therefore, if φ is not a constant, then we have

$$(d_1 - 2)\left(1 - \frac{1}{m}\right) \leq 2$$

by Corollary 2.8. This contradicts the assumption and we conclude that φ is a constant. We write $f_{n+1} = c_1 f_0$ for some constant c_1. Then, we have

$$c_2 f_0^{md_0} = Q(f_0, f_1, \ldots, f_n)^m$$

and so

$$c_3 f_0^{d_0} = Q(f_0, f_1, \ldots, f_n)$$

for some nonzero constants c_2 and c_3. By the condition (H1) for Q, we can conclude that f is a constant. Therefore, R satisfies the condition (H1).

Now, to check the condition (H2) we consider a holomorphic map f of $\mathbb{C}$ into $P^n(\mathbb{C})$ with a reduced representation $f = (f_1 : \cdots : f_{n+1})$ and a holomorophic function f_{n+2} satisfying the identity

$$R(0, f_1, \ldots, f_{n+1}) = P(0, f_{n+1}) - Q(0, f_1, \ldots, f_n)^m = e_1 f_{n+2}^{d_1}$$

for some constant e_1. Here, we may assume that $e_1 \neq 0$ and $f_{n+2} \neq 0$. Because, otherwise, the condition (H2) is reduced to (H1). We now use the assumption (P1), whence we have

$$e_1 f_{n+2}^{d_1} - e_2 f_{n+1}^{d_1} = Q(0, f_1, \ldots, f_n)^m$$

for some constant e_2. Consider the polynomial $F(X,Y) := e_1 X^{d_1} - r_2 Y^{d_1}$. Obviously, F can be factored into the product of d_1 linearly independent linear forms with multiplicity one. In the similar manner as in the check of (H1), we can show that there is a nonzero constant e_3 such that

$$e_3 f_{n+1}^{d_0} = Q(0, f_1, \ldots, f_n).$$

By the assumption (H2) for Q, we can conclude that f is a constant. The proof of Theorem 2.6 is completed. □

Theorems 2.3 and 2.5 give the following:

THEOREM 2.9. *For any $m \geq 2$ and $d \geq 4$, there is a family of hyperbolic hypersurfaces of degrees $md (\geq 8)$ in $P^3(\mathbb{C})$, which is parametrized with $md - 1 + (d+1)(d+2)/2$ analytically independent parameters.*

Proof. By Theorem 2.3 there are H-polynomials of degree $d \geq 4$ in variables u_0, u_1, u_2, which are parametrized with $(d+1)(d+2)/2$ analytically independent parameters. On the other hand, generic homogeneous polynomials in variables u_0, u_{n+1} satisfying the condition (P1), (P2) and (P3) of degree md are parametrized by $md + 1$ analytically independent parameters. Therefore, the homogeneous polynomial R given in Theorem 2.6 are parametrized by $md + (d+1)(d+2)/2$ analytically independent parameters, because the terms of degree md with respect to u_0 are duplicate. By Theorem 2.6, these polynomials are H-polynomials and by Theorem 2.2 the zero locus of these polynomials are hyperbolic hypersurfaces in $P^3(\mathbb{C})$. Since two homogeneous polynomials define the same hypersurface if and only if one is identical with a nonzero constant multiple of the other, a family of hyperbolic hypersurfaces in $P^3(\mathbb{C})$ obtained in this way is parametrized with $md - 1 + (d+1)(d+2)/2$ analytically independent parameters. The proof of Theorem 2.9 is complete. □

Using Theorem 2.6 repeatedly and with similar arguments as above, we have the following theorem:

THEOREM 2.10 ([5, Theorem 2.4]). *For each $n \geq 3$ there is a family of hyperbolic hypersurfaces of degree 2^n in $P^n(\mathbb{C})$, which is parametrized with $2^{n+1} + 6$ analytically independent parameters, and a hypersurface W of degree 2^n in $P^{n-1}(\mathbb{C})$ such that $P^{n-1}(\mathbb{C}) \setminus W$ is Brody hyperbolic.*

We can also construct many hyperbolic hypersurfaces in the complex projective space. For example, by Theorem 2.3, we can construct an H-polynomial of degree 5 in three variables and, by the use of the case $m = 3$ of Theorem 2.6 repeatedly, hyperbolic hypersurfaces of degree $5 \times 3^{n-2}$ in $P^n(\mathbb{C})$, which are used later.

§3. Hyperbolic hypersurfaces of high degrees

In this section, we construct some examples of hyperbolic hypersurfaces in $P^n(\mathbb{C})$ of high degrees.

A polynomial $F(x_0, x_1, \dots, x_m)$ in $x_0, x_1, \dots, x_m$ is called a weighted homogeneous polynomial with weights $(d_0, d_1, \dots, d_m)$ if $F(t_0^{d_0}, t_1^{d_1}, \dots, t_m^{d_m})$ is a homogeneous polynomial in $t_0, \dots, t_m$.

PROPOSITION 3.1. *With a given polynomial*

$$F := \sum_{i_1, \dots, i_m} a_{i_1 \cdots i_m} x_1^{i_1} \cdots x_m^{i_m}$$

in $x_1, \dots, x_m$ associate the weighted homogeneous polynomial

$$F^*(x_0, x_1, \dots, x_m) := \sum_{i_1, \dots, i_m} a_{i_1 \cdots i_m} x_0^{d - i_1 d_1 - \cdots - i_m d_m} x_1^{i_1} \cdots x_m^{i_m}$$

in $(x_0, x_1, \dots, x_n)$ with weights $(1, d_1, \dots, d_m)$ for some positive integers d_i, where $d := \max\{i_1 d_1 + \cdots + i_m d_m : a_{i_1 \cdots i_m} \neq 0\}$. Assume that

(i) *$F^*(0, x_1, \dots, x_m)$ consists of only one nonzero monomial, namely, we can write*

$$F^*(0, x_1, \dots, x_m) = c x_1^{j_1} \dots x_m^{j_m}$$

for some nonzero constant c and nonnegative integers $j_1, \dots, j_m$.

(ii) *if $F(\varphi_1, \dots, \varphi_m) = 0$ for meromorphic functions φ_i on $\mathbb{C}$, then at least one of the φ_i's is a constant.*

Then, for arbitrary H-polynomials $Q_i(w_0, \dots, w_n)$ of degree d_i ($1 \leq i \leq m$), the hypersurface

$$V := \left\{ w = (w_0 : \dots : w_n) : w_0^d F\left(Q_1(w)/w_0^{d_1}, \dots, Q_m(w)/w_0^{d_m}\right) = 0 \right\}$$

in $P^n(\mathbb{C})$ is hyperbolic.

Proof. Consider a holomorphic map f of $\mathbb{C}$ into $V(\subset P^n(\mathbb{C}))$ with a reduced representation $f := (f_0 : f_1 : \cdots : f_n)$. If $f_0 \equiv 0$, then we have

$$Q_1(0, f_1, \dots, f_n)^{j_1} \cdots Q_m(0, f_1, \dots, f_n)_m^{j_m} = 0.$$

We note here that $d = j_1 d_1 + \cdots + j_m d_m (> 0)$. Therefore, $Q_{j_k}(0, f_1, \ldots, f_n) \equiv 0$ for some i_k by the assumption (i), whence f is a constant by the assumption (H1) for Q_{j_k}. We now assume that $f_0 \not\equiv 0$. Then,

$$F(\varphi_1, \ldots, \varphi_n) = 0$$

for meromorphic functions $\varphi_i := Q_i(f_0, f_1, \ldots, f_n)/f_0^{d_i}$, whence some φ_{i_0} is a constant by the assumption (ii). So, we can write

$$Q_{i_0}(f_0, f_1, \ldots, f_n) = c f_0^{d_{i_0}}$$

for some constant c. This concludes that f is a constant by (H1). The proof of Proposition 3.1 is completed. □

We give an example satisfying the assumptions of Proposition 3.1.

PROPOSITION 3.2. *Set $F(x, y) := x^p + y^p + x^r y^s + 1$ for positive integers p, r, s. Assume that*

$$p \le t := \min(r, s), \quad \frac{6}{p} + \frac{2}{t} < 1. \tag{2}$$

Then, $F(x, y)$ satisfies the assumptions (i) *and* (ii) *of Proposition 3.1 for arbitrary positive integers d_1 and d_2.*

For the proof, we use the following consequence of Theorem 2.6:

THEOREM 3.3. *Let $f_0, f_1, \ldots, f_n$ be nonzero holomorphic functions on $\mathbb{C}$ which satisfy the identity*

$$f_0^p + f_1^p + \cdots + f_n^p = 0$$

for some integer $p \ge n^2$. Consider the partition

$$\{0, 1, \ldots, n\} = I_1 \cup I_2 \cup \cdots \cup I_k$$

such that i and j are in the same class I_ℓ if and only if f_i/f_j is a constant. Then

$$\sum_{i \in I_\ell} f_i^p = 0$$

for every ℓ.

For the proof, refer to [5, Proposition 3.4.7].

Proof of Proposition 3.2. By definition, the weighted homogeneous polynomial with weights (d_1, d_2) associated with F is given by

$$F^*(x_0, x_1, x_2) := x_0^{d - pd_1} x_1^p + x_0^{d - pd_2} x_2^p + x_1^r x_2^s + x_0^d,$$

where $d := rd_1 + sd_2 (> \max(pd_1, pd_2))$. Since

$$F^*(0, x_1, x_2) = x_1^r x_2^s,$$

the assumption (i) holds. To see (ii), take nonconstant meromorphic functions φ, ψ with

$$F(\varphi,\psi) = \varphi^p + \psi^p + \varphi^r\psi^s + 1 = 0. \tag{3}$$

We write

$$\varphi := \frac{f_1}{f_0},\ \psi := \frac{f_2}{f_0}$$

with entire functions f_0, f_1 and f_2 having no common zero. Consider the holomorphic map

$$\Phi := (f_0^p : f_1^p : f_2^p) : \mathbb{C} \to P^2(\mathbb{C})$$

and hyperplanes

$$H_1 := \{w_0 = 0\}, H_2 := \{w_1 = 0\}, H_3 := \{w_2 = 0\}, H_4 := \{w_0 + w_1 + w_2 = 0\}$$

in $P^2(\mathbb{C})$, which are located in general position. Then, the pull-backs $\Phi^*(H_j)$ of H_j $(j = 1, 2, 3)$, considered as divisors, are given by zeros of entire functions f_j^p counted with multiplicities. So, the multiplicities of zeros are divided by p, whence they have no positive multiplicities smaller than p. Now, consider the divisor $\Phi^*(H_4)$, which is given by zeros of the function

$$H := f_0^p + f_1^p + f_2^p.$$

On the other hand, we can rewrite the identity (3) as

$$H = -f_1^r f_2^s f_0^{p-(r+s)}. \tag{4}$$

Take a point $z_0 \in f^{-1}(H_4)$. If $f_0(z_0) \neq 0$, then we have $f_1(z_0) = 0$ or $f_2(z_0) = 0$. In any case, H has a zero with multiplicity at least t at z_0. Assume that $f_0(z_0) = 0$. If $f_1(z_0) = 0$, then $f_2(z_0) \neq 0$, because f_0, f_1, f_2 have no common zero. This implies $\sum_{j=0}^2 f_j(z_0)^p \neq 0$, which contradicts $z_0 \in f^{-1}(H_4)$. Then, $f_1(z_0) \neq 0$ and, similarly, $f_2(z_0) \neq 0$. This is impossible. In fact, the left hand side of (4) is holomorphic but the right hand side is not holomorphic by the assumption $p < r + s$. In conclusion, $\Phi^*(H_4)$ does not have positive multiplicity smaller than t. Then, there are constants c_0, c_1, c_2 with $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that

$$c_0\varphi^p + c_1\psi^p + c_2 = 0.$$

Because, otherwise, Theorem 3.3 gives

$$3\left(1 - \frac{2}{p}\right) + \left(1 - \frac{2}{t}\right) \leq 3,$$

which contradicts the assumption. If $c_2 = 0$, then φ is a constant multiple of ψ. Combining this with the identity (3), we can conclude φ and ψ are constants. Otherwise, we have $c_0 f_0^p + c_1 f_1^p + c_2 f_2^p = 0$. Since

$p \geq 4$ by the assumption, Φ is a constant by Theorem 3.3. The proof of Proposition 3.2 is completed. □

By Propositions 3.1 and 3.2, we have the following:

PROPOSITION 3.4. *Let $Q_i(w)$ be H-polynomials of degree d_i ($i = 1, 2$) in $n+1$ variables $w = (w_0, w_1, \dots, w_n)$ and p, r, s positive integers satisfying the condition* (2). *Then, the zero locus of the polynomial*

$$R(w) := Q_1(w)^p w_0^{d-pd_1} + Q_2(w)^p w_0^{d-pd_2} + w_0^d - Q_1(w)^r Q_2(w)^s$$

is a hyperbolic hypersurface in $P^n(\mathbb{C})$ of degree $d := rd_1 + sd_2$.

This gives the following detailed statement of Theorem 1.2:

THEOREM 3.5. *For each $n \geq 3$ we can take a positive integer $d(n)$ such that there are hyperbolic hypersurfaces of degree d for every $d \geq d(n)$ in $P^n(\mathbb{C})$. Here, for example, we can take*

$$(5) \quad d(n) := 9(2^n + 5 \times 3^{n-2}) + 2^n(5 \times 3^{n-2} - 1) + 5 \times 3^{n-2}(2^n - 1),$$

which is not larger than 2×6^n.

For the proof of Theorem 3.5, we give the following Lemma:

LEMMA 3.6. *Let d_1 and d_2 be mutually prime positive integers. For an arbitrarily given positive integer m_0, every integer d with*

$$(6) \qquad d \geq m_0(d_1 + d_2) + d_1(d_2 - 1) + d_2(d_1 - 1)$$

can be written as $d = rd_1 + sd_2$ with $r, s \geq m_0$.

Proof. We denote by n_0 the right hand side of (6) and consider an arbitrary integer $d \geq n_0$. Take integers t, ℓ such that $d - n_0 = td_1 + \ell$ and $0 \leq \ell < d_1$. We claim that ℓ can be written as

$$(7) \qquad \ell = rd_1 + sd_2$$

with integers r, s satisfying the conditions $|r| < d_2$ and $|s| < d_1$. To see this, we take integers r' and s' such that $\ell = r'd_1 + s'd_2$ by the use of the assumption that d_1 and d_2 are mutually prime. We write these numbers as

$$r' = ud_2 + r_1, \; s' = vd_1 + s_1$$

with integers u, v, r_1 and s_1, where $0 \leq r_1 < d_2$ and $0 \leq s_1 < d_1$. Here, we may assume that $\ell > 0$ and $(r_1, s_1) \neq (0, 0)$, because otherwise (7) is obvious. Then, we have

$$(8) \qquad \ell = (u + v)d_1 d_2 + r_1 d_1 + s_1 d_2.$$

Here, since $0 \leq \ell < d_1$, we easily see $u + v \leq 0$. For the case $u + v = 0$, we have (7) for $r := r_1$ and $s := s_1$. If $u + v \leq -2$, then

$$\ell = |\ell| \geq 2d_1d_2 - (r_1d_1 + r_2d_2) \geq 2d_1d_2 - (d_2 - 1)d_1 - (d_1 - 1)d_2 > d_1.$$

This is a contradiction. For the case $u + v = -1$, we rewrite (8) as

$$\ell = r_1d_1 + (s_1 - d_1)d_2 = (r_1 - d_2)d_1 + s_1d_2.$$

Since $|s_1 - d_1| < d_1$ when $s_1 > 0$ and $|r_1 - d_2| < d_2$ when $r_1 > 0$, we have (7).

By (7), we have

$$\begin{aligned} d &= n_0 + td_1 + rd_1 + sd_2 \\ &= m_0(d_1 + d_2) + d_1(d_2 - 1) + d_2(d_1 - 1) + rd_1 + sd_2 \\ &= (m_0 + d_2 - 1 + r)d_1 + (m_0 + d_1 - 1 + s)d_2. \end{aligned}$$

Since $r := m_0 + d_2 - 1 + r$ and $s := m_0 + d_1 - 1 + s$ are not less than m_0, we have Lemma 3.6.

Proof of Theorem 3.5. To this end, for each $n(\geq 3)$ we set $d_1(n) := 2^n$ and $d_2(n) := 5 \times 3^{n-2}$. As is mentioned in the previous section, we can find H-polynomials Q_1 and Q_2 of degree $d_1(n)$ and $d_2(n)$ respectively. Define $d(n)$ by (5). By Lemma 3.6, we can write every $d \geq d(n)$ as $d = rd_1(n) + sd_2(n)$ with $r, s \geq m_0 := 9$, because $d_1(n)$ and $d_2(n)$ are mutually prime. Set $p := 8$ and consider the polynomial R defined in Proposition 3.4. Then, these r, s, p satisfy the condition (2) and hence we can apply Proposition 3.1 to find a homogeneous polynomial R of degree d such that

$$V := \{R = 0\}$$

is a hyperbolic hypersurface in $P^n(\mathbb{C})$.

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