

ON THE GENERALIZED SET-VALUED MIXED VARIATIONAL INEQUALITIES

YALI ZHAO, ZEQING LIU AND SHIN MIN KANG

ABSTRACT. In this paper, we introduce and study a new class of the generalized set-valued mixed variational inequalities. Using the resolvent operator technique, we construct a new iterative algorithm for solving this class of the generalized set-valued mixed variational inequalities. We prove the existence of solutions for the generalized set-valued mixed variational inequalities and the convergence of the iterative sequences generated by the algorithm.

1. Introduction

Variational inequality theory provides us a unified framework for dealing with a wide class of problems arising in elasticity, structural analysis, economics, physical and engineering sciences, etc. (see [1]-[3], [5]-[7], [9] and the references therein). A useful and important generalization of variational inequalities is a mixed variational inequality containing a nonlinear term. Inspired and motivated by recent research work in [1]-[3], [5]-[7], [9] in this paper, we introduce and study the generalized set-valued mixed variational inequalities and construct a new iterative algorithm. We prove the existence of solutions for our variational inequalities and the convergence of the iterative sequences generated by the algorithm. Among the special cases of the obtained results are the corresponding results in [1], [2], [5]-[7], [9] and others.

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2. Preliminaries

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $\partial\phi$ denote the subdifferential of a proper, convex and lower function $\phi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$. Given multivalued mappings $M, S, T : H \rightarrow 2^H$, where 2^H denotes the family of nonempty subsets of H and single-valued mapping $g : H \rightarrow H$ with $g(H) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ for each $y \in H$ and a nonlinear mapping $N(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow H$, we consider the following problem:

Find $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ such that $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ and

$$(2.1) \quad \langle N(u, w, z), y - g(x) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad y \in H.$$

Problem (2.1) is called the *generalized nonlinear set-valued mixed variational inequality*.

If $N(u, w, z) = u - (w - z)$ for all $u, w, z \in H$, then problem (2.1) is equivalent to finding $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ such that $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ and

$$(2.2) \quad \langle u - (w - z), y - g(x) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad y \in H.$$

Problem (2.2) is called the *nonlinear mixed variational inequality*, which appears to be a new one.

If M, S and T are single-valued mappings, then problem (2.1) is equivalent to finding $x \in H$ such that $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ and

$$(2.3) \quad \langle N(Mx, Sx, Tx), y - g(x) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad y \in H.$$

This problem is called the *generalized mixed variational inequality*.

If $N(x, y, z) = N(x, y)$ and $\phi(x, y) = \phi(x)$ for all $x, y, z \in H$, then problem (2.1) reduces to finding $x \in H$, $u \in Mx$ and $w \in Sx$ such that $g(x) \cap \text{dom } \partial\phi(x) \neq \emptyset$ and

$$(2.4) \quad \langle N(u, w), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \quad y \in H.$$

Problem (2.4) is called the *generalized set-valued mixed variational inequality*, which was studied by Noor, Noor and Rassias [5]. It is known that a number of problems involving mechanics, economics and optimization theory can be studied via problem (2.4), see for example [5], [8] and the references therein.

In a brief, for a suitable choice of the mappings M, S, T, N, g, ϕ and the space H , one can obtain a number of known and new classes of variational inequalities and related problems from the generalized non-linear set-valued mixed variational inequality (2.1). Further, these types of variational inequalities enable us to study many important problems arising in mathematical, regional, physical and engineering sciences in a general and unified framework.

DEFINITION 2.1. ([1], [8]) If $G : H \rightarrow 2^H$ is a maximal monotone multivalued mapping, then for any fixed $\rho > 0$, the mapping $J_\rho^G : H \rightarrow H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \quad x \in H$$

is said to be the *resolvent operator* of index ρ of G , where I is the identity mapping on H . Furthermore, the resolvent operator J_ρ^G is single-valued and nonexpansive, that is,

$$\|J_\rho^G(x) - J_\rho^G(y)\| \leq \|x - y\|, \quad x, y \in H.$$

Since the subdifferential $\partial\phi$ of a proper, convex and lower semicontinuous function $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is a maximal monotone multivalued mapping, it follows that the resolvent operator $J_\rho^{\partial\phi}$ of index ρ of $\partial\phi$ is given by

$$J_\rho^{\partial\phi}(x) = (I + \rho\partial\phi)^{-1}(x), \quad x \in H.$$

DEFINITION 2.2. A set-valued mapping $S : H \rightarrow CB(H)$ is said to be (i) *H-Lipschitz continuous* if there exists a constant $h > 0$ such that

$$H(Sx_1, Sx_2) \leq h\|x_1 - x_2\|, \quad x_i \in H, i = 1, 2,$$

where $CB(H)$ is the family of all nonempty closed bounded subsets of H , and $H(\cdot, \cdot)$ denotes the Hausdorff metric.

(ii) *strongly monotone with respect to the first argument* of $N : H \times H \times H \rightarrow H$ if there exists a constant $c > 0$ such that

$$\begin{aligned} & \langle x_1 - x_2, N(w_1, \cdot, \cdot) - N(w_2, \cdot, \cdot) \rangle \\ & \geq c\|x_1 - x_2\|^2, \quad x_i \in H, w_i \in Sx_i, i = 1, 2. \end{aligned}$$

DEFINITION 2.3. A mapping $g : H \rightarrow H$ is said to be

(i) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \geq r \|x_1 - x_2\|^2, \quad x_i \in H, i = 1, 2.$$

(ii) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq s \|x_1 - x_2\|, \quad x_i \in H, i = 1, 2.$$

DEFINITION 2.4. A mapping $N(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow H$ is said to be *Lipschitz continuous with respect to the first argument* if there exists a constant $\alpha > 0$ such that

$$\|N(x_1, \cdot, \cdot) - N(x_2, \cdot, \cdot)\| \leq \alpha \|x_1 - x_2\|, \quad x_i \in H, i = 1, 2.$$

In a similar way, we can define the Lipschitz continuity of the mapping $N(\cdot, \cdot, \cdot)$ with respect to the second argument and third arguments respectively.

The following lemma plays a crucial role in the proof of our result.

LEMMA 2.1. Elements $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ with $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ are a solution of problem (2.1) if and only if $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ satisfy the following relation

$$(2.5) \quad g(x) = J_\rho^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)],$$

where $\rho > 0$ is a constant, $J_\rho^{\partial\phi(\cdot, x)} = (I + \rho \partial\phi(\cdot, x))^{-1}$ is the resolvent operator of index ρ of $\partial\phi(\cdot, x)$ and I is the identity operator on H .

PROOF. From the definition of the resolvent operator $J_\rho^{\partial\phi(\cdot, x)}$ of index ρ of $\partial\phi(\cdot, x)$ it follows that relation (2.5) with $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ holds if and only if

$$g(x) - \rho N(u, w, z) \in g(x) + \rho \partial\phi(g(x), x),$$

which is equivalent to

$$-N(u, w, z) \in \partial\phi(g(x), x).$$

From the definition of $\partial\phi(\cdot, x)$, we know that the above relation is satisfied if and only if $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ with $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ and (2.1) holds. This completes the proof. \square

REMARK 2.1. Lemma 2.1 extends Lemma 3.1 in [1], Lemma 2.1 in [2] and Lemma 3.1 in [6].

REMARK 2.2. From Lemma 2.1, we see that problem (2.1) is equivalent to the fixed point problem of the type

$$(2.6) \quad x \in F(x),$$

where

$$F(x) = \cup_{u \in Mx} \cup_{w \in Sx} \cup_{z \in Tx} \{x - g(x) + J_{\rho}^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)]\}.$$

Based on (2.5) and (2.6) and Nadler's result, we suggest the following iterative algorithm.

ALGORITHM 2.1. Let $g : H \rightarrow H$ be single-valued mapping, $S, T, M : H \rightarrow CB(H)$ be multivalued mappings and $N(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow H$ be a nonlinear mapping. For given $x_0 \in H$, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{z_n\}$ as

$$(2.7) \quad \begin{aligned} x_{n+1} &= x_n - g(x_n) + J_{\rho}^{\partial\phi(\cdot, x_n)}[g(x_n) - \rho N(u_n, w_n, z_n)], \\ u_n &\in Mx_n, \quad \|u_n - u_{n-1}\| \leq (1 + n^{-1})H(Mx_n, Mx_{n-1}), \\ w_n &\in Sx_n, \quad \|w_n - w_{n-1}\| \leq (1 + n^{-1})H(Sx_n, Sx_{n-1}), \\ z_n &\in Tx_n, \quad \|z_n - z_{n-1}\| \leq (1 + n^{-1})H(Tx_n, Tx_{n-1}) \end{aligned}$$

for $n \geq 0$, where $\rho > 0$ is a constant.

REMARK 2.3. For appropriate and suitable choice of the mappings M, S, T, N, g, ϕ and the space H , one can obtain a number of new and known iterative algorithms from Algorithm 2.1, for example, see [1]-[3], [5]-[7], [9] and the references therein.

3. Convergence result

In this section, we study the existence of solutions for the generalized nonlinear set-valued mixed variational inequality (2.1) and the convergence of the iterative sequences generated by Algorithm 2.1.

THEOREM 3.1. Let $g : H \rightarrow H$ be strongly monotone and Lipschitz continuous with constants r and s , respectively and $M, S, T : H \rightarrow$

$CB(H)$ be H -Lipschitz continuous with constants q, h and d , respectively. Let $N(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow H$ be Lipschitz continuous with respect to the first, second and third arguments with constants α, β and γ , respectively. Let M be strongly monotone with respect to the first argument of N with constant c , and let $\phi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous function on H , $g(H) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ and for each $x, y, z \in H$,

$$(3.1) \quad \|J_{\rho}^{\partial\phi(\cdot, x)}(z) - J_{\rho}^{\partial\phi(\cdot, y)}(z)\| \leq \mu\|x - y\|,$$

where $\mu > 0$ is a constant. Suppose that there exists a constant $\rho > 0$ such that

$$(3.2) \quad \rho(\beta h + \gamma d) < 1 - t, \quad t = 2\sqrt{1 - 2r + s^2}$$

and one of the following relations

$$(3.3) \quad \begin{aligned} & \alpha q > \beta h + \gamma d, \\ & |c - (\beta h + \gamma d)(1 - t)| > \sqrt{t(2 - t)(\alpha^2 q^2 - (\beta h + \gamma d)^2)}, \\ & \left| \rho - \frac{c - (\beta h + \gamma d)(1 - t)}{\alpha^2 q^2 - (\beta h + \gamma d)^2} \right| \\ & < \frac{\{[c - (\beta h + \gamma d)(1 - t)]^2 - t(2 - t)[\alpha^2 q^2 - (\beta h + \gamma d)^2]\}^{1/2}}{\alpha^2 q^2 - (\beta h + \gamma d)^2}; \end{aligned}$$

$$(3.4) \quad \begin{aligned} & \alpha q = \beta h + \gamma d, \\ & 2\rho[c - (\beta h + \gamma d)(1 - t)] > t(2 - t), \quad c > (\beta h + \gamma d)(1 - t); \end{aligned}$$

$$(3.5) \quad \begin{aligned} & \alpha q < \beta h + \gamma d, \\ & \left| \rho - \frac{(\beta h + \gamma d)(1 - t) - c}{(\beta h + \gamma d)^2 - \alpha^2 q^2} \right| \\ & > \frac{\{[(\beta h + \gamma d)(1 - t) - c]^2 + t(2 - t)[(\beta h + \gamma d)^2 - \alpha^2 q^2]\}^{1/2}}{(\beta h + \gamma d)^2 - \alpha^2 q^2} \end{aligned}$$

is satisfied. Then there exist $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$, which are a solution of problem (2.1) and $x_n \rightarrow x$, $u_n \rightarrow u$, $w_n \rightarrow w$ and $z_n \rightarrow z$ as $n \rightarrow \infty$, where $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{z_n\}$ are the sequences defined in Algorithm 2.1.

PROOF. From (2.7), we have

$$(3.6) \quad \|x_{n+1} - x_n\| = \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) + J_\rho^{\partial\phi(\cdot, x_n)}(h(x_n)) - J_\rho^{\partial\phi(\cdot, x_{n-1})}(h(x_{n-1}))\|,$$

where $h(x_n) = g(x_n) - \rho N(u_n, w_n, z_n)$. Since the resolvent operator $J_\rho^{\partial\phi}$ is nonexpansive, by (3.1) we have

$$(3.7) \quad \begin{aligned} & \|J_\rho^{\partial\phi(\cdot, x_n)}(h(x_n)) - J_\rho^{\partial\phi(\cdot, x_{n-1})}(h(x_{n-1}))\| \\ & \leq \|J_\rho^{\partial\phi(\cdot, x_n)}(h(x_n)) - J_\rho^{\partial\phi(\cdot, x_n)}(h(x_{n-1}))\| \\ & \quad + \|J_\rho^{\partial\phi(\cdot, x_n)}(h(x_{n-1})) - J_\rho^{\partial\phi(\cdot, x_{n-1})}(h(x_{n-1}))\| \\ & \leq \|h(x_n) - h(x_{n-1})\| + \mu\|x_n - x_{n-1}\| \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} & \|h(x_n) - h(x_{n-1})\| \\ & = \|g(x_n) - \rho N(u_n, w_n, z_n) - g(x_{n-1}) + \rho N(u_{n-1}, w_{n-1}, z_{n-1})\| \\ & \leq \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ & \quad + \|x_n - x_{n-1} - \rho(N(u_n, w_n, z_n) - N(u_{n-1}, w_n, z_n))\| \\ & \quad + \rho\|N(u_{n-1}, w_n, z_n) - N(u_{n-1}, w_{n-1}, z_n)\| \\ & \quad + \rho\|N(u_{n-1}, w_{n-1}, z_n) - N(u_{n-1}, w_{n-1}, z_{n-1})\|. \end{aligned}$$

By the Lipschitz continuity and strong monotonicity of g , we get that

$$(3.9) \quad \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \leq (1 - 2r + s^2)\|x_n - x_{n-1}\|^2.$$

Since M, S and T are H -Lipschitz continuous, and $N(\cdot, \cdot, \cdot)$ is Lipschitz continuous with respect to the first, second and third arguments, respectively, we obtain that

$$(3.10) \quad \begin{aligned} & \|N(u_{n-1}, w_n, z_n) - N(u_{n-1}, w_{n-1}, z_n)\| \\ & \leq \beta h(1 + n^{-1})\|x_n - x_{n-1}\|, \\ & \|N(u_{n-1}, w_{n-1}, z_n) - N(u_{n-1}, w_{n-1}, z_{n-1})\| \\ & \leq \gamma d(1 + n^{-1})\|x_n - x_{n-1}\|, \end{aligned}$$

$$\begin{aligned} & \|N(u_n, w_n, z_n) - N(u_{n-1}, w_n, z_n)\| \\ & \leq \alpha q(1 + n^{-1})\|x_n - x_{n-1}\|. \end{aligned}$$

Further, by the strong monotonicity of M with respect to the first argument of $N(\cdot, \cdot, \cdot)$, we have

$$(3.11) \quad \begin{aligned} & \|x_n - x_{n-1} - \rho(N(u_n, w_n, z_n) - N(u_{n-1}, w_n, z_n))\|^2 \\ & \leq (1 - 2\rho c + \rho^2 \alpha^2 q^2 (1 + n^{-1})^2) \|x_n - x_{n-1}\|^2. \end{aligned}$$

It follows from (3.6)-(3.11) that

$$(3.12) \quad \|x_{n+1} - x_n\| \leq \theta_n \|x_n - x_{n-1}\|,$$

where

$$\theta_n = t + \sqrt{1 - 2\rho c + \rho^2 \alpha^2 q^2 (1 + n^{-1})^2} + \rho(\beta h + \gamma d)(1 + n^{-1}).$$

Let $\theta = t + \sqrt{1 - 2\rho c + \rho^2 \alpha^2 q^2} + \rho(\beta h + \gamma d)$. We know that $\theta_n \downarrow \theta$ as $n \rightarrow \infty$. It follows from (3.2) and one of the relations (3.4)-(3.6) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (3.12) implies that $\{x_n\}$ is a Cauchy sequence in H and we can suppose that $x_n \rightarrow x \in H$. In addition, we can easily obtain that $\{u_n\}$, $\{w_n\}$ and $\{z_n\}$ are Cauchy sequences in H from Algorithm 2.1. Let $u_n \rightarrow u$, $w_n \rightarrow w$, $z_n \rightarrow z$ as $n \rightarrow \infty$. It is easy to see that

$$\begin{aligned} d(u, Mx) &= \inf\{\|u - y\| : y \in Mx\} \leq \|u - u_n\| + d(u_n, Mx) \\ &\leq \|u - u_n\| + H(Mx_n, Mx) \leq \|u - u_n\| + q\|x - x_n\| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence $u \in Mx$. Similarly, we have $w \in Sx$, $z \in Tx$. Since

$$\begin{aligned} & \|J_\rho^{\partial\phi(\cdot, x_n)}[g(x_n) - \rho N(u_n, w_n, z_n)] - J_\rho^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)]\| \\ & \leq \|J_\rho^{\partial\phi(\cdot, x_n)}[g(x_n) - \rho N(u_n, w_n, z_n)] - J_\rho^{\partial\phi(\cdot, x)}[g(x_n) \\ & \quad - \rho N(u_n, w_n, z_n)]\| + \|J_\rho^{\partial\phi(\cdot, x)}[g(x_n) - \rho N(u_n, w_n, z_n)] \\ & \quad - J_\rho^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)]\| \\ & \leq \mu\|x_n - x\| + \|g(x_n) - g(x)\| + \rho\|N(u_n, w_n, z_n) - N(u, w_n, z_n)\| \\ & \quad + \rho\|N(u, w_n, z_n) - N(u, w, z_n)\| + \rho\|N(u, w, z_n) - N(u, w, z)\| \\ & \leq (\mu + s)\|x_n - x\| + \rho\alpha\|u_n - u\| + \rho\beta\|w_n - w\| + \rho\gamma\|z_n - z\| \\ & \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. It follows that

$$(3.13) \quad \begin{aligned} & \lim_{n \rightarrow \infty} J_{\rho}^{\partial\phi(\cdot, x_n)}[g(x_n) - \rho N(u_n, w_n, z_n)] \\ &= J_{\rho}^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)]. \end{aligned}$$

By virtue of (2.7) and (3.13), we have

$$x = x - g(x) + J_{\rho}^{\partial\phi(\cdot, x)}[g(x) - \rho N(u, w, z)] \in N(x).$$

From the above equality and Remark 2.1 it follows that $x \in H$, $u \in Mx$, $w \in Sx$ and $z \in Tx$ with $g(x) \cap \text{dom } \partial\phi(\cdot, y) \neq \emptyset$ are a solution of problem (2.1). This completes the proof. \square

REMARK 3.1. Theorem 3.1 extends and improves Theorem 4.1 in [1], Theorem 3.1 in [2], Theorem 3.1 in [5], Theorems 3.1 and 3.2 in [6], Theorem 3.1 in [7] and Theorem 3.1 in [9].

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Yali Zhao
Department of Mathematics
Chaoyang Junior Normal College
Liaoning, Chaoyang 122000
People's Republic of China

Zeqing Liu
Department of Mathematics
Liaoning Normal University
P. O. Box 200, Dalian, Liaoning 116029
People's Republic of China
E-mail: zeqingliu@sina.com.cn

Shin Min Kang
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr