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ON FUZZY SUBALGEBRAS IN B-ALGEBRAS

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Abstract. In this paper, we classify the subalgebras by their family of level subalgebras in B-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([9]) introduced the notion of d-algebras, i.e., (I) $x * x = 0$; (V) $0 * x = x$; (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras, and then investigated other relations between d-algebras and oriented digraphs. On the while, Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced a new notion, called a BH-algebra, i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras. J. Neggers and H. S. Kim ([10]) introduced a new notion, called a B-algebra which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. J. R. Cho and H. S. Kim ([1]) discussed further relations between B-algebras and other topics, especially quasigroups. H. K. Park and H. S. Kim ([12]) obtained that every quadratic B-algebra on a field X with $|X| \geq 3$ is a BCI-algebra. Y. B. Jun et al. ([7]) fuzzyfied (normal) B-algebras and gave a characterization of a fuzzy B-algebras. In this paper, we classify the subalgebras by their family of level subalgebras in B-algebras.

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2. Preliminaries

A *B-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:
(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$, for all $x, y, z \in X$.
For brevity we also call X a B-algebra. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset I of a B-algebra X is called a *subalgebra* of X if $x * y \in I$ for any $x, y \in I$.

EXAMPLE 2.1. ([10]) Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra.

EXAMPLE 2.2. ([10]) Let $F < x, y, z >$ be the free group on three elements. Define $u * v := vuv^{-2}$. Thus $u * u = e$ and $u * e = u$. Also $e * u = u^{-1}$. Now, given $a, b, c \in F < x, y, z >$, let

$$\begin{aligned} w(a,b,c) &= ((a*b)*c)(a*(c*(e*b))^{-1} \\ &= (cbab^{-2}c^{-2})(b^{-1}cb^2a^{-1}cbcb^2)^{-1} \\ &= cbab^{-2}c^{-2}b^{-2}c^{-1}b^{-1}c^{-1}ba^{-1}b^{-2}c^{-1}b. \end{aligned}$$

Let $\mathcal{N}(*)$ be the normal subgroup of $F < x, y, z >$ generated by the elements $w(a, b, c)$. Let $G = F < x, y, z > /\mathcal{N}(*)$. On G define the operation "$\cdot$" as usual and define

$$(u\mathcal{N}(*)) * (v\mathcal{N}(*)) := (u * v)\mathcal{N}(*).$$

It follows that $(u\mathcal{N}(*)) * (u\mathcal{N}(*)) = e\mathcal{N}(*)$, $(u\mathcal{N}(*)) * (e\mathcal{N}(*)) = u\mathcal{N}(*)$ and

$$w(a\mathcal{N}(*), b\mathcal{N}(*), c\mathcal{N}(*)) = w(a,b,c)\mathcal{N}(*) = e\mathcal{N}(*).$$

Hence $(G; *, e\mathcal{N}(*))$ is a B-algebra.

DEFINITION 2.3. ([7]) Let μ be a fuzzy set in a B-algebra. Then μ is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in X.$$

DEFINITION 2.4. Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set

$$\mu_t := \{x \in X | \mu(x) \geq t\}$$

is called a *level subset* of μ.

3. Fuzzy subalgebras of B-algebras

In this section we classify the subalgebras by their family of level subalgebras in B-algebras.

THEOREM 3.1. *A fuzzy set μ of a B-algebra X is a fuzzy subalgebra if and only if for every $t \in [0,1]$, μ_t is either empty or subalgebra of X.*

PROOF. If μ is a fuzzy subalgebra of X and $\mu_t \neq \emptyset$, then for any $x, y \in \mu_t$

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t,$$

which implies $x * y \in \mu_t$ and so μ_t is a subalgebra of X.

Conversely, for any $x, y \in X$, denote by $t = \min\{\mu(x), \mu(y)\}$. Then by the assumption μ_t is a subalgebra of X, which implies $x * y \in \mu_t$. Hence $\mu(x * y) \geq t = \min\{\mu(x), \mu(y)\}$. Thus μ is a fuzzy subalgebra of X. □

THEOREM 3.2. *Let X be a B-algebra and let μ be a fuzzy set in X such that μ_t is a subalgebra for all $t \in [0,1]$, $t \leq \mu(0)$. Then μ is a fuzzy subalgebra of X.*

PROOF. Let $x, y \in X$ and let $\mu(x) = t_1$ and $\mu(y) = t_2$. Then $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$. Assume that $t_1 \leq t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$ and so $y \in \mu_{t_1}$. Since μ_{t_1} is a subalgebra of X, we have $x * y \in \mu_{t_1}$. Thus $\mu(x * y) \geq t_1 = \min\{\mu(x), \mu(y)\}$. This completes the proof. □

DEFINITION 3.3. Let X be a B-algebra and μ be a fuzzy subalgebra of X. The subalgebras $\mu_t, t \in [0,1]$ and $t \leq \mu(0)$, are called a *level subalgebra* of μ.

THEOREM 3.4. *Any subalgebra of a B-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.*

PROOF. Let A be a subalgebra of a given B-algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A, \end{cases}$$

where $t \in (0, 1)$ is fixed. It is clear that $\mu_t = A$. We prove that such defined μ is a fuzzy subalgebra of X. If $x, y \in A$, then also $x * y \in A$. Hence $\mu(x) = \mu(y) = \mu(x * y) = t$ and $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$ and so $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} = 0$. If at most one of x, y belong to A, then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0. Therefore $\min\{\mu(x), \mu(y)\} = 0$ and $\mu(x * y) \geq 0$, which completes the proof. □

As a generalization of Theorem 3.4, we prove the following theorem:

THEOREM 3.5. *Let X be a B-algebra. The given any chain of subalgebras*

$$A_0 \subset A_1 \subset \cdots \subset A_r = X,$$

there exists a fuzzy subalgebra of X whose level subalgebras are exactly the subalgebras of this chain.

PROOF. Consider a set of numbers

$$t_0 > t_1 > \cdots > t_r,$$

where each t_i is in $[0, 1]$. Let $\mu : X \to [0, 1]$ be a fuzzy set defined by $\mu(A_0) = t_0$ and $\mu(A_i - A_{i-1}) = t_i$, $0 < i \leq r$.

We claim that μ is a fuzzy subalgebra of X. Let $x, y \in X$. Then we distinguish two cases as follows:

Case 1: Let $x, y \in A_i - A_{i-1}$. Then by the definition of μ,

$$\mu(x) = t_i = \mu(y).$$

Since A_i is a subalgebra, it follows that $x * y \in A_i$, and so either $x * y \in A_i - A_{i-1}$ or $x * y \in A_{i-1}$. In any case we conclude that

$$\mu(x * y) \geq t_i = \min\{\mu(x), \mu(y)\}.$$

Case 2: For $i > j$, $x \in A_i - A_{i-1}$ and $y \in A_j - A_{j-1}$. Then $\mu(x) = t_i$, $\mu(y) = t_j$, and $x * y \in A_i$ because A_i is a subalgebra and $A_j \subset A_i$. Hence

$$\mu(x * y) \geq t_j = \min\{\mu(x), \mu(y)\}.$$

Thus μ is a fuzzy subalgebra of X. From the definition of μ, it follows that $Im(\mu) = \{t_0, t_1, \cdots, t_r\}$. Hence the level subalgebras of μ are given by the chain of subalgebras

$$\mu_{t_0} \subset \mu_{t_1} \subset \cdots \subset \mu_{t_r} = X.$$

Now $\mu_{t_0} = \{x \in X | \mu(x) \geq t_0\} = A_0$. Finally we prove that $\mu_{t_i} = A_i$ for $0 < i \leq r$. Clearly $A_i \subseteq \mu_{t_i}$. If $x \in \mu_{t_i}$, then $\mu(x) \geq t_i$ which implies that $x \notin A_j$ for $j > i$. Hence $\mu(x) \in \{t_1, t_2, \cdots, t_i\}$, and so $x \in A_k$ for some $k \leq i$. As $A_k \subseteq A_i$, it follows that $x \in A_i$. Therefore we obtain $\mu_{t_i} = A_i$ for $0 \leq i \leq r$. This completes the proof. □

Note that if X is a finite B-algebra, then the number of subalgebras of X is finite whereas the number of level subalgebras of a fuzzy subalgebra μ appears to be infinite. But since every level subalgebra is indeed a subalgebra of X, not all these subalgebras are distinct. The next theorem characterizes this aspect.

THEOREM 3.6. *Let μ be a fuzzy subalgebra of a B-algebra X. Two level subalgebras μ_{t_1}, μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu(x) < t_2$.*

PROOF. Assume that $\mu_{t_1} = \mu_{t_2}$ for $t_1 < t_2$ and that there exists $x \in X$ such that $t_1 \leq \mu(x) < t_2$. Then μ_{t_2} is a proper subset of μ_{t_1}, which is a contradiction.

Conversely, suppose that there is no $x \in X$ such that $t_1 \leq \mu(x) < t_2$. Since $t_1 < t_2$, we have $\mu_{t_2} \subseteq \mu_{t_1}$. If $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$ and so $\mu(x) \geq t_2$, because $\mu(x)$ does not lie between t_1 and t_2. Hence $x \in \mu_{t_2}$, which implies that $\mu_{t_1} \subseteq \mu_{t_2}$. This completes the proof. □

REMARK 3.7. As a consequence of Theorem 3.6, the level subalgebras of a fuzzy subalgebra μ of a finite B-algebra X form a chain. But $\mu(x) \leq \mu(0)$ for all $x \in X$. Therefore μ_{t_0}, where $t_0 = \mu(0)$, is the smallest level subalgebra but not always μ_{t_0} as shown in the following example, and so we have the chain $\mu_{t_0} \subset \mu_{t_1} \subset \cdots \subset \mu_{t_r} = X$, where $t_0 > t_1 > \cdots > t_r$.

EXAMPLE 3.8. Let A be a non-trivial subalgebra of a B-algebra X. Let μ be a fuzzy subalgebra in Theorem 3.4. Then $Im(\mu) = \{0, t\}$. Further, the two level subalgebra of μ are $\mu_0 = X$ and $\mu_t = A \neq \{0\}$.

COROLLARY 3.9. *Let X be a finite B-algebra and μ be a fuzzy subalgebra of X. If $Im(\mu) = \{t_1, \cdots, t_n\}$, then the family of subalgebras $\mu_{t_i}, 1 \leq i \leq n$, constitutes all the level subalgebras of μ.*

PROOF. Let $t \in [0,1]$ and $t \notin Im(\mu)$. Suppose $t_1 < t_2 < \cdots < t_n$ without loss of generality. If $t \leq t_1$, then $\mu_{t_1} = X = \mu_t$. If $t > t_n$, then $\mu_t = \emptyset$ obviously. If $t_{i-1} < t < t_i$, then $\mu_t = \mu_{t_i}$ by Theorem 3.6. Thus for any $t \in [0,1]$, the level subalgebra is one of $\{\mu_{t_i} | i = 1, \cdots, n\}$. □

The following examples show that two fuzzy subalgebras of a B-algebra may have an identical family of level subalgebras but the fuzzy subalgebras may not be equal.

EXAMPLE 3.10. Let $X := \{0,1,2,3\}$ be a set with the following table:

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a B-algebra ([11]). Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = \mu(3) = t_0$ and $\mu(1) = \mu(2) = t_1$ for $t_0, t_1 \in [0,1]$ with $t_0 > t_1$. Then μ is a fuzzy subalgebra of X. The level subalgebras of μ are $\mu_{t_0} = \{0,3\}$, $\mu_{t_1} = X$. Now let $s_0, s_1 \in [0,1]$ be such that $s_0 > s_1$ and $s_i \neq t_j$ for $i = 0,1; j = 0,1$. Define $\nu : X \to [0,1]$ by $\nu(0) = \nu(3) = s_0$ and $\nu(1) = \nu(2) = s_1$. It is clear that ν is a fuzzy subalgebra of X. The family of level subalgebras of ν are $\nu_{s_0} = \{0,3\}$, $\nu_{s_1} = X$. Thus the two fuzzy subalgebras μ and ν have the same family of level subalgebras. However it is clear that μ is not equal to ν.

LEMMA 3.11. *Let X be a B-algebra and μ a fuzzy subalgebra of X. If $Im(\mu)$ is finite, say $\{t_1, \cdots, t_n\}$, then for any $t_i, t_j \in Im(\mu)$, $\mu_{t_i} = \mu_{t_j}$ implies $t_i = t_j$.*

PROOF. Assume that $t_i \neq t_j$, say $t_i < t_j$. If $x \in \mu_{t_j}$, then $\mu(x) \geq t_j > t_i$, which implies that $x \in \mu_{t_i}$. Let $x \in X$ be such that $t_i < \mu(x) < t_j$. Then $x \in \mu_{t_i}$, but $x \notin \mu_{t_j}$. Hence $\mu_{t_j} \subset \mu_{t_i}$ and $\mu_{t_j} \neq \mu_{t_i}$, a contradiction. □

THEOREM 3.12. *Let μ and ν be two fuzzy subalgebras of a finite B-algebra X with identical family of level subalgebras. If $Im(\mu) = \{t_0, t_1, \cdots, t_r\}$ and $Im(\nu) = \{s_0, s_1, \cdots, s_k\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \cdots > s_k$ then we have*

(i) $r = k$

(ii) $\mu_{t_i} = \nu_{s_i},\ 0 \le i \le k,$

(iii) *if $x \in X$ such that $\mu(x) = t_i$, then $\nu(x) = s_i,\ 0 \le i \le k.$*

PROOF. (i) By Corollary 3.9, the only subalgebras of μ and ν are the two families μ_{t_i} and ν_{s_i}. Since μ and ν have the same family of level subalgebras, it follows that $r = k$.

(ii) Using (i) and Remark 3.7 we have chains of level subalgebras

$$\mu_{t_0} \subset \mu_{t_1} \subset \cdots \mu_{t_k} = X$$

and

$$\nu_{t_0} \subset \nu_{t_1} \subset \cdots \nu_{t_k} = X.$$

It follows clearly that if $t_i, t_j \in Im(\mu)$ such that $t_i > t_j$ then

$$(*1) \qquad \mu_{t_i} \subset \mu_{t_j}.$$

If $s_i s_j \in Im(\nu)$ such that $s_i > s_j$ then

$$(*2) \qquad \nu_{s_i} \subset \nu_{s_j}.$$

Since the two families of level subalgebras are identical, it is clear that $\mu_{t_0} = \nu_{s_0}$. By hypothesis $\mu_{t_1} = \nu_{s_j}$ for some $j > 0$. Assume that $\mu_{t_1} \neq \nu_{s_1}$. Then $\mu_{t_1} = \nu_{s_j}$ for some $j > 1$, and $\nu_{s_1} = \mu_{t_i}$ for some $t_i < t_1$. Thus by $(*1)$ and $(*2)$ we get that $\nu_{s_j} = \mu_{t_1} \subset \mu_{t_i}$ and $\mu_{t_i} = \nu_{s_1} \subset \nu_{s_j}$. This is a contradiction. Hence $\mu_{t_1} = \nu_{s_1}$. By induction on i, $0 \le i \le k$, we finally obtain that $\mu_{t_i} = \nu_{s_i},\ 0 \le i \le k$.

(iii) Let $x \in X$ be such that $\mu(x) = t_i$ and let $\nu(x) = s_j$, where $0 \le i \le k$ and $0 \le j \le k$. It is sufficient to show that $s_j = s_i$. Now $x \in \mu_{t_i} = \nu_{s_i}$ implies that $\nu(x) = s_j \ge s_i$. This gives from $(*2)$ that $\nu_{s_j} \subseteq \nu_{s_i}$. Since $x \in \nu_{s_j}$, it follows from (ii) that $x \in \mu_{t_j}$ and so $\mu(x) = t_i \ge t_j$, which implies that $\mu_{t_i} \subseteq \mu_{t_j}$ by $(*1)$. Using (ii), we have $\nu_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \nu_{s_j}$. Thus $\nu_{s_i} = \nu_{s_j}$, and by Lemma 3.11, $s_j = s_i$. This completes the proof. □

THEOREM 3.13. *Let μ and ν be two fuzzy subalgebras of a finite B-algebra X such that the families of level subalgebras of μ and ν are identical. Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$.*

PROOF. If $\mu = \nu$ then clearly $Im(\mu) = Im(\nu)$. Conversely assume that $Im(\mu) = Im(\nu)$. For convenience, let us denote $Im(\mu) = \{t_0, t_1, \cdots, t_r\}$ and $Im(\nu) = \{s_0, s_1, \cdots s_r\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \cdots s_r$. Then $s_0 \in Im(\nu) = Im(\mu)$. Thus $s_0 = t_{k_0}$ for some k_0. Assume that $t_{k_0} \neq t_0$. So $t_{k_0} < t_0$. Now $s_1 \in Im(\mu)$, and hence $s_1 = t_{k_1}$ for some k_1. Since $s_0 > s_1$, we have $t_{k_0} > t_{k_1}$. Continuing in this way, we have $t_{k_0} > t_{k_1} > \cdots > t_{k_r}$. Since $s_0 = t_{k_0} < t_0$, this contradicts the fact that $Im(\mu) = Im(\nu)$. Hence we must have $s_0 = t_0$. Proceeding this manner, we get that $s_i = t_i$, $0 \leq i \leq r$. Now let $x_0, x_1, \cdots, x_r$ be distinct elements of X such that $\mu(x_i) = t_i$, $0 \leq i \leq r$. By Theorem 3.12, $\nu(x_i) = s_i$, $0 \leq i \leq r$. Since $s_i = t_i$, it follows that $\mu(x) = \nu(x)$ for each $x \in X$. Therefore $\mu = \nu$ and therefore the proof is complete. □

Let F denote the class of fuzzy subalgebras of a finite B-algebra X. Define a relation " $\sim$ " on F as follows: for any $\mu, \nu \in F$, $\mu \sim \nu$ if and only if μ and ν have the identical family of level subalgebras. We note that by Theorem 3.10, two elements μ and ν of F may be such that $\mu \sim \nu$ but μ and ν need not be equal. The following lemma is easy to prove.

LEMMA 3.14. *The relation $\sim$ is an equivalence relation.*

If $\mu \in F$, let $[\mu]$ denote the equivalence class determined by μ. Since X is a finite B-algebra, the number of distinct level subalgebras of X is finite because each level subalgebras is a subalgebra of X. From Theorem 3.4, it follows that the number of possible chains of level subalgebras is also finite. Since each equivalence class is characterized completely by its chain of level subalgebras, we have the following theorem:

THEOREM 3.15. *If X is a finite B-algebra, then the number of distinct equivalence classes in F is finite.*

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