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Estimation of P(Y<X) in the Exponential Distribution with Censored Data Using Minimax Regret Significance Levels

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Abstract

Given a prior guess for the unknown value of P(Y<X) when X and Y are independently exponentially distributed, the preliminary test estimator based on the maximum likelihood estimator for the exponential scale parameter with censored data is developed. The optimal significance levels based on the minimax regret criterion and the corresponding critical values are numerically obtained.

Key Words: Censored data, Exponential Distribution, Maximum likelihood estimator, Minimax regret criterion, Preliminary test estimator, Optimum significance levels, Rayleigh distribution.

1. Introduction

The problem of estimating $R = p(Y < X)$ when X and Y are independently exponentially distributed has received prominent attention in the literature. This probability may be interpreted as a measure of the reliability or performance of an item of strength Y subject to a stress X, or the probability that one component fails prior to another component of some device.

Tong (1974,1975) obtained the uniformly minimum variance unbiased (UMVU) estimator for R when X and Y are independently exponentially distributed, Beg (1980) obtained the UMVU estimator for R when X and Y have two parameter exponential distributions with unequal scale and location parameters. Gupta and Gupta (1988) obtained the maximum likelihood estimator, the UMVU estimator, and a Bayesian estimator for R when

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the location parameters are unequal but there is a common scale parameter. Bai and Hong (1992) obtained the UMVU estimator of R with unequal sample sizes when X and Y are independent two parameter exponential random variables with an unknown common location parameter.

In some applications, the experimenter possesses some knowledge about the parameter. This knowledge may obtained from past experience, or from the aquaintance with similar situations. Thus he is in position to give an educated guess or prior estimate. This prior information may be incorporated in the estimation process using a preliminary test estimator (Ohtani and Toyoda, 1978; Toyoda and Wallace, 1975; Sawa and Hiromatsu, 1973). Thus improving the estimation process. In this paper we present a preliminary test estimator for the parameter of the Rayleigh distribution . The procedure for obtaining the optimum values of the significance levels using the minimax regret criterion of Brook (1976) is developed in section 2. The results are given in the final section.

2. Preliminary test estimation

Consider the two failure censored random samples $x_{(1)},\ldots,x_{(r_1)}$ and $y_{(1)},\ldots,y_{(r_2)}$ obtained from putting n_1 and n_2 units on test. Their lifetimes are assumed to be exponentially distributed with parameters θ_1 and θ_2 respectively. The corresponding probability density functions are;

$$f_X(t)=\frac{1}{\theta_1}e^{-\frac{t}{\theta_1}}, \qquad t,\theta_1>0.$$

$$f_Y(t)=\frac{1}{\theta_2}e^{-\frac{t}{\theta_2}}, \qquad t,\theta_2>0. \tag{1}$$

Based on the available samples, our aim is to estimate the quantity

$$R=P(Y<X)=\frac{\theta_1}{\theta_1+\theta_2} \tag{2}$$

The maximum likelihood estimators of θ_1 and θ_2 are given respectively by (Lawless, 1982);

$$\hat{\theta}_1=\frac{1}{r_1}\left(\sum_{i=1}^{r_1}x_{(i)}+(n_1-r_1)x_{(r_1)}\right), \quad \hat{\theta}_2=\frac{1}{r_2}\left(\sum_{i=1}^{r_2}y_{(i)}+(n_2-r_2)y_{(r_2)}\right) \tag{3}$$

It can be shown that $2r_i\hat{\theta}_i/\theta_i \sim \chi^2_{2r_i}$, $i=1,2$. (Lawless, 1982). The maximum likelihood estimator of R is given by $\hat{R}=\frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2}=\frac{1}{1+\hat{\theta}_2/\hat{\theta}_1}$. Let $\tau=\frac{\theta_2}{\theta_1}$, then $\hat{\tau}=\frac{\hat{\theta}_2}{\hat{\theta}_1}$ Assume that τ_0 is a prior guess of τ, or equivalently R_0 is a prior guess of R (Notice that $R=\frac{1}{1+\tau}$). Consider testing $H_0:\tau=\tau_0$ against $H_1:\tau\neq\tau_0$, it can be shown that the likelihood ratio test rejects H_0 when either $\frac{\tau_0}{\hat{\tau}}<c_1$ or $\frac{\tau_0}{\hat{\tau}}>c_2$ where c_1 and c_2 are suitably chosen constants. a preliminary test estimator $\tilde{\tau}$ of τ may be obtained as follows

$$\tilde{\tau}=\begin{cases}\tau_0, & c_1<\frac{\tau_0}{\hat{\tau}}<c_2\\ \hat{\tau}, & \text{Otherwise.}\end{cases} \tag{4}$$

where c_1 and c_2 are such that $p_{\tau_0}(W<c_1)=p_{\tau_0}(W>c_2)=\frac{\alpha}{2}$, where $W\sim F_{2r_1,2r_2}$. Our aim is to find the optimum values of α according to the minimax regret criterion. The mean of $\tilde{\tau}$ is given by

$$E(\tilde{\tau})=\tau_0E\left[I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right]+E\left[\hat{\tau}\left\{1-I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right\}\right] \tag{5}$$

Now, $E\left[I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right]=p\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)=\int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}}g(w)dw$

where $g(w)$ is the pdf of an F random variable with $(2r_1,2r_2)$ degrees of freedom. Also;

$$E\left[\hat{\tau}\left\{1-I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right\}\right]=\tau\,E\left(\frac{\hat{\tau}}{\tau}\right)-\tau\,E\left(\frac{\hat{\tau}}{\tau}I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right)$$

Notice that

$$\tau\,E\left(\frac{\hat{\tau}}{\tau}\right)=\tau\frac{r_1}{r_1-1}\ \text{ and }\ \tau\,E\left(\frac{\hat{\tau}}{\tau}I\left(\frac{c_1\tau}{\tau_0}<\frac{\tau}{\hat{\tau}}<\frac{c_2\tau}{\tau_0}\right)\right)=\tau\int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}}\frac{1}{w}g(w)dw.$$

Thus, $E(\tilde{\tau})=\tau_0\int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}}g(w)dw+\tau\frac{r_1}{r_1-1}-\tau\int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}}\frac{1}{w}g(w)dw$

Proceeding in a similar way, the second moment of $\tilde{\tau}$ is given by

$$E\left(\tilde{\tau}^2\right) = \tau_0^{\,2} \int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}} g(w)dw + \tau^2\left(\frac{r_1^{\,2}(r_1+r_2-1)}{r_2(r_1-1)(r_1-2)} + \left(\frac{r_1}{r_1-1}\right)^2\right) - \tau^2 \int_{\frac{c_1\tau}{\tau_0}}^{\frac{c_2\tau}{\tau_0}} \frac{1}{w^2} g(w)dw \qquad (6)$$

The mean squared error of $\tilde{\tau}$ is given by

$$MSE(\tilde{\tau}) = E\left(\tilde{\tau}^2\right) - \left(E(\tilde{\tau})\right)^2 + \left(E(\tilde{\tau}) - \tau\right)^2 = E\left(\tilde{\tau}^2\right) - 2\tau E(\tilde{\tau}) + \tau^2 \qquad (7)$$

Now, the quantity $\dfrac{MSE(\tilde{\tau})}{\tau^2}$ can be considered as a risk function (Chiou, 1988). Let $\psi = \dfrac{\tau_0}{\tau}$ we get the risk function

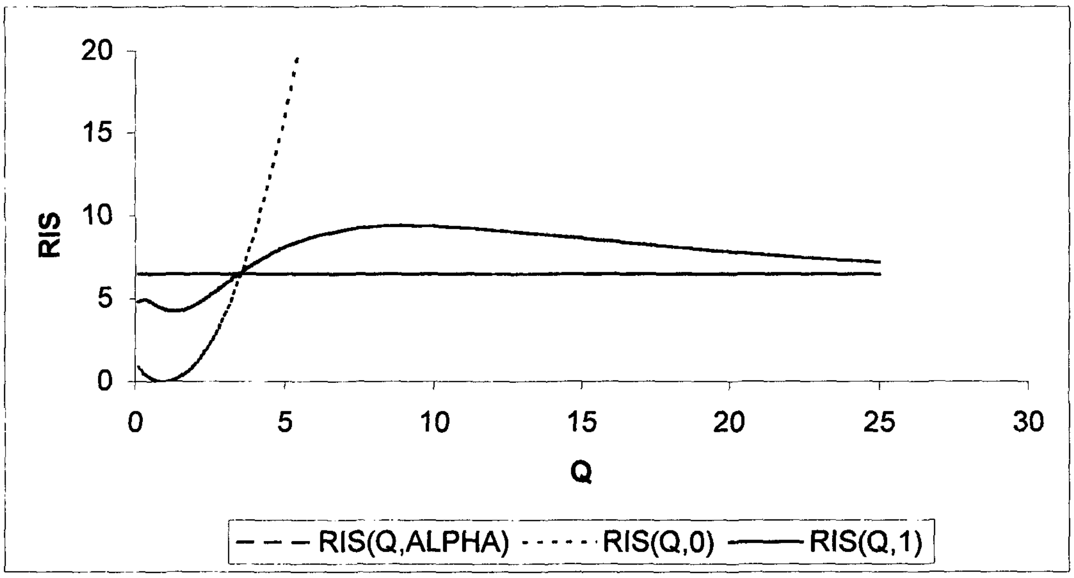
$$RIS(\psi,\alpha) = \psi^2 \int_{\frac{c_1}{\psi}}^{\frac{c_2}{\psi}} g(w)dw + \left(\frac{r_1^{\,2}(r_1+r_2-1)}{r_2(r_1-1)(r_1-2)} + \left(\frac{r_1}{r_1-1}\right)^2\right) - \int_{\frac{c_1}{\psi}}^{\frac{c_2}{\psi}} \frac{1}{w^2} g(w)dw$$

$$-2\left(\psi \int_{\frac{c_1}{\psi}}^{\frac{c_2}{\psi}} g(w)dw + \frac{r_1}{r_1-1} - \int_{\frac{c_1}{\psi}}^{\frac{c_2}{\psi}} \frac{1}{w} g(w)dw\right) + 1, \qquad (8)$$

Notice that the risk function depends on α through c_1 and c_2 which are determined such that $p_{\tau_0}(W < c_1) = p_{\tau_0}(W > c_2) = \frac{\alpha}{2}$, where $W \sim F_{2r_1,2r_2}$.

If $\psi \to 0$ or ∞, then $RIS(\psi,\alpha)$ converges to $RIS(\psi,1)$ which is the risk of the maximum likelihood estimator $\hat{\sigma}$. The shape of $RIS(\psi,\alpha)$ is given in Figure 1. The risk functions in this figure correspond to $r_1 = r_2 = 4$. In this figure the horizontal line corresponds to $\alpha = 1$, while for $\alpha = 0$ the risk function takes the value zero at the true value of the parameter an then increases sharply in both directions. The stepped line corresponds to $\alpha = 0.1071$ which is the "optimal" according to the criterion adopted.

Figure 1: Risk functions coresponding to $r_1 = r_2 = 4$ and $\alpha = 0, 0.1071, 1$.



An optimal value of α is $\alpha = 1$ if $\psi \leq \psi_1$ or $\psi \geq \psi_2$ and $\alpha = 0$ otherwise (Sawa and Hiromatsu,1973), where ψ_1 and ψ_2 are intersections of $RIS(\psi,0) = (\psi - 1)^2$ with

$$RIS(\psi,1) = \frac{r_1^2(r_1 + r_2 - 1)}{r_2(r_1 - 1)(r_1 - 2)} + \frac{1}{(r_1 - 1)^2}$$

The intersections are

$$\psi_1 = 1 - \left(\frac{r_1^2(r_1 + r_2 - 1)}{r_2(r_1 - 1)(r_1 - 2)} + \frac{1}{(r_1 - 1)^2} \right)^{1/2}$$

$$\psi_2 = 1 + \left(\frac{r_1^2(r_1 + r_2 - 1)}{r_2(r_1 - 1)(r_1 - 2)} + \frac{1}{(r_1 - 1)^2} \right)^{1/2}, \quad (9)$$

Since ψ is unknown we seek an optimal value $\alpha = \alpha^*$ which gives a reasonable risk for all values of ψ. Going along the lines of Sawa and Hiromatsu (1973), the regret function is

$$REG(\psi,\alpha) = RIS(\psi,\alpha) - \inf_{\alpha} RIS(\psi,\alpha), \quad (10)$$

where

$$\inf_{\alpha} RIS(\psi,\alpha) = \begin{cases} RIS(\psi,1), & \psi \leq \psi_1 \text{ or } \psi \geq \psi_2 \\ RIS(\psi,0), & \text{otherwise.} \end{cases}$$

For $\psi \leq \psi_2$ $REG(\psi,\alpha)$ takes a maximum value at ψ_L. For $\psi > \psi_2$, $REG(\psi,\alpha)$ takes a maximum value at ψ_U, see (Chiou, 1988; figure 1). Thus the minimax regret criterion determines α^* such that $REG(\psi_L,\alpha^*) = REG(\psi_U,\alpha^*)$. An estimator for R that uses the minimax regret significance levels now can be defined as

$$\widetilde{R} = \begin{cases} \dfrac{1}{1+\tau_0}, & c_1 < \dfrac{\tau_0}{\hat{\tau}} < c_2 \\ \dfrac{1}{1+\hat{\tau}}, & \text{otherwise.} \end{cases} \tag{11}$$

where c_1 and c_2 are such that $p_{\tau_0}(W < c_1) = p_{\tau_0}(W > c_2) = \dfrac{\alpha^*}{2}$, where $W \sim F_{2r_1,2r_2}$.

3. Results

We found numerically the optimum significance levels α^* and the corresponding critical values for all combinations of $r_1 = 3,\ldots,9$ and $r_2 = 3,\ldots,9$. The results are given in table 1.

Table 1: Optimum significance levels and the corresponding critical values.

r_1	r_2	α^*	c_1	c_2	r_1	r_2	α^*	c_1	c_2
3	3	1464	0.2797	3.5748	6	7	0426	0.2989	3.1750
	4	1038	0.2452	3.5280		8	0356	0.2904	3.1294
	5	0790	0.2221	3.4964		9	0304	0.2833	3.0953
	6	0631	0.2053	3.4736	7	3	1182	0.3706	3.6753
	7	0524	0.1924	3.4563		4	0807	0.3478	3.4879
	8	0446	0.1821	3.4428		5	0592	0.3323	3.3747
	9	0389	0.1738	3.4319		6	0457	0.3205	3.2839
4	3	1498	0.3299	3.4377		7	0366	0.3111	3.2139
	4	1071	0.2987	3.3477		8	0302	0.3034	3.1600
	5	0820	0.2775	3.2862		9	0255	0.2967	3.1170
	6	0658	0.2618	3.2414	8	3	1083	0.3738	3.7885

	7	0547	0.2496	3.2072		4	0726	0.3530	3.5877
	8	0466	0.2397	3.1803		5	0524	0.3388	3.4477
	9	0407	0.2315	3.1584		6	0399	0.3281	3.3442
5	3	1405	0.3527	3.4786		7	0315	0.3196	3.2643
	4	0994	0.3248	3.3552		8	0256	0.3124	3.2024
	5	0752	0.3058	3.2704		9	0214	0.3063	3.1538
	6	0597	0.2915	3.2083	9	3	0996	0.3753	3.9032
	7	0492	0.2802	3.1608		4	0656	0.3561	3.6805
	8	0416	0.2710	3.1230		5	0466	0.3432	3.5249
	9	0359	0.2634	3.0951		6	0349	0.3334	3.4096
6	3	1291	0.3644	3.5683		7	0272	0.3256	3.3204
	4	0898	0.3393	3.4164		8	0219	0.3190	3.2488
	5	0670	0.3221	3.3114		9	0180	0.3119	3.1988
	6	0525	0.3092	3.2342					

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