Free Vibration Analysis of Non-symmetric Thin-Walled Curved Beams with Shear Deformation

전단변형을 고려한 비대칭 박벽 곡선보의 자유진동 해석

김 남 일*　　김 문 영**　　민 병 철***

Kim, Nam Il　　Kim, Moon Young　　Min Byoung Cheol

국문요약

본 연구에서는 전단변형을 고려한 비대칭 박벽 곡선보의 자유진동해석을 수행할 수 있는 일반이론을 제시하기 위하여, 3차원 연속체에 대한 가상일의 원리로부터 전단변형 효과를 고려하고 비대칭 박벽단면과 뒴(warping)을 포함하는 변위장을 도심 축에 대해 정의한 후 곡선보의 변형도-변위관계로부터 공간 박벽 곡선보의 일반화된 탄성변형에너지와 운동에너지를 새롭게 유도한다. 또한, 전단변형이 고려된 곡선보의 총포텐셜에너지에 대해 변분을 취함으로써 평형방정식과 힘-변위관계를 제시한다. 한편, 제시된 이론에 대해 등매개 보요소를 도입하여 유한요소 정식화를 수행하였으며 곡선보의 동적 거동특성을 조사하기 위하여 전단변형, 곡률효과 그리고 진동모드에 대한 매개변수 연구를 수행한다. 마지막으로, 본 연구의 타당성을 입증하기 위하여, 다양한 해석예제에 대한 3차원 고유진동수를 산정하고 타 연구자들의 결과 및 ABAQUS의 쉘요소를 이용한 해석결과와 비교·검증한다.

주요어 : 곡선보, 박벽 단면, 뒴, 전단변형, 자유진동, 등매개 보요소

ABSTRACT

For spatial free vibration of non-symmetric thin-walled curved beams with shear deformation, an improved formulation is proposed in the present study. The elastic strain and the kinetic energies are first derived by considering constant curvature and shear deformation effects due to shear forces and restrained warping torsion. Next equilibrium equations and force-deformation relations are obtained using a stationary condition of total potential energy. And the finite element procedures are developed by using isoparametric curved beam element with arbitrary thin-walled sections. Particularly not only shear deformation and thickness-curvature effects on vibration behaviors of curved beams but also mode transition and crossover phenomena with change in curvatures of beams are parametrically investigated. In order to illustrate the accuracy and the reliability of this study, various numerical solutions for spatial free vibration are compared with results by available references and ABAQUS's shell element.

Key words : curved beam, thin-walled, warping, shear deformation, free vibration, isoparametric beam element

1. Introduction

The spatially vibrational behavior of thin-walled curved beam structures is very complex due to the coupling effect of extensional, bending and torsional deformation. Investigation into the free vibration for thin-walled curved members with open and closed cross sections has been carried out extensively since the early works of Vlasov[(1)] and Timoshenko and Gere.[(2)]

Many researches[(3)-(21)] on the free in-plane vibration of curved beam have been done considering various parameters such as boundary conditions, shear deformation, rotary inertia, variable curvatures and variable cross sections. Particularly several authors[(3)-(13)] have reported on the mode transition phenomena which are characterized by sharp increase in frequencies of some modes that occurs at combinations of curvature and length of curved beam. Tarnopolskaya et. al[(3),(4)] examined the phenomenon of mode transition from extensional to inextensional that accompanies an increase in beam curvature. Chidamparam and Leissa[(5)] investigated the influence of extensibility on the in-plane free vibration frequencies of loaded circular arches and the crossover phenomenon which occur at the crossings of the frequency curves under the extensional condition. Also Charpie and Burroughs[(6)] presented an analytical model for the free in-plane vibration of curved beam with variable curvature and depth and concluded that the vibration frequencies and mode shapes are far more dependent upon variable depth than upon curvature. Scott and Woodhouse[(7)] studied the vibration of S-shaped strip of uniform cross-section and interpreted the variation of frequency with change in curvature by membrane and bending theories. Petyt and Fleischer[(8)] examined the transformation of the mode shape at the stage of the increase in eigenvalues using the finite element analysis. Even though a significant amount of research has been conducted on the frequency crossover phenomenon for the vibration of curved beam, it is judged that most of these studies was limited to only the in-plane

* 성균관대학교 토목환경공학과 박사과정

** 정회원 · 성균관대학교 토목환경공학과 교수(대표저자 : kmye@skku.ac.kr)

*** 인덕대학 건설환경설계전공 전임강사

본 논문에 대한 토의를 2003년 10월 31일까지 학회로 보내 주시면 그 결과를 게재하겠습니다.　(논문접수일 : 2003. 3. 21 / 심사종료일 : 2003. 5. 7)

free vibration behavior and did not analyze the effect of shear deformation in connection with frequency crossover.

On the other hand, the research for spatial free vibrations including the out-of-plane vibration behavior of curved beam has been performed by several authors[(22)-(34)]. Cortinez et al.[(22)] and Piovan et al.[(23)] investigated the out-of-plane vibration of simply and continuous supported thin-walled curved beam with shear deformation but they did not take fully into account the effect of thickness-curvature and shear deformation due to shear forces and restrained torsional moment of curved beam. Interestingly for the spatially coupled vibrational behavior of curved beam, Gendy and Saleeb[(33)] presented an effective formulation on spatial free vibration of arbitrary thin-walled curved beam by including the shear deformation and rotary inertia. However, they partially considered the effect of thickness-curvature and shear deformation. Recently Kim et al.[(34)] presented an analytical and numerical solutions on a spatial free vibration of thin-walled curved beam with non-symmetric section neglecting shear deformation effects.

Despite those extensive studies, it is judged that there still remain some margins to improve in formulating coupled vibration problems of curved beams having non-symmetric thin-walled cross sections. Now, in this paper, more improved formulations than previous studies are introduced as follows :

i) The shear deformable displacement field for non-symmetric thin-walled curved beams having constant curvature is first introduced in which all parameters including the normalized warping function are defined at the centroid.

ii) Force-deformation relationships due to simple shear and *warping-torsional shear* stresses as well as normal stresses considering the thickness-curvature effects are presented in the general form.

iii) The elastic strain and the kinetic energies of curved beams considering shear deformation effects due to shear forces and warping-torsion are formulated and the resulting equilibrium equations of curved beams are derived using the stationary principle of the total potential energy.

iv) Finite element procedures are presented by developing isoparametric curved beam elements with arbitrary thin-walled sections.

v) Spatial free vibration behavior of curved beams is investigated through the parametric studies. Particularly, not only shear deformation and thickness-curvature effects but also mode transition and crossover phenomena with change in curvature are investigated on spatial free vibration for non-symmetric curved beams.

2. Free Vibration Theory of Thin-walled Curved Beams with Shear Deformation

To degenerate a spatial free vibrational theory for the continuum to that for the thin-walled curved beam with shear deformation, the following assumptions are adopted:

1. The thin-walled curved beam is linearly elastic and prismatic.
2. The cross section is rigid with respect to in-plane deformation except for warping.
3. The axis of curvature does not necessarily coincide with one of the principal axes.
4. The effects of distortional deformations are negligible but shear deformations due to shear forces and restrained warping torsion are taken into account.

2.1 Kinematics and force-deformation relations of thin-walled curved beams

To develop a general theory for free vibration analysis of shear deformable thin-walled curved beams consistently, a curvilinear coordinate system (x_1 , x_2 , x_3) shown in Fig. 1 is adopted in which the x_1 axis coincides with a centroidal axis having the radius of curvature R but x_2 , x_3 are not necessarily principal inertia axes according to assumption 3.

Now to introduce the displacement field for the non-symmetric thin-walled cross section, seven displacement parameters are shown in Fig. 2(a). U_x , U_y , U_z and ω_1, ω_2, ω_3 are rigid body translations and rotations of the cross section with respect to x_1, x_2 and x_3 axes, respectively. f is a displacement parameter measuring warping deformations. In addition, (x_2^p, x_3^p) means principal axes defined at the centroid where α is the angle between x_2^p and x_2 axes in the counterclockwise direction. Assuming that the cross section is rigid with respect to in-plane deformation, the total displacement field can be written as follows:

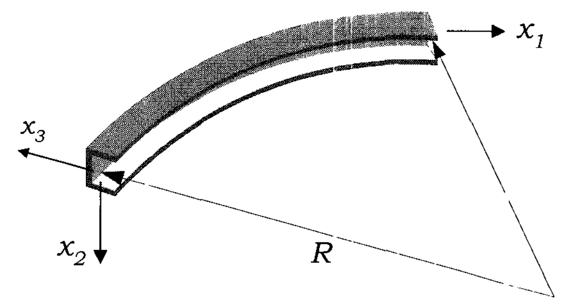
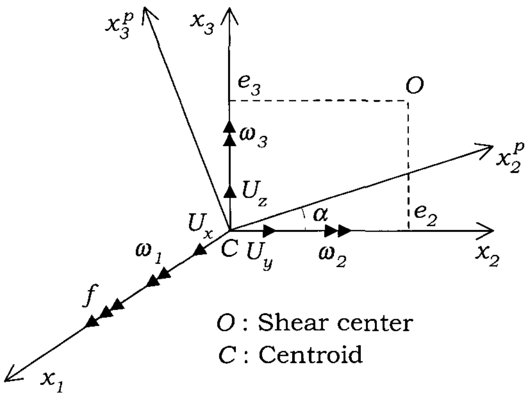
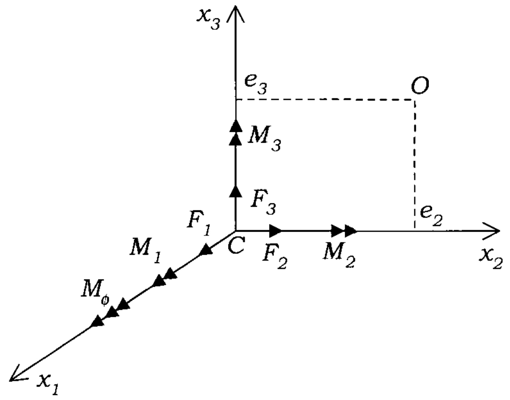


Fig. 1 Coordinate system of non-symmetric thin-walled curved beam



(a) Displacement parameters



(b) Stress resultants

Fig. 2 Notation for Displacement Parameters and Stress Resultants

$$U_1 = U_x - x_2\,\omega_3 + x_3\,\omega_2 + f\,\phi(x_2, x_3) \tag{1a}$$

$$U_2 = U_y - x_3\,\omega_1 \tag{1b}$$

$$U_3 = U_z + x_2\,\omega_1 \tag{1c}$$

where

ϕ : the normalized warping function defined at the centroid

Stress resultants with respect to the centroid are defined as follows:

$$F_1 = \int_A \tau_{11}\,dA, \qquad F_2 = \int_A \tau_{12}\,dA,$$
$$F_3 = \int_A \tau_{13}\,dA, \qquad M_1 = \int_A (\tau_{13}x_2 - \tau_{12}x_3)\,dA,$$
$$M_2 = \int_A \tau_{11}x_3\,dA, \qquad M_3 = -\int_A \tau_{11}x_2\,dA,$$
$$M_\phi = \int_A \tau_{11}\,\phi\,dA,$$
$$M_R = \int_A \left[\tau_{12}\,\phi_{,2} + \tau_{13}\left(\phi_{,3} - \frac{\phi}{R+x_3}\right)\right]\frac{R+x_3}{R}\,dA \tag{2a-h}$$

where

F_1 : the axial force

F_2 and F_3 : shear forces acting at the centroid

M_1 : the total twist moment with respect to the centroid axis

M_2 and M_3 : bending moments with respect to x_2 and x_3 axes

M_R and M_ϕ : restrained torsional moment and the bi-moment about the x_1 axis

In addition, sectional properties are defined by

$$I_2 = \int_A x_3^2\,dA, \qquad I_3 = \int_A x_2^2\,dA, \qquad I_{23} = \int_A x_2x_3\,dA,$$
$$I_\phi = \int_A \phi^2\,dA, \qquad I_{\phi2} = \int_A \phi\,x_3\,dA, \quad I_{\phi3} = \int_A \phi\,x_2\,dA,$$
$$I_{222} = \int_A x_3^3\,dA, \qquad I_{223} = \int_A x_2\,x_3^2\,dA, \quad I_{233} = \int_A x_2^2\,x_3\,dA,$$
$$I_{333} = \int_A x_2^3\,dA, \qquad I_{\phi22} = \int_A \phi\,x_3^2\,dA, \quad I_{\phi33} = \int_A \phi\,x_2^2\,dA,$$
$$I_{\phi23} = \int_A \phi\,x_3x_2\,dA, \quad I_{\phi\phi2} = \int_A \phi^2x_3\,dA,$$
$$I_{\phi\phi3} = \int_A \phi^2x_2\,dA \tag{3a-o}$$

where

A , I_2 , I_3 and I_{23}: the cross sectional area, second moments of inertia and product moment of inertia about x_2 and x_3 axes, respectively

I_ϕ : the warping moment of inertia. Also, normal and shear strain-displacement relations may be expressed as follows

$$e_{11} = \left[\left(U_x{}' + \frac{U_z}{R}\right) - x_2\left(\omega_3{}' - \frac{\omega_1}{R}\right) + x_3\,\omega_2{}' + \phi f'\right]\frac{R}{R+x_3} \tag{4a}$$

$$2e_{12} = (U_y{}' - \omega_3)\frac{R}{R+x_3} - \left(\phi_{,2} + \frac{x_3R}{R+x_3}\right)\left(\omega_1{}' + \frac{\omega_3}{R}\right) + \left(f + \omega_1{}' + \frac{\omega_3}{R}\right)\phi_{,2} \tag{4b}$$

$$2e_{13} = \left(U_z{}' + \omega_2 - \frac{U_x}{R}\right)\frac{R}{R+x_3} + \left(\frac{x_2R}{R+x_3} - \phi_{,3} + \frac{\phi}{R+x_3}\right)\left(\omega_1{}' + \frac{\omega_3}{R}\right) + \left(f + \omega_1{}' + \frac{\omega_3}{R}\right)\left(\phi_{,3} - \frac{\phi}{R+x_3}\right) \tag{4c}$$

Now substituting (4a) into (2a), (2e), (2f), (2g) and integrating over the cross section leads to

$$\begin{Bmatrix} F_1 \\ M_2 \\ M_3 \\ M_\phi \end{Bmatrix} = E \begin{bmatrix} A + \frac{\hat{I}_2}{R^2} & -\frac{\hat{I}_2}{R} & \frac{\hat{I}_{23}}{R} & -\frac{\hat{I}_{\phi2}}{R} \\ -\frac{\hat{I}_2}{R} & \hat{I}_2 & -\hat{I}_{23} & \hat{I}_{\phi2} \\ \frac{\hat{I}_{23}}{R} & -\hat{I}_{23} & \hat{I}_3 & -\hat{I}_{\phi3} \\ -\frac{\hat{I}_{\phi2}}{R} & \hat{I}_{\phi2} & -\hat{I}_{\phi3} & \hat{I}_\phi \end{bmatrix} \begin{Bmatrix} U_x{}' + \frac{U_z}{R} \\ \omega_2{}' \\ \omega_3{}' - \frac{\omega_1}{R} \\ f' \end{Bmatrix} \tag{5a-d}$$

where

E : Young's modulus and

$$\hat{I}_2 = I_2 - \frac{I_{222}}{R}, \quad \hat{I}_3 = I_3 - \frac{I_{233}}{R},$$
$$\hat{I}_{23} = I_{23} - \frac{I_{223}}{R}, \quad \hat{I}_\phi = I_\phi - \frac{I_{\phi\phi2}}{R},$$
$$\hat{I}_{\phi3} = I_{\phi3} - \frac{I_{\phi23}}{R}, \quad \hat{I}_{\phi2} = I_{\phi2} - \frac{I_{\phi22}}{R} \tag{6a-f}$$

and also, a following approximation is used.

$$\frac{R}{R+x_3} \cong 1 - \frac{x_3}{R} + \left(\frac{x_3}{R}\right)^2 \tag{7}$$

On the other hand, it may be assumed that the force-deformation relations for shear forces, restrained torsional moment and St. Venant torsion in curved beams take the same form as those in straight beams[35] except that the curvature effect terms should be added to average shear deformations. Accordingly force-deformation relations due to shear stresses are given by

$$\begin{Bmatrix} F_2 \\ F_3 \\ M_R \end{Bmatrix} = G \begin{bmatrix} A_2 & A_{23} & A_{2r} \\ A_{23} & A_3 & A_{3r} \\ A_{2r} & A_{3r} & A_r \end{bmatrix} \begin{Bmatrix} U_y' - \omega_3 \\ U_z' - \frac{U_x}{R} + \omega_2 \\ \omega_1' + \frac{\omega_3}{R} + f \end{Bmatrix} \tag{8a-c}$$

$$M_{st} = M_1 - M_R = M_1 - M_\phi' = GJ\left(\omega_1' + \frac{\omega_3}{R}\right) \tag{8d}$$

where

G : the shear modulus

J : torsional constant and

$$A_2 = A_2^s \cos^2\alpha + A_3^s \sin^2\alpha,$$
$$A_3 = A_3^s \cos^2\alpha + A_2^s \sin^2\alpha,$$
$$A_r = A_r^s + A_2^s e_3^2 + A_3^s e_2^2,$$
$$A_{23} = (A_2^s + A_3^s)\cos\alpha \sin\alpha,$$
$$A_{2r} = -A_2^s e_3 \cos\alpha - A_3^s e_2 \sin\alpha,$$
$$A_{3r} = -A_2^s e_3 \sin\alpha + A_3^s e_2 \cos\alpha \tag{9a-f}$$

where

A_2^s, A_3^s and A_r^s : effective shear areas defined by

$$\frac{1}{A_2^s} = \frac{1}{I_{3p}^2}\int_A Q_3^2 \frac{ds}{t},$$
$$\frac{1}{A_3^s} = \frac{1}{I_{2p}^2}\int_A Q_2^2 \frac{ds}{t},$$
$$\frac{1}{A_r^s} = \frac{1}{(I_\phi^s)^2}\int_A Q_r^2 \frac{ds}{t} \tag{10a-c}$$

and

$$I_{2p} = \int_A (x_3^p)^2\, dA, \quad I_{3p} = \int_A (x_2^p)^2\, dA,$$
$$I_\phi^s = \int_A (\phi^s)^2\, dA, \quad Q_2 = \int_0^s x_3^p t\, ds,$$
$$Q_3 = \int_0^s x_2^p t\, ds, \quad Q_r = \int_0^s \phi^s t\, ds \tag{11a-f}$$

Consequently (5) and (8) constitute force-deformation relations of shear deformable thin-walled curved beams. In reference to (8d), it may be demonstrated that a supplementary equilibrium equation ($M_R = M_\phi'$) is satisfied between the bimoment M_ϕ and the restrained warping torsion M_R.

2.2 Principle of linearized virtual work for thin-walled curved beams

The principle of linearized virtual work for the general continuum vibrating harmonically is expressed as

$$\int_V \tau_{ij}\delta e_{ij}\, dV - \omega^2 \int_V \rho U_i \delta U_i\, dV = \int_S T_i \delta U_i\, dS \tag{12}$$

where

τ_{ij} and e_{ij} : stress and linear strain, respectively

ρ : density

ω : circular frequency

T_i : surface force

U_i : displacement.

In case of the thin-walled circular beam, (12) may be transformed to principle of the total potential energy Π as follows:

$$\Pi = \Pi_E - \Pi_M - \Pi_{ext} \tag{13}$$

where the detailed expressions for each term of Π are

$$\Pi_E = \frac{1}{2}\int_0^L \int_A \left[\tau_{11}e_{11} + 2\tau_{12}e_{12} + 2\tau_{13}e_{13}\right]\frac{R+x_3}{R}\, dA\, dx_1 \tag{14a}$$

$$\Pi_M = \frac{1}{2}\rho\omega^2 \int_0^L \int_A \left[U_1^2 + U_2^2 + U_3^2\right]\frac{R+x_3}{R}\, dA\, dx_1 \tag{14b}$$

$$\Pi_{ext} = \frac{1}{2}\int_S T_i U_i\, dS \tag{14c}$$

Substituting the linear strain (4) into (14a) and integrating over the cross sectional area, (14a) are reduced to the following equation.

$$\Pi_E=\frac{1}{2}\int_o^L\Big[F_1\Big(U_x'+\frac{U_z}{R}\Big)+M_2\,\omega_2'+M_3\Big(\omega_3'-\frac{\omega_1}{R}\Big)$$
$$+M_\phi f'+F_2(U_y'-\omega_3)+F_3\Big(U_z'-\frac{U_x}{R}+\omega_2\Big)$$
$$+(M_1-M_R)\Big(\omega_1'+\frac{\omega_3}{R}\Big)+M_R\Big(\omega_1'+\frac{\omega_3}{R}+f\Big)\Big]dx_1 \qquad (15)$$

Now substitution of the force-deformation relations (5) and (8) into (15) leads to the elastic strain energy:

$$\Pi_E=\frac{1}{2}\int_0^L\Big[EA\Big(U_x'+\frac{U_z}{R}\Big)^2+E\hat{I}_2\Big(\omega_2'-\frac{U_x'}{R}-\frac{U_z}{R^2}\Big)^2$$
$$+E\hat{I}_3\Big(\omega_3'-\frac{\omega_1}{R}\Big)^2+E\hat{I}_\phi f'^2$$
$$+2E\widehat{I_{\phi2}}\Big(\omega_2'-\frac{U_x'}{R}-\frac{U_z}{R^2}\Big)f'-2E\widehat{I_{\phi3}}\Big(\omega_3'-\frac{\omega_1}{R}\Big)f'$$
$$-2E\widehat{I_{23}}\Big(\omega_3'-\frac{\omega_1}{R}\Big)\Big(\omega_2'-\frac{U_x'}{R}-\frac{U_z}{R^2}\Big)$$
$$+GJ\Big(\omega_1'+\frac{\omega_3}{R}\Big)^2+GA_2(U_y'-\omega_3)^2$$
$$+GA_3\Big(U_z'-\frac{U_x}{R}+\omega_2\Big)^2+GA_r\Big(\omega_1'+\frac{\omega_3}{R}+f\Big)^2$$
$$+2GA_{23}(U_y'-\omega_3)\Big(U_z'-\frac{U_x}{R}+\omega_2\Big)$$
$$+2GA_{2r}(U_y'-\omega_3)\Big(\omega_1'+\frac{\omega_3}{R}+f\Big)$$
$$+2GA_{3r}\Big(U_z'-\frac{U_x}{R}+\omega_2\Big)\Big(\omega_1'+\frac{\omega_3}{R}+f\Big)\Big]dx_1 \qquad (16)$$

Also substituting the displacement field (1) into Eq. (14b), the kinetic energy Π_M including shear deformation and rotary inertia can be obtained as

$$\Pi_M=\frac{1}{2}\rho\omega^2\int_0^L[A(U_x^2+U_y^2+U_z^2)+\tilde{I}_o\,\omega_1^2$$
$$+\tilde{I}_2\,\omega_2^2+\tilde{I}_3\,\omega_3^2+\tilde{I}_\phi f^2+2\frac{I_2}{R}(U_x\omega_2-U_y\omega_1)$$
$$-2\frac{I_{23}}{R}(U_x\omega_3-U_z\omega_1)-2\widetilde{I_{23}}\,\omega_2\,\omega_3$$
$$+2\widetilde{I_{\phi2}}\omega_2 f-2\widetilde{I_{\phi3}}\omega_3 f+2\frac{I_{\phi2}}{R}U_x f\Big]dx_1 \qquad (17)$$

where

$$\tilde{I}_o=I_2+I_3+\frac{I_{222}+I_{233}}{R},$$
$$\tilde{I}_2=I_2+\frac{I_{222}}{R},\quad \tilde{I}_3=I_3+\frac{I_{233}}{R},\quad \widetilde{I_{23}}=I_{23}+\frac{I_{223}}{R},$$
$$\tilde{I}_\phi=I_\phi+\frac{I_{\phi\phi2}}{R},\quad \widetilde{I_{\phi2}}=I_{\phi2}+\frac{I_{\phi22}}{R},\quad \widetilde{I_{\phi3}}=I_{\phi3}+\frac{I_{\phi23}}{R} \qquad (18a\text{-}g)$$

Finally invoking the stationary condition of the total potential energy, equations of motion and boundary conditions for curved beams are obtained as

$$-EA\Big(U_x''+\frac{1}{R}U_z'\Big)+\frac{1}{R}E\hat{I}_2\Big(\omega_2''-\frac{1}{R}U_x''-\frac{1}{R^2}U_z'\Big)$$
$$+\frac{1}{R}E\widehat{I_{\phi2}}f''-\frac{1}{R}E\widehat{I_{23}}\Big(\omega_3''-\frac{1}{R}\omega_1'\Big)$$
$$-\frac{1}{R}GA_3\Big(U_z'+\omega_2-\frac{1}{R}U_x\Big)-\frac{1}{R}GA_{23}(U_y'-\omega_3)$$
$$-\frac{1}{R}GA_{3r}\Big(\omega_1'+f+\frac{1}{R}\omega_3\Big)$$
$$=\rho\omega^2\Big(AU_x+\frac{1}{R}I_2\omega_2-\frac{1}{R}I_{23}\omega_3+\frac{1}{R}I_{\phi2}f\Big)$$

$$-GA_2(U_y''-\omega_3')-GA_{23}\Big(U_z''+\omega_2'-\frac{1}{R}U_x'\Big)$$
$$-GA_{2r}\Big(\omega_1''+f'+\frac{1}{R}\omega_3'\Big)$$
$$=\rho\omega^2\Big(AU_y-\frac{1}{R}I_2\omega_1\Big)$$

$$\frac{1}{R}EA\Big(U_x'+\frac{1}{R}U_z\Big)-\frac{1}{R^2}E\hat{I}_2\Big(\omega_2'-\frac{1}{R}U_x'-\frac{1}{R^2}U_z\Big)$$
$$-\frac{1}{R^2}E\widehat{I_{\phi2}}f'+\frac{1}{R^2}E\widehat{I_{23}}\Big(\omega_3'-\frac{1}{R}\omega_1\Big)$$
$$-GA_3\Big(U_z''+\omega_2'-\frac{1}{R}U_x'\Big)-GA_{23}(U_y''-\omega_3')$$
$$-GA_{3r}\Big(\omega_1''+f'+\frac{1}{R}\omega_3'\Big)$$
$$=\rho\omega^2\Big(AU_z+\frac{1}{R}I_{23}\omega_1\Big)$$

$$-\frac{1}{R}E\hat{I}_3\Big(\omega_3'-\frac{1}{R}\omega_1\Big)+\frac{1}{R}E\widehat{I_{\phi3}}f'$$
$$+\frac{1}{R}E\widehat{I_{23}}\Big(\omega_2'-\frac{1}{R}U_x'-\frac{1}{R^2}U_z\Big)-GJ\Big(\omega_1''+\frac{1}{R}\omega_3'\Big)$$
$$-GA_r\Big(\omega_1''+f'+\frac{1}{R}\omega_3'\Big)-GA_{2r}(U_y''-\omega_3')$$
$$-GA_{3r}\Big(U_z''+\omega_2'-\frac{1}{R}U_x'\Big)$$
$$=\rho\omega^2\Big(\tilde{I}_o\omega_1-\frac{1}{R}I_2U_y+\frac{1}{R}I_{23}U_z\Big)$$

$$-E\hat{I}_2\Big(\omega_2''-\frac{1}{R}U_x''-\frac{1}{R^2}U_z'\Big)-E\widehat{I_{\phi2}}f''$$
$$+E\widehat{I_{23}}\Big(\omega_3''-\frac{1}{R}\omega_1'\Big)+GA_3\Big(U_z'+\omega_2-\frac{1}{R}U_x\Big)$$
$$+GA_{23}(U_y'-\omega_3)+GA_{3r}\Big(\omega_1'+f+\frac{1}{R}\omega_3\Big)$$
$$=\rho\omega^2\Big(\tilde{I}_2\omega_2+\frac{1}{R}I_2U_x-\widetilde{I_{23}}\omega_3+\widetilde{I_{\phi2}}f\Big)$$

$$-E\hat{I}_3\Big(\omega_3''-\frac{1}{R}\omega_1'\Big)+E\widehat{I_{\phi3}}f''$$
$$+E\widehat{I_{23}}\Big(\omega_2''-\frac{1}{R}U_x''-\frac{1}{R^2}U_z'\Big)+\frac{1}{R}GJ\Big(\omega_1'+\frac{1}{R}\omega_3\Big)$$
$$-GA_2(U_y'-\omega_3)-GA_{23}\Big(U_z'+\omega_2-\frac{1}{R}U_x\Big)$$
$$+\frac{1}{R}GA_r\Big(\omega_1'+f+\frac{1}{R}\omega_3\Big)+\frac{GA_{3r}}{R}\Big(U_z'+\omega_2-\frac{1}{R}U_x\Big)$$
$$-GA_{2r}\Big(\omega_1'+f-\frac{1}{R}U_y'+\frac{2}{R}\omega_3\Big)$$
$$=\rho\omega^2\Big(\tilde{I}_3\omega_3-\frac{1}{R}I_{23}U_x-\widetilde{I_{23}}\omega_2-\widetilde{I_{\phi3}}f\Big)$$

$$- E\widehat{I}_\phi f'' - E\widehat{I}_{\phi 2}\left(\omega_2'' - \frac{1}{R}U_x'' - \frac{1}{R^2}U_z'\right) + E\widehat{I}_{\phi 3}\left(\omega_3'' - \frac{1}{R}\omega_1'\right) + GA_r\left(\omega_1' + f + \frac{1}{R}\omega_3\right) + GA_{2r}(U_y' - \omega_3) + GA_{3r}\left(U_z' + \omega_2 - \frac{1}{R}U_x\right) = \rho\omega^2\left(\widetilde{I}_\phi f + \widetilde{I}_{\phi 2}\omega_2 - \widetilde{I}_{\phi 3}\omega_3 + \frac{1}{R}I_{\phi 2}U_x\right) \qquad (19a\text{-}g)$$

and

$$\delta U_x(o) = \delta U_x^p \ \text{ or } \ F_1(o) = -F_1^p\ ; \quad \delta U_x(l) = \delta U_x^q \ \text{ or } \ F_1(l) = F_1^q \qquad (20a,b)$$

$$\delta U_y(o) = \delta U_y^p \ \text{ or } \ F_2(o) = -F_2^p\ ; \quad \delta U_y(l) = \delta U_y^q \ \text{ or } \ F_2(l) = F_2^q \qquad (20c,d)$$

$$\delta U_z(o) = \delta U_z^p \ \text{ or } \ F_3(o) = -F_3^p\ ; \quad \delta U_z(l) = \delta U_z^q \ \text{ or } \ F_3(l) = F_3^q \qquad (20e,f)$$

$$\delta \omega_1(o) = \delta \omega_1^p \ \text{ or } \ M_1(o) = -M_1^p\ ; \quad \delta \omega_1(l) = \delta \omega_1^q \ \text{ or } \ M_1(l) = M_1^q \qquad (20g,h)$$

$$\delta \omega_2(o) = \delta \omega_2^p \ \text{ or } \ M_2(o) = -M_2^p\ ; \quad \delta \omega_2(l) = \delta \omega_2^q \ \text{ or } \ M_2(l) = M_2^q \qquad (20i,j)$$

$$\delta \omega_3(o) = \delta \omega_3^p \ \text{ or } \ M_3(o) = -M_3^p\ ; \quad \delta \omega_3(l) = \delta \omega_3^q \ \text{ or } \ M_3(l) = M_3^q \qquad (20k,l)$$

$$\delta f(o) = \delta f^p \ \text{ or } \ M_\phi(o) = -M_\phi^p\ ; \quad \delta f(l) = \delta f^q \ \text{ or } \ M_\phi(l) = M_\phi^q \qquad (20m,n)$$

In neglecting the shear deformations and putting θ in place of ω_1 , Eqs. (16) and (17) are reduced to

$$\Pi_E = \frac{1}{2}\int_0^L \Big[EA\left(U_x' + \frac{U_z}{R}\right)^2 + E\widehat{I}_2\left(U_z'' + \frac{U_z}{R^2}\right)^2 + E\widehat{I}_3\left(U_y'' - \frac{\theta}{R}\right)^2 + E\widehat{I}_\phi\left(\theta'' + \frac{U_y''}{R}\right)^2 + GJ\left(\theta' + \frac{U_y'}{R}\right)^2 + 2E\widehat{I}_{\phi 2}\left(U_z'' + \frac{U_z}{R^2}\right)\left(\theta'' + \frac{U_y''}{R}\right) + 2E\widehat{I}_{\phi 3}\left(U_y'' - \frac{\theta}{R}\right)\left(\theta'' + \frac{U_y''}{R}\right) + 2E\widehat{I}_{23}\left(U_y'' - \frac{\theta}{R}\right)\left(U_z'' + \frac{U_z}{R^2}\right)\Big] dx_1 \qquad (21)$$

$$\Pi_M = \frac{1}{2}\rho\omega^2\int_0^L \Big[A(U_x^{\,2} + U_y^{\,2} + U_z^{\,2}) + \widetilde{I}_o\theta^2 + \widetilde{I}_2\left(U_z' - \frac{U_x}{R}\right)^2 - 2\frac{I_2}{R}U_y\theta - 2\frac{I_2}{R}U_x\left(U_z' - \frac{U_x}{R}\right) + \widetilde{I}_3 U_y'^{\,2} + \widetilde{I}_\phi\left(\theta' + \frac{U_y'}{R}\right)^2 + 2\widetilde{I}_{\phi 3}U_y'\left(\theta' + \frac{U_y'}{R}\right) + 2\widetilde{I}_{\phi 2}\left(U_z' - \frac{U_x}{R}\right)\left(\theta' + \frac{U_y'}{R}\right) - 2\frac{I_{\phi 2}}{R}U_x\left(\theta' + \frac{U_y'}{R}\right) + 2\frac{I_{223}}{R}U_y'\left(U_z' - \frac{U_x}{R}\right) + 2I_{23}\left(U_y'U_z' - \frac{2}{R}U_xU_y' + \frac{1}{R}U_z\theta\right)\Big] dx_1 \qquad (22)$$

which are identical to Eq. (10) and (12) in Kim et al.[34].

3. Isoparametric Thin-walled Curved Beam Elements

In this Section, isoparametric curved beam elements having arbitrary thin-walled cross sections are presented. The element has seven degrees of freedom per a node. Generally the reduced integration scheme is adopted to avoid the shear locking phenomena.

Fig. 3 shows the nodal displacement vector of 3-noded isoparametric thin-walled curved beam element. In this study, 2-, 3-, and 4-noded isoparametric curved beam element are introduced to interpolate displacement parameters that are defined at the centroid axis. Resultantly, the coordinate and all the displacement parameters of the *curved* beam element can be interpolated with respect to the nodal coordinates and displacements, respectively, as follows:

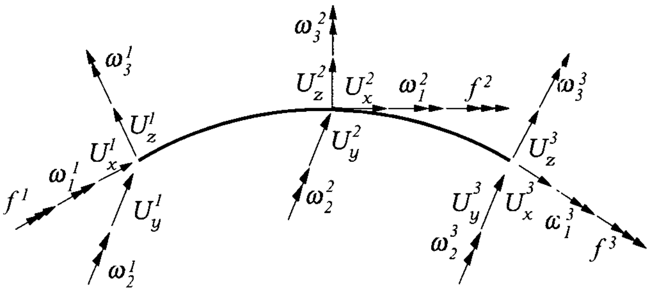


Fig. 3 Nodal displacement vector of a 3-noded isoparametric curved beam element

$$x_1 = \frac{L}{2}(1 + r) \qquad (23a)$$

$$U_i = \sum_{\alpha=1}^{n} N_\alpha(r)\, U_i^\alpha \qquad i = x, y, z \qquad (23b)$$

$$\omega_i = \sum_{\alpha=1}^{n} N_\alpha(r)\, \omega_i^\alpha \qquad i = 1, 2, 3 \qquad (23c)$$

$$f = \sum_{\alpha=1}^{n} N_\alpha(r)\, f^\alpha \qquad (23d)$$

where n is the total number of node a element; U_i^α, ω_i^α and f^α are the translational and rotational displacements in the x_i direction and warping parameter at node α , respectively; N_α is the isoparametric interpolation function whose the detailed expression is presented in Bathe[36]; r is a natural coordinate that varies from -1 to +1.

Substituting the shape functions, cross-sectional properties into Eqs. (16) and (17) and integrating along the element length, the total potential energy of thin-walled curved beam element is obtained in matrix form as

$$\Pi=\frac{1}{2}\,\boldsymbol{U}_e^T\,(\boldsymbol{K}_e-\omega^2\,\boldsymbol{M}_e)\,\boldsymbol{U}_e \tag{24}$$

where $\boldsymbol{K}_e$ and $\boldsymbol{M}_e$ are element elastic stiffness and mass matrices in local coordinate, respectively. $\boldsymbol{U}_e$ is nodal displacement vector which is defined as

$$\boldsymbol{U}_e=[\,U^1\,,U^2\,,\cdots,\,U^n\,] \tag{25a}$$

$$\boldsymbol{U}^\alpha=[\,U_x^\alpha\,,U_y^\alpha\,,U_z^\alpha\,,\,\omega_1^\alpha\,,\,\omega_2^\alpha\,,\,\omega_3^\alpha\,,f^\alpha\,]^T,\quad \alpha=1\,,2\,,\cdots,\,n \tag{25b}$$

where elastic stiffness matrix is evaluated using a reduced Gauss numerical integration scheme.

Now using direct stiffness method, the matrix equilibrium equation for the free vibration analysis of non-symmetric thin-walled curved beam is obtained as

$$\boldsymbol{K}_E\,\boldsymbol{U}=\omega^2\,\boldsymbol{M}_E\,\boldsymbol{U} \tag{26}$$

where $\boldsymbol{K}_E$ and $\boldsymbol{M}_E$ are global elastic stiffness and mass matrices, respectively.

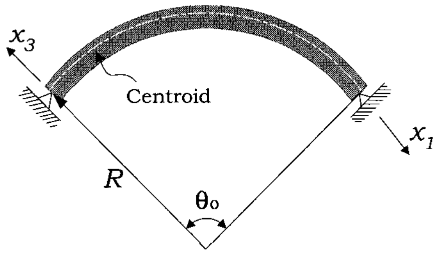
4. Numerical Examples

Numerical results analyzed by this curved beam element are presented and compared with other researchers' analytical solutions and results by shell element of ABAQUS[37]. Also, parametric studies for spatial free vibration of curved beams with respect to various subtended angle and boundary conditions are performed in this Section.

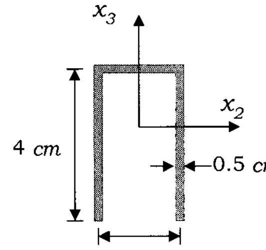
4.1 Convergence study

To examine the convergence properties of the isoparametric thin-walled curved beams developed by this study, we consider a simply supported curved beam with monosymmetric cross section for the x_3 axis. The geometric and material data are given in Fig. 4, in which the subtended angle θ_o is taken to be 90°and the length of beam l is 250cm.

The convergence study using two, three and four-noded isoparametric curved beam elements is performed for in-plane and out-of-plane free vibrational cases and Figs. 5 and 6 show plots of a number of elements versus fundamental in-plane and out-of-plane normalized frequencies, respectively, where ω_{30}^2 denotes the frequency obtained by 30 four-noded isoparametric curved beam elements. It may be noticed that the convergence speeds of both three and four-noded elements are much higher than those of two-noded element for two cases. Furthermore, convergence speeds for out-of-plane frequency are higher than those for in-plane frequency. With 5 elements, the ratios ω^2/ω_{30}^2



(a) Simply supported curved beam



(b) Monosymmetric cross section

$A=5.0\text{cm}^2$, $E=73000\text{kg/cm}^2(71613\times10^5\text{Pa})$,
$G=28000\text{kg/cm}^2(27468\times10^5\text{Pa})$, $R=127.324\text{cm}$,
$\rho=0.00785\text{kg/cm}^3$, $J=0.41667\text{cm}^4$, $e_2=0$ cm,
$e_3=3.44615\text{cm}$, $I_2=8.53333$ cm^4, $I_3=4.33333\text{cm}^4$,
$I_\phi=58.02667\text{cm}^6$, $I_{\phi3}=-14.9333$ cm^5, $I_{222}=-2.56\text{cm}^5$,
$I_{233}=-1.06667\text{cm}^5$, $I_{\phi\phi2}=-59.904\text{cm}^7$, $I_{\phi23}=10.24\text{cm}^6$,
$A_2^s=0.46788\text{cm}^2$, $A_3^s=3.41344\text{cm}^2$, $A_r^s=3.47515\text{cm}^4$

(c) Material and section properties

Fig. 4 Circular curved beam with monosymmetric cross section

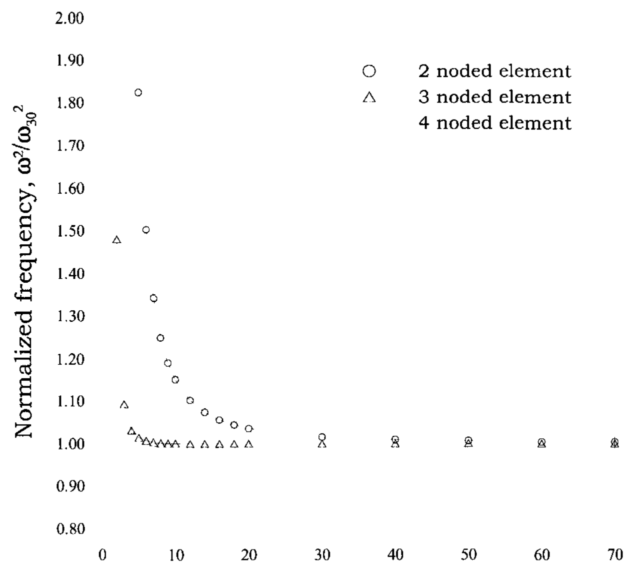


Fig. 5 Normalized in-plane frequencies

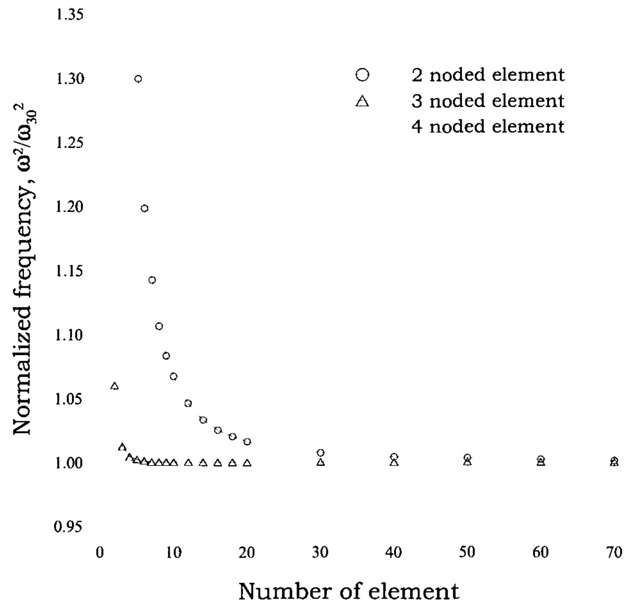


Fig. 6 Normalized out-of-plane frequencies

are 1.823, 1.013 and 1.000 for two, three and four-noded elements, respectively for in-plane frequencies and 1.299, 1.002 and 1.000 for out-of-plane frequencies. Based on results of this convergence study, in subsequent examples on the free vibrational problems of thin-walled curved beam, a curved beam is modeled by 20 three-noded curved beam elements.

4.2 Simply supported curved beam with doubly symmetric cross sections

To compare the result by present theory with that by other researcher, in-plane vibration behaviors of the simply supported curved beam with doubly symmetric cross sections are examined. Consider in-plane vibration of curved beams with square cross section whose subtended angle is 10° and 100° with the constant radius. Material and geometric data used for analysis are follows:

$E = 73000\text{kgf/cm}^2(71613\times10^5\text{Pa})$,
$G = 28000\text{kgf/cm}^2(27468\times10^5\text{Pa})$,
$\rho = 0.00785\text{kgf/cm}^3$,
$b = h = 1\text{cm}$, $R = 100\text{cm}$

Numerical solutions by this study for the lowest four frequencies are presented in Table 1 with solutions by the previous study[34] neglecting only shear deformation and results by Chidamparam and Leissa[5] neglecting effects of shear deformation, rotary inertia and thickness-curvature. Table 1 shows that the maximum difference of the present results with those by the previous study[34] and with Chidamparam and Leissa's results are 12.4% and 17.2%, respectively, at the fourth natural frequency for the angle is 10°.

Table 1 In-plane natural frequencies of doubly symmetric beam (radian/sec)^2

Angle(θ_o)	Mode	Present study	Reference[34]	Chidamparam[5]
10	1	1550.05	1556.79	1564.99
	2	12416.9	12826.5	12963.4
	3	60005.7	64335.5	65897.8
	4	177374.	199312.	207855.
100	1	0.89954	0.89976	0.89978
	2	5.20058	5.20297	5.20349
	3	19.0033	19.0120	19.0183
	4	46.7508	46.7534	46.7779

4.3 Thin-walled curved beam with monosymmetric section

In this example, in-plane and out-of-plane vibrational behaviors of monosymmetric curved beams are investigated through the various parametric studies. The same geometric and material data of curved beam as the one used in the example 4.1 are adopted(see Fig. 4).

Table 2 shows the first five in-plane natural frequencies with respect to the two subtended angles for simply supported curved beams whose length is 50 and 200cm, respectively. For comparison, the results by 20 cubic Hermitian beam elements without shear deformation effect are together presented.

And in Table 3 is listed the first five out-of-plane natural frequencies by this study with shear deformation and by Hermitian beam element without shear deformation. It can be noticed that relative differences of frequencies due shear deformation effects are large in higher vibrational modes.

Table 2 In-plane natural frequencies of monosymmetric beam(radian/sec)^2

l	θ_o (R)	Mode	With shear def.	Without shear def.
50	30(95.493)	1	1041.96	1048.64
		2	3398.32	3731.69
		3	15613.7	18850.2
		4	36737.3	37264.3
		5	43238.4	57386.0
	180(15.916)	1	1147.45	1265.21
		2	8981.06	10787.5
		3	31214.9	34479.5
		4	33400.2	40603.0
		5	68574.1	73140.7
200	30(381.972)	1	14.8261	14.9206
		2	44.3843	44.5777
		3	89.7269	90.6257
		4	237.870	243.762
		5	576.534	598.278
	180(63.662)	1	4.94910	4.98098
		2	45.6069	46.2587
		3	183.525	188.096
		4	476.864	494.805
		5	1038.69	1093.71

Table 3 Out-of-plane natural frequencies of monosymmetric beam (radian/sec)^2

l	θ_o (R)	Mode	With shear def.	Without shear def.
50	30(95.493)	1	102.591	111.133
		2	516.130	558.077
		3	553.315	576.594
		4	1475.67	1789.61
		5	3300.75	4507.84
	180(15.916)	1	0.00000	0.00000
		2	763.956	946.746
		3	2617.51	3353.05
		4	3162.79	3490.99
		5	4589.14	6456.94
200	30(381.972)	1	0.34352	0.34495
		2	7.44025	7.61160
		3	28.8735	30.0129
		4	38.6432	38.7100
		5	65.6371	68.9167
	180(63.662)	1	0.00000	0.00000
		2	0.87050	0.87389
		3	17.0841	17.5055
		4	98.3089	106.156
		5	222.994	248.870

The fundamental in-plane symmetric and antisymmetric frequencies for the simply supported(S-S) and clamped(C-C) curved beams of length l=100cm versus various subtended angles have been plotted in Fig. 7. As shown in Fig. 7, the antisymmetric frequencies decrease slowly as the subtended angle increases whereas the symmetric frequencies experience a sharp increase. Also this Figure exhibits the phenomenon of mode crossover, which occurs at the crossing of symmetric and antisymmetric natural frequency curves. The crossover occurs at the subtended angle around θ_o= 31°, 50° for simply supported and clamped conditions, respectively. Furthermore, after the crossover point is passed, it is observed that frequencies corresponding to the symmetric mode increase rapidly and then change very slowly as the subtended angle increases further. In this range, it turns out that the sym- metric mode having one half-wave is transformed into the symmetric mode with three half-waves (see Fig. 8). This is called mode transition phenomenon for the symmetric in-plane vibrational mode of simply supported curved beams. A similar situation occurs for the clamped ends.

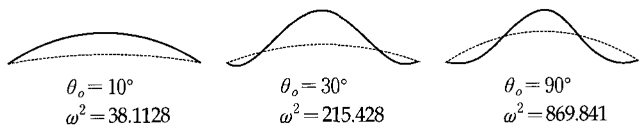


Fig. 8 Mode transition of fundamental symmetric modes for the simply supported beam

Now, effects of shear deformation on the in-plane natural frequencies are examined. Fig. 9 shows relative differences of the fundamental in-plane frequencies due to shear deformation with the increase of subtended angles for simply supported and clamped ends, respectively, in which $\omega*^2$ denotes the natural frequency neglecting shear deformation. It is interesting to note that effect of shear deformation jumps upward at the point of crossover for simply supported and clamped ends. In the low range of subtended angle, effect of shear deformation is small because the fundamental vibrational mode is symmetric mode with one half-wave as can be seen Fig. 8. However after the crossover point, the fundamental mode changes into the antisymmetric mode with two half-waves, in which shear deformation effect is large. Also it can be

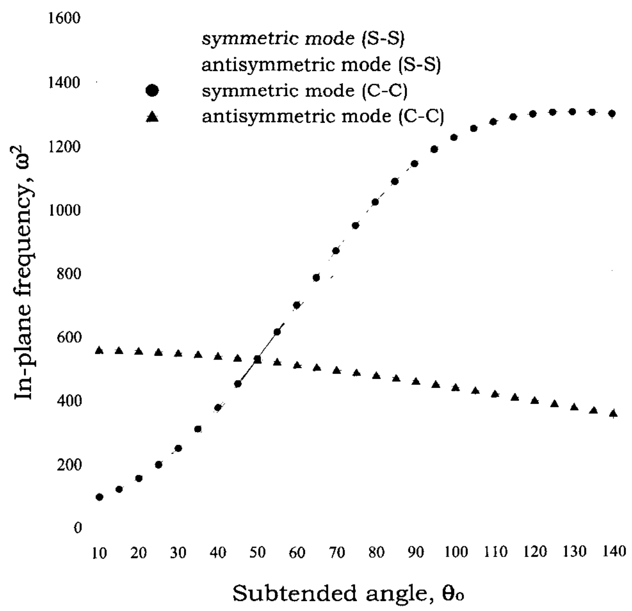


Fig. 7 In-plane frequency with subtended angle

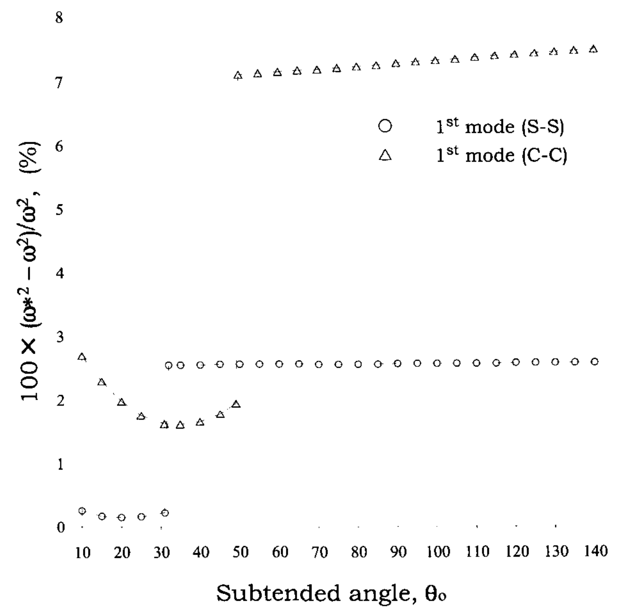
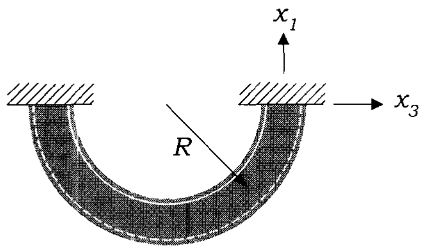


Fig. 9 Effect of shear deformation of fundamental in-plane frequency

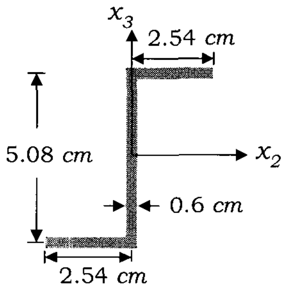
found that the shear deformation effect of curved beam with clamped ends is not always larger than that of simply supported curved beam in the whole range of subtended angles due to the phenomenon of crossover mentioned above. In addition, shear deformation effect of the antisymmetric mode is almost constant with increase of subtended angle for both boundary conditions.

4.4 Clamped semicircular beams with Z-sections

This example considers the clamped semicircular beam with Z-section of equal flanges as shown in Fig. 10. Table 4 shows the lowest five spatially coupled natural frequencies by present study. For comparison, the numerical solutions[34] neglecting shear deformation and the results by the Gendy and Saleeb[33] and those by the shell element model of Noor *et al.*[38] are together presented. From Table 4, it is observed that the results by the present study are better than those by Gendy and Saleeb' model when comparing with those using the shell element. Also the maximum difference of results by this study and those by Gendy and Saleeb's model is 6.4% at third vibrational mode.



(a) Clamped curved beam



(b) Non-symmetric Z cross section

$A=6.096\text{cm}^2$, $E=730887\text{kgf/cm}^2(717\times10^8\text{Pa})$,
$G=281110\text{kgf/cm}^2(275769\times10^5\text{Pa})$, $\rho=0.002768\text{kgf/cm}^3$,
$J=0.73152\text{cm}^4$, $e_2=0\text{cm}$, $e_3=0\text{cm}$, $I_2=26.2193\text{cm}^4$,
$I_3=6.55483\text{cm}^4$, $I_\phi=26.4307\text{cm}^6$, $I_{\phi2}=-21.1446\text{cm}^5$,
$I_{\phi3}=-21.1446\text{cm}^5$, $I_{\phi23}=-26.4307\text{cm}^6$, $R=100\text{cm}$,
$A_2^s=2.32404\text{cm}^2$, $A_3^s=2.89801\text{cm}^2$, $A_r^s=12.0494\text{cm}^4$

(c) Material and section properties

Fig. 10 Clamped semicircular curved beam with Z cross section

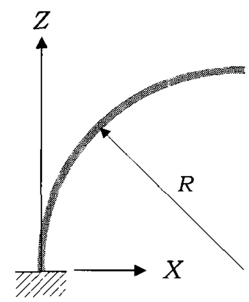
Table 4 Natural frequencies of clamped semicircular beam(radian/sec)²

Mode	Present study	Reference[34]	Gendy & Saleeb[33]	Shell[38]
1	4.8198	4.8963	4.8993	4.5199
2	23.119	23.252	23.852	22.777
3	107.22	107.90	114.06	105.60
4	162.93	166.50	168.55	156.79
5	333.93	336.97	335.82	329.87

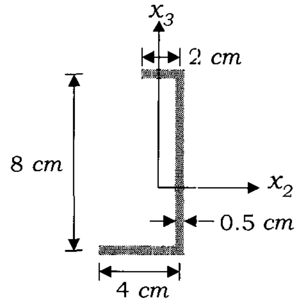
4.5 Cantilever and clamped curved beams with non-symmetric cross sections

In this example, the spatially coupled free vibration analysis of non-symmetric curved beams with clamped-free and clamped-clamped ends is performed for various subtended angles. Fig. 11 shows a non-symmetric curved cantilever beam and its material and sectional properties. In Tables 5 and 6, the lowest ten frequencies of curved cantilever and clamped-clamped curved beams of length l=200cm are presented with respect to the various subtended angles.

The solutions by present study, the solutions[34] neglecting shear deformation and the results by 300 shell elements of ABAQUS which is the commercial F. E. analysis program are presented for comparison. From Tables, it is shown that the results by present method are in a good agreement with those by ABAQUS's shell elements. As shown in Table 5, for the curved cantilever beam, the maximum difference of frequencies due to shear deformation is 6.0%



(a) Geometry of curved beam



(b) Cross section

$A=7\text{cm}^2$, $E=30000\text{kgf/cm}^2(2943\times10^6\text{Pa})$,
$G=11500\text{kgf/cm}^2(112815\times10^4\ \text{Pa})$, $\rho=0.00785\text{kgf/cm}^3$,
$J=0.5833\text{cm}^4$, $I_2=67.0476\text{cm}^4$, $I_3=8.4286\text{cm}^4$,
$I_{23}=9.1429\text{cm}^4$, $I_{222}=52.2449\text{cm}^5$, $I_{223}=-20.0272\text{cm}^5$,
$I_{233}=-17.4150\text{cm}^5$, $I_{333}=-13.3878\text{cm}^5$, $I_\phi=272.544\text{cm}^6$,
$I_{\phi2}=115.810\text{cm}^5$, $I_{\phi3}=30.4762\text{cm}^5$, $I_{\phi22}=59.2109\text{cm}^6$,
$I_{\phi23}=-107.102\text{cm}^6$, $I_{\phi33}=-63.1293\text{cm}^6$, $I_{\phi\phi2}=-67.1720\text{cm}^7$,
$I_{\phi\phi3}=-388.727\text{cm}^7$, $A_2^s=1.6935\text{cm}^2$, $A_3^s=3.4815\text{cm}^2$,
$A_r^s=26.708\text{cm}^4$

(c) Material and section properties

Fig. 11 Cantilever curved beam with non-symmetric cross section

Table 5 Natural frequencies of non-symmetric cantilever curved beam(radian/sec)^2

θ_o (R)	Method	Vibration mode									
		1	2	3	4	5	6	7	8	9	10
10(1145.916)	Present study	0.0289	0.2672	0.5938	1.5157	5.0785	7.5997	16.973	20.192	26.497	50.875
	Reference[34]	0.0290	0.2686	0.5963	1.5252	5.1373	7.7438	17.386	20.623	27.159	52.344
30(381.972)	Present study	0.0211	0.2798	0.3737	2.2526	4.9884	7.2984	19.109	20.051	27.454	47.676
	Reference[34]	0.0212	0.2815	0.3747	2.2666	5.0554	7.4328	19.493	20.511	28.177	49.067
60(190.986)	Present study	0.0107	0.2470	0.3067	2.3617	5.6800	7.0548	17.886	27.702	30.422	43.573
	Reference[34]	0.0107	0.2480	0.3084	2.3788	5.8332	7.1242	18.221	28.219	31.322	44.828
90(127.324)	Present study	0.0062	0.2051	0.2883	2.0111	5.0525	7.2818	17.197	31.627	37.201	46.719
	Reference[34]	0.0062	0.2061	0.2901	2.0272	5.2139	7.3646	17.473	32.844	37.949	47.721
	ABAQUS[37]	0.0060	0.2043	0.2779	1.9714	5.0293	7.1815	17.079	32.233	36.624	43.574
120(95.493)	Present study	0.0042	0.1598	0.2929	1.6744	4.2126	6.9482	16.484	32.775	35.555	66.874
	Reference[34]	0.0043	0.1608	0.2945	1.6893	4.3515	7.0502	16.728	34.383	36.373	67.954
150(76.394)	Present study	0.0034	0.1213	0.3127	1.3800	3.5354	6.3441	15.567	31.995	33.572	66.454
	Reference[34]	0.0034	0.1222	0.3141	1.3938	3.6529	6.4512	15.798	33.813	34.377	68.559
180(63.662)	Present study	0.0030	0.0922	0.3447	1.1265	3.0410	5.6769	14.500	29.706	31.903	62.142
	Reference[34]	0.0030	0.0929	0.3462	1.1389	3.1403	5.7825	14.724	31.503	32.765	64.026

Table 6 Natural frequencies of non-symmetric clamped curved beam(radian/sec)^2

θ_o (R)	Method	Vibration mode									
		1	2	3	4	5	6	7	8	9	10
10(1145.916)	Present study	0.9388	4.3755	6.2020	16.887	18.361	20.637	47.850	57.241	95.128	105.91
	Reference[34]	0.9488	4.4120	6.3262	17.732	18.778	21.295	49.634	59.534	99.775	119.58
30(381.972)	Present study	0.8264	5.2791	10.645	17.672	21.490	30.441	43.626	65.695	88.794	110.16
	Reference[34]	0.8338	5.3737	10.799	18.125	22.087	31.469	45.206	68.388	93.079	123.91
60(190.986)	Present study	0.7681	4.4271	15.044	23.538	26.510	38.338	58.008	80.708	101.98	117.80
	Reference[34]	0.7753	4.4992	15.392	24.041	27.515	39.667	60.538	84.626	104.35	131.16
90(127.324)	Present study	0.7134	3.9278	13.272	30.219	34.103	41.075	67.752	76.159	128.49	137.44
	Reference[34]	0.7223	3.9916	13.570	31.829	35.223	41.852	71.047	80.658	138.20	148.88
	ABAQUS[37]	0.7020	3.9088	13.388	30.838	34.855	37.792	69.831	78.659	115.15	140.53
120(95.493)	Present study	0.6344	3.5618	11.986	30.249	31.713	63.380	66.200	79.918	124.23	151.78
	Reference[34]	0.6446	3.6238	12.257	31.523	33.616	64.519	68.869	86.169	130.81	165.18
150(76.394)	Present study	0.5420	3.2285	10.998	27.214	30.484	60.262	83.318	94.335	115.54	180.24
	Reference[34]	0.5522	3.2906	11.259	28.378	32.733	62.680	90.153	96.772	121.55	193.33
180(63.662)	Present study	0.4495	2.8955	10.162	24.061	28.316	55.990	84.329	107.63	129.94	188.48
	Reference[34]	0.4589	2.9571	10.422	25.347	30.400	58.225	93.334	113.14	131.99	200.85

at the eighth mode for the subtended angle θ_o=180°. Also it is observed that the shear deformation effects in the clamped-clamped curved beam are relatively larger than those in the cantilever beam and the maximum difference is 12.9% at the tenth mode for the subtended angle θ_o=10°.

5. Conclusion

For spatial free vibration analysis of shear deformable curved beams having arbitrary thin-walled cross sections, an improved theory is formulated. And the isoparametric curved beam elements are developed. The F. E. solutions by the present study are compared with other researchers' results and numerical results using ABAQUS's shell elements and the parametric study is performed. Consequently, conclusions drawn from this study are as follows:

i) *Mode transition* phenomena from the symmetric mode having one half-wave to the symmetric mode with three half-waves are observed in case of in-plane free vibration of simply supported and clamped curved beams. In this case, shear deformation effects increase sharply around the mode transition point due to the increase of the half-wave number.

ii) For spatially coupled free vibration problem of non-symmetric thin-walled cantilever and clamped curved beams, it is shown that the numerical solutions by the isoparametric curved beam element considering shear effects are in a good agreement with the results by shell elements of ABAQUS.

Acknowledgements

The authors are grateful for the support provided by a grant(R01-2002-000-00265-0) from the Korea Science and Engineering Foundation(KOSEF).

References

1. Vlasov, V. Z., *Thin-walled elastic beams*, National Science Foundation, Washington, D.C, 1961.
2. Timoshenko, S. P. and Gere, J. M., *Theory of elastic stability*, McGraw-Hill, N.Y., 1961.
3. Tarnopolskaya, T., Hoog, F. R. De, and Fletcher, N. H., "Low-frequency mode transition in the free in-plane vibration of curved beams," *Journal of Sound and Vibration* 228, 1999, pp. 69-90.
4. Tarnopolskaya, T., Hoog, F. R. De, Fletcher, N. H., and Thwaites, S., "Asymptotic analysis of the free in-plane vibrations of beam with arbitrarily varying curvature and cross-section," *Journal of Sound and Vibration* 196, 1996, pp. 659-680.
5. Chidamparam, P. and Leissa, A. W., "Influence of centerline extensibility on the in-plane free vibrations of loaded circular arches," *Journal of Sound and Vibration* 183, 1995, pp. 779-795.
6. Charpie, J. P. and Burroughs, C. B., "An analytic model for the free in-plane vibration of beams of variable curvature and depth," *Journal of the Acoustical Society of America* 94, 1993, pp. 866-879.
7. Scott, J. F. M. and Woodhouse, J., "Vibration of an elastic strip with varing curvature," *Philosophical Transactions of the Royal Society of London, Physical Sciences and Engineering* 339, 1992, pp. 587-625.
8. Petyt, M. and Fleischer, C. C., "Free vibration of a curved beam," *Journal of Sound and Vibration* 18, 1971, pp. 17-30.
9. Krishnan, A. and Suresh, Y. J., "A simple cubic linear element for static and free vibration analysis of curved beams," *Computers & Structures* 68, 1998, pp. 473-489.
10. Huang, C. S., Tseng, Y. P., Leissa, A. W., and Nieh, K. Y., "An exact solution for in-plane vibrations of an arch having variable curvature and cross section," *International Journal of Mechanical Sciences* 40, 1998, pp. 1159-1173.
11. Tseng, Y. P., Huang, C. S., and Lin, C. J., "Dynamic stiffness analysis for in-plane vibrations of arches with variable curvature," *Journal of Sound and Vibration* 207, 1997, pp. 15-31.
12. Suzuki, K. and Takahashi, S., "In-plane vibration of curved bars with varying cross-section," *Bulletin of the Japan Society of Mechanical Engineers* 25, 1982, pp. 1100-1107.
13. Irie, T., Yamada, G., and Takahashi, I., "In-plane vibration of a free-clamped slender arc of varying cross-section," *Bulletin of the Japan Society of Mechanical Engineers* 23, 1980, pp. 567-573.
14. Raveendranath, P., Singh, Gajbir, and Pradhan, B., "Free vibration of arches using a curved beam element based on a coupled polynomial displacement field," *Computers & Structures* 78, 2000, pp. 583-590.
15. Oh, S. J., Lee, B. K., and Lee, I. W., "Natural frequencies of non-circular arches with rotatory inertia and shear deformation," *Journal of Sound and Vibration* 219, 1999, pp. 23-33.
16. Gupta, A. K. and Howson, W. P., "Exact natural frequencies of plane structures composed of slender elastic curved members," *Journal of Sound and Vibration* 175, 1994, pp. 145-157.
17. Lee, B. K. and Wilson, J. F., "Free vibrations of arches with variable curvature," *Journal of Sound and Vibration* 136, 1989, pp. 75-89.
18. Issa, M. S., Wang, T. M., and Hsiao, B. T., "Extensional vibrations of continuous circular curved beams with rotary inertia and shear deformation, I: free vibration," *Journal of Sound and Vibration* 114, 1987, pp. 297-308.
19. Wang, T. M. and Guilbert, M. P., "Effects of rotary inertia and shear on natural frequencies of continuous circular curved beams," *International Journal of Solids and Structures* 17, 1981, pp. 281-289.
20. Wang, T. M., "Lowest natural frequency of clamped parabolic arcs," *Journal of Structural Division* 98, 1972, pp. 407-411.
21. Anestis S. Veletsos, Austin, Walter J., Lopes, Carlos A., and Wung, Shyr-jen, "Free in-plane vibration of circular arches," *Journal of Engineering Mechanics Division* 98, 1972, pp. 311-329.
22. Cortinez, V. H., Piovan, M. T., and Rossi, R. E., "Out of plane vibrations of thin walled curved beams considering shear flexibility," *Structural Engineering and Mechanics* 8, 1999, pp. 257-272.
23. Piovan, M. T., Cortinez, V. H., and Rossi, R. E., "Out-of-plane vibrations of shear deformable continuous horizontally curved thin-walled beams," *Journal of Sound and Vibration* 237, 2000, pp. 101-118.
24. Howson, W. P. and Jemah, A. K., "Exact out-of-plane natural frequencies of curved Timoshenko beams," *Journal of Engineering Mechanics* 125, 1999, pp. 19-25.
25. Howson, W. P., Jemah, A. K., and Zhou, J. Q., "Exact natural fre- quencies for out-of-plane motion of plane

structures composed of curved beam members," *Computers & Structures* 55, 1995, pp. 989-995.
26. Snyder, J. M. and Wilson, J. F., "Free vibrations of continuous horizontally curved beams," *Journal of Sound and Vibration* 157, 1992, pp. 345-355.
27. Wang, T. M. and Brannen, W. F., "Natuarl frequencies for out-of-plane vibrations of curved beams on elastic foundations," *Journal of Sound and Vibration* 84, 1982, pp. 241-246.
28. Wang, T. M., Nettleton, R. H., and Keita, B., "Natural frequencies for out-of-plane vibrations of continuous curved beams," *Journal of Sound and Vibration* 68, 1980, pp. 427-436.
29. Suzuki, K., Aida, H., and Takahashi, S., "Vibrations of curved bars perpendicular to their planes," *Bulletin of the Japan Society of Mechanical Engineers* 21, 1978, pp. 162-173.
30. Takahashi, S. and Suzuki, K., "Vibrations of elliptic arc bar perpendicular to its plane," *Bulletin of the Japan Society of Mechanical Engineers* 20, 1977, pp. 1409-1416.
31. Kawakami, M., Sakiyama, T., Matsuda, H., and Morita, C., "In-plane and out-of-plane free vibrations of curved beams with variable sections," *Journal of Sound and Vibration* 187, 1995, pp. 381-401.
32. Kang, K., Bert, C. W., and Striz, A. G., "Vibration analysis of shear deformable circular arches by the differential quadrature method," *Journal of Sound and Vibration* 181, 1995, pp. 353-360.
33. Gendy, A. S. and Saleeb, A. F., "Vibration analysis of coupled extensional/flexural/torsional modes of curved beams with arbitrary thin-walled sections," *Journal of Sound and Vibration* 174, 1994, pp. 261-274.
34. Kim, M. Y., Kim, N. I., and Min, B. C., "Analytical and numerical study on spatial free vibration of non-symmetric thin-walled curved beams," *Journal of Sound and Vibration* 258, 2002, pp. 595-618.
35. Kim, M. Y., Kim, N. I., and Yun, H. T., "Exact dynamic and static stiffness matrices of shear deformable thin-walled beam-columns(accepted for publication)," *Journal of Sound and Vibration*, 2003.
36. Bathe, K. J., *Finite Element Procedures*, Prentice- Hall, Englewood Cliffs, N. J., 1996.
37. ABAQUS, *User's Manual Vol. I and Vol. II. Ver. 5.2*, Hibbit, Karlsson, 1992.
38. Noor, A. K., Peters, J. M., and Min, B. J., "Mixed finite element models for free vibrations of thin-walled beams," *Finite Elements in Analysis and Design* 5, 1989, pp. 291-305.