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COMPACT INTERPOLATION FOR VECTORS IN TRIDIAGONAL ALGEBRA

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ABSTRACT. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for n vectors satisfies the equation $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this article, we investigate compact interpolation problems in tridiagonal algebra : Given vectors x and y in a Hilbert space, when is there a compact operator A in a tridiagonal algebra such that $Ax = y$?

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the 'n-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [10]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [11]: his result was extended by Hopenwasser [5] to the case that $\mathcal{U}$ is a CSL-algebra. Recently, Munch [12] obtained conditions for interpolation in case T is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [6] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra.

First, we establish some notations and conventions. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify

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projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If each pair of projections in $\mathcal{L}$ commutes, then $\mathcal{L}$ is called a commutative subspace lattice, or CSL. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ is the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3].

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$. Let D=$\{A :\ A$ is a dia -gonal operator on$\mathcal{H}\}$. Then D is a masa(maximal abelian subalgebra) of Alg$\mathcal{L}$ and $\mathcal{D}$=(Alg$\mathcal{L}$)$\cap$ (Alg$\mathcal{L}$)*, where (Alg$\mathcal{L}$)$^* = \{A^* : A \in$ Alg$\mathcal{L}\}$.

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on $\mathcal{H}$.

In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

The following theorem is well-known.

THEOREM 1 [4]. *Let A be a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal $\{a_n\}$. Then A is compact if and only if $a_n \to 0$ as $n \to \infty$.*

THEOREM 2. *Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. Then the following statements are equivalent.*

(1) *There exists an operator* A *in* Alg$\mathcal{L}$ *such that* $Ax = y$, A *is compact and every* E *in* $\mathcal{L}$ *reduces* A.

(2) $\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$ *and* $y_n x_n^{-1} \to 0$ *as* $n \to \infty$.

PROOF. If $\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$, then, there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A by Theorem 1 ([9]). Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $A = (a_{ii})$ is diagonal and $Ax = y$, $a_{ii}x_i = y_i$ for all $i = 1, 2, \cdots$. Since $y_n x_n^{-1} \to 0$ as $n \to \infty$, A is compact.

Conversely, since $Ax = y$ and every E in $\mathcal{L}$ reduces A, $AEx = Ey$ for every E in $\mathcal{L}$. So $A(\sum_{k=1}^{l}\alpha_k E_k x) = \sum_{k=1}^{l}\alpha_k E_k y$ for every $l \in \mathbb{N}$, every $\alpha_k \in \mathbb{C}$ and every $E_k \in \mathcal{L}$. Thus $\|\sum_{k=1}^{l}\alpha_k E_k y\| \leq \|A\|\|\sum_{k=1}^{l}\alpha_k E_k x\|$. If $\|\sum_{k=1}^{l}\alpha_k E_k x\| \neq 0$, then $\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} \leq \|A\|$. $\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax = y$, $y_i = a_{ii}x_i$ and hence $a_{ii} = y_i x_i^{-1}$ for all $i = 1, 2, \cdots$. Since A is compact, $y_i x_i^{-1} \to 0$ as $i \to \infty$. □

THEOREM 3. *Let* $x_p = (x_{pi})$ *and* $y_p = (y_{pi})$ *be vectors in* $\mathcal{H}$ *such that* $x_{qi} \neq 0$ *for some fixed* q, *all* $i = 1, 2, \cdots$ *and all* $p = 1, 2, \cdots, n$. *If there is an operator* A *in* Alg$\mathcal{L}$ *such that* $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$, *every* E *in* $\mathcal{L}$ *reduces* A *and* A *is compact, then*

$$\sup\left\{\frac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$$

and $y_{qi}x_{qi}^{-1} \to 0$ *as* $i \to \infty$.

PROOF. Since $Ax_p = y_p$ and every E in $\mathcal{L}$ reduces A, $AEx_p = Ey_p$ for every $p = 1, 2, \cdots, n$. So $A(\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p) = \sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p$, $m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L}$ and $\alpha_{k,p} \in \mathbb{C}$. Thus

$$\left\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\right\| \leq \|A\|\left\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\right\|.$$

If $\|\sum_{p=1}^{l}\sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p\| \neq 0$, then $\dfrac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} \leq$ $\|A\|$. Hence

$$\sup\left\{\frac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty.$$

Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax_p = y_p$, $y_{pi} = a_{ii}x_{pi}$ for all $p = 1, 2, \cdots, n$ and all $i = 1, 2, \cdots$. Since $x_{qi} \neq 0$, $a_{ii} = y_{qi}x_{qi}^{-1}$ $(i = 1, 2, \cdots)$. Since A is compact, $y_{qi}x_{qi}^{-1} \to 0$ as $i \to \infty$. □

THEOREM 4. *Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}$ such that $x_{qi} \neq 0$ for some fixed q, all $i = 1, 2, \cdots$ and all $p = 1, 2, \cdots, n$.*

If $\sup\left\{\dfrac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ *and $y_{qi}x_{qi}^{-1} \to 0$ as $i \to \infty$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for all $p = 1, 2, \cdots, n$, every E in $\mathcal{L}$ reduces A and A is compact.*

PROOF. Without loss of generality, we may assume that

$\sup\left\{\dfrac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} = 1$. So

$$\left\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\right\| \leq \left\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\right\|, m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \cdots (*).$$

$$\text{Let } \mathcal{M} = \left\{\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p : m_p \in \mathbb{N}, l \leq n, \alpha_{k,p} \in \mathbb{C} \text{ and } E_{k,p} \in \mathcal{L}\right\}.$$

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{p=1}^{l}\sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p) = \sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^{\perp}} = 0$. Clearly $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$ and $\|A\| \leq 1$. By an argument similar to that of the proof of Theorem 2, every E in $\mathcal{L}$ reduces A. So A is a diagonal operator. Let $A = (a_{ii})$. Since $y_p = Ax_p$, $a_{ii} = y_{pi}x_{pi}^{-1}$ $(i = 1, 2, \cdots)$. Since $y_{qi}x_{qi}^{-1} \to 0$ as $i \to \infty$, A is compact. □

If we summarize Theorems 3 and 4, then we can get the following theorem.

THEOREM 5. *Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}$ such that $x_{qi} \neq 0$ for some fixed q and all $i = 1, 2, \cdots$. Then the following statements are equivalent.*

(1) *There exists an operator A in $Alg\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, \cdots, n)$, every E in $\mathcal{L}$ reduces A and A is compact.*

(2) $$\sup\left\{\frac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$$
and $y_{qi}x_{qi}^{-1} \to 0$ as $i \to \infty$.

If we modify the proof of Theorems 3 and 4, then we can get the following theorem.

THEOREM 6. *Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}(p = 1, 2, \cdots)$ such that $x_{qi} \neq 0$ for all i and for some fixed q. Then the following statements are equivalent.*

(1) *There exists an operator A in $Alg\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, \cdots)$ every E in $\mathcal{L}$ reduces A and A is compact.*

(2) $$\sup\left\{\frac{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{p=1}^{l}\sum_{k=1}^{m_p}\alpha_{k,p}E_{k,p}x_p\|} : m_p,\ l \in \mathbb{N}, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$$
and $y_{qi}x_{qi}^{-1} \to 0$ as $i \to \infty$.

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