J. Korean Math. Soc. **40** (2003), No. 4, pp. 577–593

COMPLEX ANALYSIS AND THE FUNK TRANSFORM

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Abstract. The Funk transform is defined by integrating a function on the two-sphere over its great circles. We use complex analysis to invert this transform.

Introduction

In 1917 Radon [19] introduced a transform $f \mapsto Rf$ for f a suitable real-valued function on $\mathbb{R}^2$ by

$$(Rf)(L) = \int_L f$$

for L a straight line in $\mathbb{R}^2$. Thus, Rf is a function defined on the set of straight lines in $\mathbb{R}^2$. See, for example, [12] for a review. There are many variations on this theme in real integral geometry.

One such variation was already introduced in 1913 by Funk [10]. Its definition is just like the Radon transform except that $\mathbb{R}^2$ is replaced by the round sphere S^2 and great circles play the role of straight lines. Funk proved that a smooth function f on S^2 lies in the kernel of this transformation if and only if f is odd (see [13] for a modern treatment and a discussion of Funk's motivation in constructing Zoll metrics on the sphere).

Our aim, in this article, is to show how the Funk transform $\mathcal{F}$ acting on smooth functions may be inverted using complex analysis. More precisely, we shall show that an inverse transform $\mathcal{T}$ arises as a result of solving a certain $\overline{\partial}$-problem on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$ (where $\overline{\partial}$ is the Cauchy-Riemann operator). We obtain the following explicit formula. Using

Received October 1, 2002.
2000 Mathematics Subject Classification: Primary 53C65; Secondary 32W05, 44A12.
Key words and phrases: Funk, Penrose, Radon, Zoll.
This research was supported by the Australian Research Council and the Isaac Newton Institute for Mathematical Sciences.

standard coördinates (x, y, z) on $\mathbb{R}^3$,

$$(1) \qquad (\mathcal{T}\phi)\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{4\pi}\int_{S^2} \log\left|\frac{x}{y}\right| \frac{\partial^2\phi}{\partial x^2},$$

where ϕ is extended off $S^2 \subset \mathbb{R}^3$ as a function homogeneous of degree -1 in the sense that $\phi(\lambda U) = \lambda^{-1}\phi(U)$, for $\lambda > 0$. There are many similar formulae in the literature (see, for example, [11]) and we are not claiming any original contribution here. From our point of view, the pleasing and original aspect of this particular formula is its unexpected derivation from complex analysis. The fact that $\mathcal{F}$ is invertible on smooth functions (without recourse to a particular inversion formula) is also well-known. It may be derived from the theory of spherical harmonics (see [13, Appendix A]).

The motivation for this article is a link between the Radon transform in some generality and a transform in complex geometry introduced, independently, around 1967 by Andreotti and Norguet [1], Penrose [17], and Schmid [20]. (A version of the transform was found in 1904 by Bateman [2].) See, for example, [8] for a review of this 'Penrose transform'. Our approach to the Funk transform has its roots in understanding this link [5]. This aspect, however, will be entirely suppressed in this article.

Given this motivation, it almost goes without saying that this simple case generalises considerably. Indeed, our investigations originally concentrated on the X-ray transform (see, for example, [14]), the real analogue of the classical Penrose transform. Much of this article applies mutatis mutandis, as discussed at the end. The X-ray transform has also been investigated from this point of view by Sparling [21] and Woodhouse [22]. A general link between real and complex integral geometry is presented in [3, 4, 9]. A link in terms of $\mathcal{D}$-modules is also available [6, 7]. Rather than embark on any general discussion, however, we prefer to concentrate on what we believe to be the simplest non-trivial example.

This article is organized as follows. In §1 we define the Funk transform and establish some of its elementary properties. In §2 we discuss the transform $\mathcal{T}$ and prove that it provides a *right* inverse: $\mathcal{F} \circ \mathcal{T} = \mathrm{Id}$. In §3 we obtain (1) by solving a $\overline{\partial}$-problem on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$. As a result of this interpretation, we prove that $\mathcal{T}$ is also a *left* inverse: $\mathcal{T} \circ \mathcal{F} = \mathrm{Id}$.

We shall use the standard index notation of differential geometry. Greek indices will run over the range $1, 2, 3$, so that X^α denotes a vector in $\mathbb{R}^3$ or $\mathbb{C}^3$. Naïvely, one may regard X^α as listing the components of X, but it is preferable to regard α as an *abstract* index in the sense of

Penrose [18]. The Einstein summation convention

$$X^\alpha U_\alpha := \sum_{\alpha=1}^{3} X^\alpha U_\alpha$$

is then interpreted as the natural pairing between vectors and covectors.

We would like to thank Robin Graham for helpful observations.

1. Definitions and preliminaries

By *smooth* we shall always mean infinitely differentiable. Suppose $f(X)$ is a smooth function on $\mathbb{R}^3 \setminus \{0\}$ homogeneous of degree -2, in the sense that

$$f(\lambda X) = \lambda^{-2} f(X), \quad \text{for } \lambda > 0.$$

For linearly independent $\xi, \eta \in \mathbb{R}^3$, consider the function

$$\phi(\xi, \eta) = \frac{1}{2\pi} \oint f(u\xi + v\eta)\,(u\,dv - v\,du), \tag{2}$$

where the integral is taken around any smooth closed curve in the (u, v)-plane of winding number one about the origin. The homogeneity of f implies that the form $f(u\xi + v\eta)\,(u\,dv - v\,du)$ is closed, so ϕ is well-defined independent of choice of curve. A change of variables shows that

$$\phi(a\xi + b\eta, c\xi + d\eta) = \frac{1}{|ad - bc|}\phi(\xi, \eta)$$

so ϕ is a well-defined function of $\xi \wedge \eta$ homogeneous in the sense that

$$\phi(\lambda \xi \wedge \eta) = |\lambda|^{-1} \phi(\xi \wedge \eta), \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\}.$$

Fix a volume form $\epsilon_{\alpha\beta\gamma}$ on $\mathbb{R}^3$ and let $U_\alpha = \epsilon_{\alpha\beta\gamma}\xi^\beta \eta^\gamma$. Then ϕ is a function of U_α and

$$\phi(\mu U_\alpha) = \frac{1}{|\mu|}\phi(U_\alpha).$$

In other words, ϕ is homogeneous of degree -1 on $(\mathbb{R}^3)^* \setminus \{0\}$ and even. On the other hand, the transform $f \mapsto \phi$ clearly kills odd functions. This is the *Funk transform*

$$\mathcal{F} : \left\{ \begin{array}{l} \text{smooth even functions on} \\ \mathbb{R}^3 \setminus \{0\}, \text{ homogeneous of} \\ \text{degree } -2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{smooth even functions on} \\ \mathbb{R}^3 \setminus \{0\}, \text{ homogeneous of} \\ \text{degree } -1 \end{array} \right\}.$$

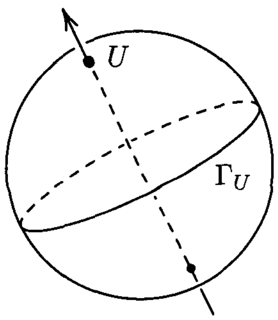
Of course, a homogeneous function on $\mathbb{R}^3$ is equivalent, by restriction, to a function on S^2, the unit sphere in $\mathbb{R}^3$. This gives the more classical form of the Funk transform:

$$\{\text{smooth even functions on } S^2\} \longrightarrow \{\text{smooth even functions on } S^2\}.$$

This formulation is more geometric—if $\epsilon_{\alpha\beta\gamma}$ is the standard volume form on $\mathbb{R}^3$, the point U is on the unit sphere, and ξ and η are also unit vectors so that U, ξ, η forms an oriented orthonormal basis, then (2) may be rewritten as

$$\phi(U) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\theta\,\xi + \sin\theta\,\eta)\,d\theta.$$

This is an integral over great circles:



$$\phi(U) = \frac{1}{2\pi}\oint_{\Gamma_U} f\,ds,$$

where ds is arclength.

In this classical form, the Funk transform is clearly SO(3)-invariant. Our formulation in terms of homogeneous functions is manifestly invariant under a larger group, namely $\mathrm{SL}(3,\mathbb{R})$. This invariance may be expressed infinitesimally—if $P_\alpha{}^\beta$ is trace-free, then

$$P_\alpha{}^\beta U_\beta \frac{\partial}{\partial U_\alpha}\mathcal{F}(f) = -\mathcal{F}\left(P_\alpha{}^\beta X^\alpha \frac{\partial}{\partial X^\beta} f\right).$$

On the other hand, by Euler's relation for homogeneous functions,

$$\begin{aligned} U_\alpha \frac{\partial}{\partial U_\alpha}\mathcal{F}(f) &= -\mathcal{F}(f) \\ -\frac{\partial}{\partial X^\alpha}(X^\alpha f) &= -3f - X^\alpha\frac{\partial}{\partial X^\alpha} f = -3f + 2f = -f \end{aligned}$$

so, in general,

$$U_\beta \frac{\partial}{\partial U_\alpha}\mathcal{F}(f) = \mathcal{F}\left(-\frac{\partial}{\partial X^\beta}(X^\alpha f)\right). \tag{3}$$

This suggests trying to define a transform $\mathcal{G}$ on smooth functions homogeneous of degree -1 by

$$U_\beta\,\mathcal{G}(g) := \mathcal{F}\left(\frac{\partial g}{\partial X^\beta}\right).$$

In other words, we are trying to define a smooth function $\psi(U)$ by

$$U_\beta\,\psi(U) = \frac{1}{2\pi}\oint \frac{\partial g}{\partial X^\beta}(u\xi + v\eta)\,(u\,dv - v\,du), \tag{4}$$

where, as always, $U_\beta = \epsilon_{\beta\gamma\delta}\xi^\gamma\eta^\delta$. To see that this is a bona fide definition, it suffices to check that the right hand side of this equation is annihilated by contraction with ξ^β and η^β. For fixed ξ and η, the function $g(u\xi + v\eta)$ is homogenous in the (u,v)-coördinates of degree -1. Using Euler's relation, it follows that

$$d(ug) = \frac{\partial g}{\partial v}(u\,dv - v\,du) = \eta^\beta\frac{\partial g}{\partial X^\beta}(u\,dv - v\,du).$$

Therefore,

$$\eta^\beta\frac{1}{2\pi}\oint \frac{\partial g}{\partial X^\beta}(u\xi + v\eta)\,(u\,dv - v\,du) = \frac{1}{2\pi}\oint d(ug) = 0,$$

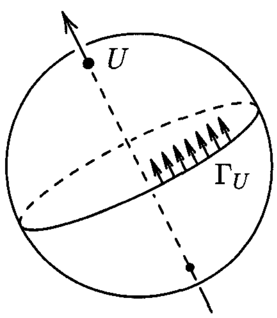
as required. Contracting the right hand side of (4) with ξ^β is similar. Clearly, (4) characterises an odd function ψ and, if g is even, then ψ vanishes. To summarise, (4) defines a transform

$$\mathcal{G} : \left\{\begin{array}{l}\text{smooth odd functions on}\\ \mathbb{R}^3\setminus\{0\}\text{, homogeneous of}\\ \text{degree } -1\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{smooth odd functions on}\\ \mathbb{R}^3\setminus\{0\}\text{, homogeneous of}\\ \text{degree } -2\end{array}\right\}.$$

The infinitesimal invariance of $\mathcal{F}$ specified by (3) can now be separated into two equations:

$$U_\beta\,\mathcal{G}(g) = \mathcal{F}\left(\frac{\partial g}{\partial X^\beta}\right) \quad\text{and}\quad \mathcal{G}(X^\alpha f) = -\frac{\partial}{\partial U_\alpha}\mathcal{F}(f). \tag{5}$$

There is also a geometric interpretation of $\mathcal{G}$:



$$\psi(U) = \frac{1}{2\pi}\oint_{\Gamma_U}\frac{\partial g}{\partial n}\,ds,$$

where $\partial g/\partial n$ is the indicated normal derivative to Γ_U and ds is arclength.

2. An inverse

Suppose $\rho(U)$ is a smooth function on $\mathbb{R}^3 \setminus \{0\}$ homogeneous of degree -3. Let d^2U denote the differential form $\epsilon^{\alpha\beta\gamma} U_\alpha \, dU_\beta \wedge dU_\gamma$ on $\mathbb{R}^3$. A calculation shows that $\rho(U)\, d^2U$ is a closed 2-form on $\mathbb{R}^3 \setminus \{0\}$. Let Σ denote a smooth embedded two-sphere in $\mathbb{R}^3 \setminus \{0\}$ of degree one. Then

$$\operatorname{mass} \rho := \frac{1}{4\pi} \int_\Sigma \rho(U)\, d^2U$$

is independent of choice of Σ. We shall suppose that this mass vanishes. Let us further restrict Σ to be everywhere transverse to the Euler field and consider the integral

$$h(X) = \frac{1}{4\pi} \int_\Sigma \log |X^\alpha U_\alpha| \rho(U)\, d^2U, \tag{6}$$

for $X \in \mathbb{R}^3$. Since $\int_0^1 \log t \, dt$ converges, $\log |X^\alpha U_\alpha|$ is an integrable function on Σ and elementary estimates show that, in fact, $h(X)$ is a continuous function of $X \in \mathbb{R}^3 \setminus \{0\}$. Also, $h(X)$ is even and homogeneous of degree zero:

$$\begin{aligned} h(\lambda X) &= \frac{1}{4\pi} \int_\Sigma \log |\lambda X^\alpha U_\alpha| \rho(U)\, d^2U \\ &= \frac{1}{4\pi} \int_\Sigma \log |X^\alpha U_\alpha| \rho(U)\, d^2U + \log|\lambda| \frac{1}{4\pi} \int_\Sigma \rho(U)\, d^2U \\ &= \frac{1}{4\pi} \int_\Sigma \log |X^\alpha U_\alpha| \rho(U)\, d^2U \;=\; h(X). \end{aligned}$$

The definition of $h(X)$ depends on Σ. However, if we consider

$$h(X) - h(Y) = \frac{1}{4\pi} \int_\Sigma \log \left| \frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha} \right| \rho(U)\, d^2U, \tag{7}$$

and notice that the integrand is Lie propagated by the Euler vector field, it follows that the right hand side does not depend on Σ. Thus, (6) or (7) defines $h(X)$ up to an additive constant.

LEMMA 1. *The function $h(X)$ defined above is smooth on $\mathbb{R}^3 \setminus \{0\}$.*

Proof. The formula (7) is manifestly $\mathrm{SL}(3, \mathbb{R})$-invariant. Infinitesimally, this invariance shows that h is differentiable and

$$X^\beta \frac{\partial h}{\partial X^\gamma}(X) - Y^\beta \frac{\partial h}{\partial Y^\gamma}(Y) = -\frac{1}{4\pi} \int_\Sigma \log \left| \frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha} \right| \frac{\partial}{\partial U_\beta}(U^\gamma \rho)\, d^2U \tag{8}$$

(compare (3)). In particular, the derivatives of h are given by integrals of the same form as is h itself. Iterating this conclusion, it follows that h is smooth. □

Suppose ψ is a smooth function homogeneous of degree -2 on $\mathbb{R}^3 \setminus \{0\}$. Then,

$$\begin{aligned} &d(\psi\epsilon^{\beta\gamma\delta}U_\beta\, dU_\gamma) \\ &= \frac{\partial\psi}{\partial U_\alpha}\epsilon^{\beta\gamma\delta}U_\beta\, dU_\alpha \wedge dU_\gamma + \psi\epsilon^{\beta\gamma\delta}dU_\beta \wedge dU_\gamma \\ &= \frac{1}{2}\frac{\partial\psi}{\partial U_\alpha}U_\alpha\epsilon^{\beta\gamma\delta}dU_\beta \wedge dU_\gamma - \frac{1}{2}\frac{\partial\psi}{\partial U_\delta}d^2U + \psi\epsilon^{\beta\gamma\delta}dU_\beta \wedge dU_\gamma \\ &= -\ \frac{1}{2}\frac{\partial\psi}{\partial U_\delta}d^2U. \end{aligned}$$

Therefore, by Stokes' theorem, $\partial\psi/\partial U_\delta$ has zero mass and we may consider

$$h^\delta(X) - h^\delta(Y) = \frac{1}{4\pi}\int_\Sigma \log\left|\frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha}\right| \frac{\partial\psi}{\partial U_\delta}\, d^2U,$$

an integral of the form (7) (of course, $h^\delta(X)$ is vector-valued but this does not affect our analysis and, in particular, Lemma 1 is still valid). Equation (8) now reads

$$X^\beta\frac{\partial h^\delta}{\partial X^\gamma}(X) - Y^\beta\frac{\partial h^\delta}{\partial Y^\gamma}(Y) = -\frac{1}{4\pi}\int_\Sigma \log\left|\frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha}\right| \frac{\partial}{\partial U_\beta}(U^\gamma\frac{\partial\psi}{\partial U_\delta})\, d^2U.$$

Contracting over γ and δ yields

(9) $$X^\beta g(X) - Y^\beta g(Y) = -\frac{1}{4\pi}\int_\Sigma \log\left|\frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha}\right| \frac{\partial\psi}{\partial U_\beta}\, d^2U,$$

where $g(X) = -\frac{1}{2}\partial h^\gamma/\partial X^\gamma$. It is clear that (9) defines g uniquely. Indeed, if we choose an inner product on $\mathbb{R}^3$, take X and Y to be orthonormal, and contract with X, then

$$g(X) = -\frac{1}{4\pi}\int_{S^2} \log\left|\frac{X^\alpha U_\alpha}{Y^\alpha U_\alpha}\right| X_\beta\frac{\partial\psi}{\partial U_\beta}\, d^2U.$$

Thus, with standard coördinates on $\mathbb{R}^3$,

(10) $$g\begin{pmatrix}1\\0\\0\end{pmatrix} = -\frac{1}{4\pi}\int_{S^2} \log\left|\frac{x}{y}\right| \frac{\partial\psi}{\partial x},$$

for example. Lemma 1 shows that g is smooth. Also notice that g is an odd function and, if ψ is even, then g vanishes. Let us write $\mathcal{S}$ for the transform $\psi \mapsto g$. Thus,

$$\mathcal{S}:\left\{\begin{array}{l}\text{smooth odd functions on}\\ \mathbb{R}^3\setminus\{0\},\text{ homogeneous of}\\ \text{degree } -2\end{array}\right\}\longrightarrow\left\{\begin{array}{l}\text{smooth odd functions on}\\ \mathbb{R}^3\setminus\{0\},\text{ homogeneous of}\\ \text{degree } -1\end{array}\right\}.$$

Invariance under $\mathrm{SL}(3,\mathbb{R})$ implies that

$$X^\alpha\frac{\partial}{\partial X^\beta}\mathcal{S}(\psi)=-\mathcal{S}\left(\frac{\partial}{\partial U_\alpha}(U_\beta\psi)\right). \tag{11}$$

Now suppose ϕ is a smooth even function homogeneous of degree -1 on $\mathbb{R}^3\setminus\{0\}$. Then, $\partial\phi/\partial U_\gamma$ is a smooth odd function, homogeneous of degree -2 and so we may consider $\mathcal{S}(\partial\phi/\partial U_\gamma)$. According to (11),

$$X^\alpha\frac{\partial}{\partial X^\beta}\mathcal{S}(\frac{\partial\phi}{\partial U^\gamma})=-\mathcal{S}\left(\frac{\partial}{\partial U_\alpha}\left(U_\beta\frac{\partial\phi}{\partial U_\gamma}\right)\right)$$

and, contracting over β and γ yields

$$X^\alpha f(X)=-\mathcal{S}\left(\frac{\partial\phi}{\partial U_\alpha}\right)$$

where $f(X)=-\partial(\mathcal{S}(\partial\phi/\partial U_\gamma))/\partial X^\gamma$. Incorporating the definition of $\mathcal{S}$ more explicitly, we have shown that

$$X^\alpha X^\beta f(X)-Y^\alpha Y^\beta f(Y)=\frac{1}{4\pi}\int_\Sigma\log\left|\frac{X^\gamma U_\gamma}{Y^\gamma U_\gamma}\right|\frac{\partial^2\phi}{\partial U_\alpha\partial U_\beta}\,d^2U, \tag{12}$$

defines a smooth function f homogeneous of degree -2. Writing $\mathcal{T}$ for the transform $\phi\mapsto f$, we have

$$\mathcal{T}:\left\{\begin{array}{l}\text{smooth even functions on}\\ \mathbb{R}^3\setminus\{0\},\text{ homogeneous of}\\ \text{degree } -1\end{array}\right\}\longrightarrow\left\{\begin{array}{l}\text{smooth even functions on}\\ \mathbb{R}^3\setminus\{0\},\text{ homogeneous of}\\ \text{degree } -2\end{array}\right\}.$$

The definition of $\mathcal{T}$ and the invariance of $\mathcal{S}$ described by (11) may be combined into the following two equations

$$X^\alpha\mathcal{T}(\phi)=-\mathcal{S}\left(\frac{\partial\phi}{\partial U_\alpha}\right)\quad\text{and}\quad\mathcal{T}(U_\beta\psi)=\frac{\partial}{\partial X^\beta}\mathcal{S}(\psi) \tag{13}$$

(compare (5)). A main result of this article is:

THEOREM 1. *The transforms $\mathcal{F}$ and $\mathcal{T}$ are mutually inverse as are the transforms $\mathcal{G}$ and $\mathcal{S}$.*

Suppose ψ is a smooth odd function on $\mathbb{R}^3\setminus\{0\}$ homogeneous of degree -2. Then, using (13) and (5),

$$(\mathcal{F}\circ\mathcal{T})(U_\beta\psi)=\mathcal{F}\left(\frac{\partial}{\partial X^\beta}\mathcal{S}(\psi)\right)=U_\beta(\mathcal{G}\circ\mathcal{S})(\psi).$$

Thus, if we can show that $\mathcal{T}$ is a right inverse for $\mathcal{F}$, then the corresponding statement for $\mathcal{G}$ and $\mathcal{S}$ will follow. Similarly, if f is a smooth even function on $\mathbb{R}^3 \setminus \{0\}$ homogeneous of degree -2, then

$$(\mathcal{S} \circ \mathcal{G})(X^\alpha f) = -\mathcal{S}\left(\frac{\partial}{\partial U_\alpha}\mathcal{F}(f)\right) = X^\alpha(\mathcal{T} \circ \mathcal{F})(f).$$

Thus, if we know that $\mathcal{S}$ is a left inverse for $\mathcal{G}$, then the corresponding statement for $\mathcal{T}$ and $\mathcal{F}$ will follow. We shall spend the rest of this section showing that $\mathcal{T}$ is indeed a right inverse for $\mathcal{F}$. That $\mathcal{S}$ is a left inverse for $\mathcal{G}$ will be a consequence of the link with complex analysis to be developed in the following section (see Proposition 2).

PROPOSITION 1. *The transform $\mathcal{T}$ is a right inverse to the Funk transform $\mathcal{F}$.*

Proof. Suppose ϕ is a smooth even function homogeneous of degree -1 on $\mathbb{R}^3 \setminus \{0\}$. Write $f = \mathcal{T}(\phi)$ as in (12). Fix an inner product on $\mathbb{R}^3$ so that we can view both and ϕ and f as smooth functions on the round sphere S^2. It suffices to show that

$$\mathcal{F} \circ \mathcal{T}(\phi)\begin{pmatrix}0\\0\\1\end{pmatrix} = \phi\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{14}$$

Both $\mathcal{F}$ and $\mathcal{T}$ are SO(3)-invariant. In particular, they are invariant under rotations about the z-axis. By averaging, it follows that it suffices to check that (14) holds for ϕ which are invariant under such rotations. Using standard coordinates on $\mathbb{R}^3$, equation (12) gives

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{4\pi}\int_{S^2} \log\left|\frac{x}{y}\right| \frac{\partial^2 \phi}{\partial x^2},$$

as in (1). In spherical polar coördinates, $\phi|_{S^2}$ is a function of latitude only. This allows one to compute explicitly and after integration by parts (twice), conclude that

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \phi\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The details of this calculation will be omitted. By rotational invariance, f has the same value at *any* point on the equator. Since the Funk transform of f at a pole is obtained by averaging f over the equator, (14) follows. □

3. A link with complex analysis

Define a mapping

$$\begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & \mathbb{R}^3 \\ \Cup & & \Cup \\ Z^\alpha = \xi^\alpha + i\eta^\alpha & \longmapsto & U_\alpha := \epsilon_{\alpha\beta\gamma}\xi^\beta\eta^\gamma. \end{array}$$

Notice that if Z is replaced by λZ, then U is replaced by $\lambda\overline{\lambda}U$. Also $U = 0$ if and only if Z is a (possibly) complex multiple of a real 3-vector. Thus, $Z^\alpha \mapsto U_\alpha$ defines a mapping

$$\pi : \mathbb{CP}_2 \setminus \mathbb{RP}_2 \longrightarrow S^2$$

where S^2 is the space of rays in $\mathbb{R}^3 \setminus \{0\}$ (and can be identified with the round sphere in the presence of an inner product).

If $\psi(U)$ is a smooth odd function homogeneous of degree -2 on $\mathbb{R}^3 \setminus \{0\}$, then we may compose with the mapping above to obtain $\psi(Z)$, homogeneous of degree $(-2, -\overline{2})$ in the sense that

$$\psi(\lambda Z) = \lambda^{-2}\overline{\lambda}^{-2}\psi(Z).$$

Thus, $\psi(Z)$ may be regarded as a smooth section of a appropriate homogeneous line bundle on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$. Now

$$\omega := \tfrac{1}{4}\psi(Z)U_\beta\, d\overline{Z}^\beta = \tfrac{i}{8}\psi(Z)\epsilon_{\alpha\beta\gamma}Z^\alpha\overline{Z}^\beta\, d\overline{Z}^\gamma$$

is a $(0,1)$-form on $\mathbb{C}^3$ homogeneous of degree -1. It descends to give a $(0,1)$-form on $\mathbb{CP}_2 \setminus \mathbb{RP}_2$ homogeneous of degree -1 which we shall also denote by ω. It is easy to check that this form is $\overline{\partial}$-closed and it is natural to ask whether it is $\overline{\partial}$-exact. Even though $H^1(\mathbb{CP}_2 \setminus \mathbb{RP}_2, \mathcal{O}(-1)) \neq 0$, this is indeed the case:

LEMMA 2. *The following integral*

$$G(Z) = \frac{1}{8\pi}\int_\Sigma \frac{\psi(V)}{Z^\alpha V_\alpha}\, d^2V \quad \textit{for } [Z] \in \mathbb{CP}_2 \setminus \mathbb{RP}_2 \tag{15}$$

defines a smooth function G homogeneous of degree -1 satisfying $\overline{\partial}G = \omega$. Here, as in §2, we have chosen Σ an arbitrary smoothly embedded two-sphere in $\mathbb{R}^3 \setminus \{0\}$ of degree one everywhere transverse to the Euler field.

Proof. As in §2, a short calculation shows that the integrand is Lie propagated by the Euler vector field. The integral is therefore independent of choice of Σ. The singularity in the integrand is like $1/(x + iy)$ in the complex plane and is therefore integrable.

The fibres of π may be parameterised by choosing linearly independent $\rho, \sigma \in \mathbb{R}^3$ and considering $Z = \rho + \zeta\sigma$ for ζ in the upper half-plane. The singularity in the integrand is like $1/(x+\zeta y)$ and its derivative with respect to ζ has a singularity like $y/(x+\zeta y)^2$ both of which are integrable. Hence, we may differentiate under the integral sign to conclude that G is holomorphic on each fibre. These fibres are hemispheres of standard $\mathbb{CP}_1$ cycles in $\mathbb{CP}_2$. All other such $\mathbb{CP}_1$'s intersect $\mathbb{RP}_2$ in a single point and may be parameterised as follows. Choose $\rho, \sigma, \tau \in \mathbb{R}^3$ linearly independent and let $Z = \rho + i\sigma - \zeta\tau$ for $\zeta \in \mathbb{C}$. If $\overline{\partial} G = \omega$, then

$$\frac{\partial G}{\partial \overline{\zeta}} = \frac{\partial G}{\partial \overline{Z}^\alpha} \frac{\partial \overline{Z}^\alpha}{\partial \overline{\zeta}} = -\tau^\alpha \frac{\partial G}{\partial \overline{Z}^\alpha} = -\tfrac{1}{4}\tau^\alpha U_\alpha \psi(U).$$

Conversely, if this equation holds for all choices of ρ, σ, and τ, then $\overline{\partial} G = \omega$. Invariance under $\mathrm{SL}(3, \mathbb{R})$ implies that it suffices to check this equation when ρ, σ, and τ are the standard basis vectors in $\mathbb{R}^3$ and Σ is the round sphere S^2. In this case, writing $\zeta = p + iq$ gives

$$U_\alpha = \epsilon_{\alpha\beta\gamma}(\rho^\beta - p\tau^\beta)(\sigma^\gamma - q\tau^\gamma) = p\rho_\alpha + q\sigma_\alpha + \tau_\alpha.$$

Thus, it suffices to show that if ψ is a smooth odd function on $\mathbb{R}^3 \setminus \{0\}$ homogeneous of degree -2, then

$$G(\zeta) = \frac{1}{8\pi} \int_{S^2} \frac{\psi(x,y,z)}{x + iy - \zeta z} \tag{16}$$

is a smooth function satisfying

$$\frac{\partial G}{\partial \overline{\zeta}} = -\tfrac{1}{4}\psi(p, q, 1). \tag{17}$$

We may parameterise a hemisphere of S^2 by setting

$$x = \frac{s}{\sqrt{1+s^2+t^2}} \qquad y = \frac{t}{\sqrt{1+s^2+t^2}} \qquad z = \frac{1}{\sqrt{1+s^2+t^2}}.$$

The standard area element is $ds\,dt/(1+s^2+t^2)^{3/2}$ in these coördinates. Since the integrand of (16) is even we may rewrite

$$\begin{aligned} G(\zeta) &= \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\psi(\frac{s}{\sqrt{1+s^2+t^2}}, \frac{t}{\sqrt{1+s^2+t^2}}, \frac{1}{\sqrt{1+s^2+t^2}})\, ds\, dt}{(s+it-\zeta)(1+s^2+t^2)} \\ &= -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\psi(s,t,1) ds\, dt}{\zeta - (s+it)}. \end{aligned}$$

The fundamental solution of the Laplacian in $\mathbb{R}^2$ is $\frac{1}{4\pi}\log(\zeta\bar{\zeta})$. Since

$$\frac{\partial}{\partial\zeta}\log(\zeta\bar{\zeta}) = \frac{1}{\zeta} \quad \text{and} \quad \frac{\partial^2}{\partial\bar{\zeta}\partial\zeta} = \frac{1}{4} \times \text{Laplacian},$$

equation (17) follows immediately. □

LEMMA 3. *The equation $\bar{\partial}G = \omega$ uniquely determines G.*

Proof. Since $\mathbb{RP}_2$ is a totally real submanifold of $\mathbb{CP}_2$, any holomorphic function defined on $\Omega \setminus \mathbb{RP}_2$, for Ω an open subset of $\mathbb{CP}_2$, extends uniquely to Ω. Since this is a local observation, the same is true for holomorphic sections of a line bundle. Thus,

$$\Gamma(\mathbb{CP}_2 \setminus \mathbb{RP}_2, \mathcal{O}(-1)) = \Gamma(\mathbb{CP}_2, \mathcal{O}(-1)) = 0$$

and the result follows. □

Now suppose that g is a smooth odd function homogeneous of degree -1 on $\mathbb{R}^3 \setminus \{0\}$ and that $\psi = \mathcal{G}(g)$ as characterised by equation (4).

LEMMA 4. *In this case,*

$$G(Z) = G(\xi + i\eta) = \frac{1}{4\pi}\oint \frac{g(u\xi + v\eta)}{u + iv}(u\,dv - v\,du) \tag{18}$$

Proof. Differentiating under the integral sign and using the chain rule,

$$\begin{aligned}\frac{\partial G}{\partial \bar{Z}^\beta} &:= \frac{1}{2}\left(\frac{\partial G}{\partial \xi^\beta} + i\frac{\partial G}{\partial \eta^\beta}\right)\\ &= \frac{1}{8\pi}\oint \frac{\partial g}{\partial X^\beta}(u\xi + v\eta)\,(u\,dv - v\,du) = \tfrac{1}{4}U_\beta\psi(U).\end{aligned}$$

Contracting both sides of this equation with $d\bar{Z}^\beta$ gives $\bar{\partial}G = \omega$ and the result follows from Lemma 3. □

PROPOSITION 2. *The transform $\mathcal{S}$ is a left inverse to the transform $\mathcal{G}$.*

Proof. Let us denote by $\mathcal{E}(-1)$, the underlying smooth bundle of the holomorphic line bundle $\mathcal{O}(-1)$ on $\mathbb{CP}_2$. Recall that g is homogeneous of degree -1 and *odd* on $\mathbb{R}^3 \setminus \{0\}$. Thus, $g(\lambda X) = \lambda^{-1}g(X)$ for all $\lambda \in \mathbb{R}^3 \setminus \{0\}$ rather than just positive λ. We may then use this equation with *complex* λ to regard g as a smooth section of $\mathcal{E}(-1)|_{\mathbb{RP}_2}$. From (18) we shall deduce that g is a suitable limit of $G(Z)$ as $Z \in \mathbb{CP}_2 \setminus \mathbb{RP}_2$ approaches $\mathbb{RP}_2$. To complete the proof we shall use (15) to obtain a formula for this limit in terms of ψ. On $\mathbb{RP}_2$ this formula will reduce to (9).

Choose linearly independent vectors $\rho, \sigma \in \mathbb{R}^3$ and consider $Z = \rho + \zeta\sigma$ for $\zeta \in \mathbb{C} \setminus \mathbb{R}$. As in the proof of Lemma 2, this parameterises the two hemispheres of a standard $\mathbb{CP}_1$ cycle in $\mathbb{CP}_2$ each of which is a fibre of $\pi : \mathbb{CP}_2 \setminus \mathbb{RP}_2 \to S^2$. Writing $\zeta = p + iq$ and $Z = \xi + i\eta$ as in (18), we obtain

$$\xi = \rho + p\sigma \quad \text{and} \quad \eta = iq\sigma.$$

Notice that ξ and η are linearly independent. Suppose $q > 0$ and define u and v by

$$u = q \quad \text{and} \quad v = t - p \quad \text{for} \; -\infty < t < \infty.$$

As a limiting case of (18), bearing in mind that g is odd, we obtain

$$\begin{aligned} G(\rho + \zeta\sigma) &= 2 \times \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{g(u\xi + v\eta)}{u + iv} (u\,dv - v\,du) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{g(q(\rho + t\sigma))}{i(t - \zeta)} q\,dt \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(\rho + t\sigma)}{t - \zeta} dt. \end{aligned}$$

This is precisely Cauchy's integral formula. If $q < 0$, the same conclusion holds save for a change of sign. It follows that $G(\rho + \zeta\sigma)$ has smooth limits as ζ approaches the real axis from either side and that $g(\rho + t\sigma)$ is the sum of these two limits. This is the sense in which g is the limit of G.

We now compute this limit from the formula (15). In terms of standard orthogonal coördinates on $\mathbb{R}^3$,

$$g\begin{pmatrix}1\\0\\0\end{pmatrix} = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi} \int_{S^2} \frac{\psi}{x + \epsilon i y} + \frac{\psi}{x - \epsilon i y} = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi} \int_{S^2} \frac{2x\psi}{x^2 + \epsilon^2 y^2}.$$

A short computation, bearing in mind that ψ is homogeneous of degree -2, gives

$$\begin{aligned} &d\left[\log\left(\frac{x^2 + \epsilon^2 y^2}{y^2}\right) \psi(z\,dy - y\,dz)\right] \\ &= \left[\frac{2x\psi}{x^2 + \epsilon^2 y^2} + \log\left(\frac{x^2 + \epsilon^2 y^2}{y^2}\right) \frac{\partial \psi}{\partial x}\right] (z\,dx \wedge dy + y\,dz \wedge dx + x\,dy \wedge dz), \end{aligned}$$

where $y \neq 0$. Let us integrate this two-form over the cap $S^2 \cap \{y > \delta\}$. By Stokes' theorem,

$$\left| \int_{S^2 \cap \{y > \delta\}} d \left[\log \left(\frac{x^2 + \epsilon^2 y^2}{y^2} \right) \psi (z\, dy - y\, dz) \right] \right|$$
$$= \delta \left| \int_{S^2 \cap \{y = \delta\}} \log \left(\frac{x^2 + \epsilon^2 \delta^2}{\delta^2} \right) \psi\, dz \right|$$
$$\leq \delta (|\log \epsilon^2| + |\log \delta^2|) \int_{S^2 \cap \{y = \delta\}} |\psi| \longrightarrow 0$$

as $\delta \to 0$. A similar conclusion applies to the antipodal cap $S^2 \cap \{y < \delta\}$. Noting that the 2-form $z\, dx \wedge dy + y\, dz \wedge dx + x\, dy \wedge dz$ restricts to the standard area form on the unit sphere, we obtain

$$\frac{1}{8\pi} \int_{S^2} \frac{2x\psi}{x^2 + \epsilon^2 y^2} = -\frac{1}{8\pi} \int_{S^2} \log \left(\frac{x^2 + \epsilon^2 y^2}{y^2} \right) \frac{\partial \psi}{\partial x}.$$

The right hand side has a limit as $\epsilon \downarrow 0$, namely

$$-\frac{1}{8\pi} \int_{S^2} \log \left(\frac{x^2}{y^2} \right) \frac{\partial \psi}{\partial x} = -\frac{1}{4\pi} \int_{S^2} \log \left| \frac{x}{y} \right| \frac{\partial \psi}{\partial x}.$$

This coincides with (10), as required. □

We conclude this article with a brief discussion of the modifications needed to approach the X-ray transform (see [16]) by similar means. Suppose f is a smooth even function on $\mathbb{R}^4 \setminus \{0\}$ homogeneous of degree -2. The definition (2) of $\phi(\xi, \eta) = \phi(\xi \wedge \eta)$ for linearly independent $\xi, \eta \in \mathbb{R}^4$ is unchanged. However, ϕ is not an arbitrary smooth function, but satisfies the differential equation

$$\frac{\partial^2 \phi}{\partial \xi^{[\alpha} \partial \eta^{\beta]}} = 0, \tag{19}$$

the corresponding equation being vacuous in the Funk case. If ϕ is regarded as an odd function on $\mathrm{Gr}_2^+(\mathbb{R}^4)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^4$, then (19) becomes a second order *scalar* differential equation called the *ultrahyperbolic wave equation.* If $g(X)$ is a smooth odd function on $\mathbb{R}^4 \setminus \{0\}$, homogeneous of degree -1, then

$$\psi_\beta(\xi, \eta) = \frac{1}{2\pi} \oint \frac{\partial g}{\partial X^\beta}(u\xi + v\eta)\, (u\, dv - v\, du)$$

is a function of $\xi\wedge\eta$ satisfying $\xi^\beta\psi_\beta = 0 = \eta^\beta\psi_\beta$ and also the differential equations

$$\frac{\partial}{\partial\xi^{[\alpha}}\,\psi_{\beta]} = 0 = \frac{\partial}{\partial\eta^{[\alpha}}\,\psi_{\beta]}. \tag{20}$$

Again, ψ_β may be interpreted as a field on $\mathrm{Gr}_2^+(\mathbb{R}^4)$, satisfying an intrinsically defined system of differential equations (an ultrahyperbolic version of the Dirac equation). The mapping

$$\begin{array}{ccc} \mathbb{C}^4 & \longrightarrow & \bigwedge^2\mathbb{R}^4 \\ \Cup & & \Cup \\ Z^\alpha = \xi^\alpha + i\eta^\alpha & \longmapsto & \xi\wedge\eta \end{array}$$

induces

$$\pi : \mathbb{CP}_3 \setminus \mathbb{RP}_3 \longrightarrow \mathrm{Gr}_2^+(\mathbb{R}^4)$$

and $\omega = \frac{1}{4}\psi_\alpha(Z)d\overline{Z}^\alpha$ defines a $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{CP}_3\setminus\mathbb{RP}_3$ (closure being equivalent to the differential equations (20) on $\mathrm{Gr}_2^+(\mathbb{R}^4)$). The formula (18) defines a smooth homogeneous function G of degree -1 on $\mathbb{CP}_3\setminus\mathbb{RP}_3$ satisfying $\overline{\partial}G = \omega$ (in fact, from [15], $H^1(\mathbb{CP}_3\setminus\mathbb{RP}_3, \mathcal{O}(-1)) = 0$). The same argument as given in the proof of Proposition 2 shows that, in the sense explained in this proof, $\lim G = g$ along the fibres of π.

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