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On the Probability Inequalities under Linearly Negatively Quadrant Dependent Condition[1)]

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Abstract

Let $X_1, X_2, \cdots$ be real valued random variables under linearly negatively quadrant dependent ($LNQD$). In this paper, we discuss the probability inequality of ennett(1962) and Hoeffding(1963) under some suitable random variables. These results are to extend Theorem A and B to $LNQD$ random variables. Furthermore, let $\zeta_p$denote the pth quantile of the marginal distribution function of the X_i's which is estimated by a smooth estima te $\hat{\zeta}_{pn}$, on the basis of $X_1, X_2, \cdots, X_n$. We establish a convergence of $\hat{\zeta}_{pn}$, under Hoeffding-type probability inequality of $LNQD$.

Keywords: Linearly negatively quadrant dependent, Hoeffding-type probability inequality
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1. Introduction

Let $X_1, X_2, \cdots$ be real valued random variables defined on the underlying probability (Ω, A, P), and set S_n for the sum of the first n random variables, $\sum_{i=1}^{n} X_i$, and $\overline{S_n}$ for S_n/n. The problem of providing exponential bounds for the probabilities $P(|S_n| \geq \varepsilon)$ is of paramount importance, both in probability and statistics. From a statistical view point, such inequalities can be used, among other things, for the purpose of providing rates of convergence for estimates of various quantities. Especially so in a nonparamet ric setting, where the advantages of a parametric structure are not available to the investigator.

Exponential probability bounds for sums of random variable are very useful in many

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probabilistic derivations, and particularly so in many aspects of parametric as well as nonparametric statistical inference. Bennett(1962) and Hoeffding(1963) have obtained the following results on independent random variables.

Theorem A(Bennett 1962). Let $X_1, \cdots, X_n$ be independent random variables almost surely bounded, $|X_i| \le C_i \ a.s. \ i=1,\cdots,n$, and without loss of generality, assume them to be centered at their expectations. Set $\sigma_i^2 = Var(X_i) = EX_i^2$, $s_n^2 = \sum_{i=1}^{n} \sigma_i^2 = Var(S_n)$, where $S_n = \sum_{i=1}^{n} X_i$. Finally, let $C_0 = \max\{C_i, i=1,\cdots,n\}$. Then for every $\varepsilon > 0$,

$$P(|S_n| \ge s_n t) \le 2e^{-t^2/(2+2/3\frac{C_0 t}{s_n})}. \tag{1.1}$$

Theorem B(Hoeffding 1963). Let $X_1, \cdots, X_n$ be independent random variables such that $a_i \le X_i \le b_i, i=1,\cdots,n$. Set $\mu_i = EX_i, i=1,\cdots,n$ and $\mu = n^{-1}\sum_{i=1}^{n}\mu_i$. Then for every $\varepsilon > 0$,

$$P(|\overline{X} - \mu| \ge \varepsilon) \le 2e^{-n^2 t^2 / \sum_{i=1}^{n}(b_i - a_i)^2}. \tag{1.2}$$

However, many variables are dependent in actual problems. One of them is negative dependence. The concepts of negative dependence have been introduced by Joag and Pro schan(1983) and have found significant applications in system reliability, statistics, and may also be appropriate to model certain biosystems and ecosystems. Thus,the following definitions of negative dependence will be used in obtaining the probability inequality.

Definition 1.1. The random variables X and Y are said to be negatively quadrant dependent (NQD), if

$$P(X > x, Y > y) - P(X > x)P(Y > y) \le 0, \quad \forall x, y \in R.$$

Definition 1.2. The random variables $\{X_i, i \ge 1\}$ are said to be linearly negatively quadrant dependent ($LNQD$), if for any nonempty disjoint subsets A and B of $\{1,2,\cdots,n\}$ and for any positive λ_i's, the random variables $\sum_{i\in A}\lambda_i X_i$ and $\sum_{j\in B}\lambda_j X_j$ are NQD.

Suppose that let $\{X_j | j \ge 1\}$ be a stationary sequence of $LNQD$ random variables with continuous distribution function F. Let ζ_p denote the quantile of the distribution funct ion $F(x)$, i.e. a root of the equation $F(\zeta_p) = p$, with $0 < p < 1$,which is assumed to be unique. Then, we define an estimate of $F_n(x)$ and ζ_p as the followings,

$$\widehat{F}_n(x) = \int_R K(\frac{x-t}{h_n}) dF_n(t) = 1/n \sum_{=0}^{n} K(\frac{x-X_j}{h_n}),$$

where K is a continuous distribution function and $\{h_n; n\geq 1\}$ is a sequence of bandwi dths with $0 < h_n \to 0$ as $n\to\infty$,

$$\hat{\zeta}_{pn} = \hat{F}_n^{-1}(p) = inf\{x\in R| \hat{F}_n(x) \geq p\}.$$

The main purpose of this paper is to extend Theorem A and Theorem B to *LNQD* random variables. Furthermore, convergence of $\hat{\zeta}_{pn}$ is established, under Hoeffding prob ability-type inequality of *LNQD* random variables.

Lemma 1. Let $\{X_i | 1\leq i\leq n\}$ be mean zero of *LNQD* random variables with $|X_i|\leq C_i$ $a.s,\ i=1,2,\cdots,n$. Let $C_0 = \max\{C_i | i=1,2,\cdots,n\}$, $\sigma_i^2 = Var(X_i) = EX_i^2$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2 = Var(S_n)$, where $S_n = \sum_{i=1}^n X_i$. Then, for every $t>0$,

$$P(|S_n|\geq s_n t)\leq 2e^{(-t^2/2+\frac{2}{3}c_0 t/s_n)}. \tag{1.3}$$

Lemma 2. Suppose that let $\{X_i | 1\leq i\leq n\}$ be *LNQD* random variables such that

$a_i\leq X_i\leq b_i,\ i=1,2,\ldots,n$. Let $\mu_i = EX_i,\ i=1,2,\ldots,n$ and $\mu = \sum_{i=1}^n \mu_i/n$.

Then for every $t>0$,

$$P(|\overline{X}-\mu|\geq t)\leq 2e^{-2n^2t^2/\sum_{i=1}^n (b_i-a_i)^2}. \tag{1.4}$$

As an application of Lemma 2, we obtain the following probability inequality for the distribution function of the difference of two sample means under *LNQD* random variabl es.
Remark 1. If $X_1,\ldots,X_m,\ Y_1,\ldots,Y_n$ are *LNQD* random variables with vales in the interval $[a,b]$, and if $\overline{X} = \sum_{i=1}^m X_i/m,\ \overline{Z} = \sum_{j=1}^n Y_j/n$, then for $t>0$

$$P(|\overline{X}-\overline{Y}-(E\overline{X}-E\overline{Y})|\geq t)\leq 2e^{-2t^2}/(m^{-1}+n^{-1})(b-a)^2.$$

Remark 2. For the sake of a comparison of Bennett and Hoeffding inequalities under *LNQD* random variables, suppose that $|X_i|\leq C,\ i=1,\cdots,n$, and in (1.3) and (1.4), repla ce t by nt/s_n. Then inequalities (1.3) and (1.4) become, respectively,

$$P(|\overline{S}_n|\geq t)\leq 2e^{-\frac{3n^2t^2}{6s_n^2+2Cnt}} \tag{1.5}$$

and

$$P(|\overline{S_n}|\geq t)\leq 2e^{-\frac{nt^2}{2C^2}}. \tag{1.6}$$

Theorem 1. Let $\{X_j|j\geq 1\}$ be a stationary sequence of *LNQD* random variables with continuous distribution function F. Then for any ε and for all sufficiently large n, we have

$$P(|\hat{\zeta}_{pn}-\zeta_p|>\varepsilon)\leq 2e^{-n\delta^2(\varepsilon)/8},$$

where $\delta(\varepsilon)=\min\{F(\zeta_p)-F(\zeta_p-\varepsilon),\ F(\zeta_p+\varepsilon)-F(\zeta_p)\}>0$.

2. Proof of probability inequalities under *LNQD*

Proof of Lemma1. Expanding e^{cX_i} according to Taylor's formula, for $c>0$,

$$e^{cX_i}=1+cX_i+\sum_{r=2}^{\infty}\frac{c^rX_i^r}{r!}$$

and
$$Ee^{cX_i}=1+cEX_i+\sum_{r=2}^{\infty}\frac{c^rE(X_i^r)}{r!}$$

$$=1+\frac{1}{2}c^2\sigma_i^2\sum_{r=2}^{\infty}\frac{c^{r-2}E(X_i^r)}{\frac{1}{2}r!\sigma_i^2}\qquad \text{since } EX_i=0$$

$$\leq 1+\frac{1}{2}c^2\sigma_i^2F_i(c)$$

$$\leq e^{c^2\sigma_i^2/2F_i(c)},$$

where $F_i(c)=\sum_{r=2}^{\infty}(c^{r-2}E|X_i|^r/\frac{1}{2}r!\sigma_i^2)$, for $i=1,2,\cdots,n$. (2.1)

Suppose that, for $i=1,2,\cdots,n$,

$$E|X_i|^r\leq\frac{1}{2}\sigma_i^2W_n^{r-2}r!\ ,\ r\geq 2,\quad W_n=C_o/3.$$

Since $EX_i^r\leq E|X_i|^r$, substituting the above inequality into equation (2.1) gives

$$F_i(c)\ \leq\ \sum_{r=2}^{\infty}\frac{c^{r-2}\frac{1}{2}\sigma_i^2W_n^{r-2}r!}{\frac{1}{2}\sigma_i^2r!}$$

$$=\ \sum_{r=2}^{\infty}(cW_n)^{r-2}$$

$$=\ \sum_{s=0}^{\infty}(cW_n)^s$$

$$\leq(1-cW_n)^{-1}\quad\text{if } cW_n<1\ .$$

In addition to satisfying this last inequality, c is also chosen so that $(1-cW_n)^{-1}\leq M_n$, where $M_n=W_ns_n^{-1}t+1=C_0t/3s_n+1$, then

$$Ee^{cX_i} \leq e^{c^2\sigma_i^2 M_n/2}, \quad i=1,2,\cdots,n.$$

For any $c>0$, by Markov's inequality, we obtain that

$$\begin{aligned} P(S_n \geq s_n t) &\leq e^{-cs_n t} Ee^{cS_n} \\ &= e^{-cs_n t} E(e^{c\sum_{i=1}^{n-1} X_i} e^{cX_n}) \\ &= e^{-cs_n t} Cov(e^{c\sum_{i=1}^{n-1} X_i}, e^{cX_n}) + e^{-cs_n t} Ee^{c\sum_{i=1}^{n-1} X_i} Ee^{cX_n} \\ &=: I_1 + I_2 \end{aligned}$$

Next, we will show that I_1 is nonpositive. To this end, and application of the Hoeffding identity (see, for example, Lemma in Lehmann, 1966) yields

$$e^{cs_n t} I_1 = \int\int P(c\sum_{i=1}^{n-1} X_i > \log x,\ cX_n > \log y) - P(c\sum_{i=1}^{n-1} X_i > \log x) P(cX_n > \log y)\, dF(x,y),$$

where $F(x,y)$ is the joint distribution function of $\sum_{i=1}^{n-1} X_i$ and X_n.

Since $\sum_{i=1}^{n-1} X_i$ and X_n are *NQD*, it follows that for all $x, y > 0$,

$$P(c\sum_{i=1}^{n-1} X_i > \log x,\ cX_n > \log y) - P(c\sum_{i=1}^{n-1} X_i > \log x) P(cX_n > \log y) \leq 0.$$

So that I_1 is nonpositive. Thus

$$P(S_n \geq s_n t) \leq I_2 = e^{-cs_n t} Ee^{c\sum_{i=1}^{n-1} X_i} Ee^{cX_n}.$$

A repetition of this argument leads to the inequality

$$\begin{aligned} P(S_n \geq s_n t) &\leq e^{-cs_n t} \prod_{i=1}^{n} Ee^{cX_i} \\ &\leq e^{-cs_n t} e^{c^2 s_n^2 M_n/2} \\ &= e^{c^2 s_n^2 M_n/2 - cs_n t}. \end{aligned}$$

Taking $c = t/s_n$, we obtain that

$$P(S_n \geq s_n t) \leq e^{-t^2/(2+\frac{2}{3}C_0 t/s_n)}. \tag{2.2}$$

By replacing S_n by $-S_n$ from the above statement, we obtain

$$P(S_n \leq -s_n t) \leq e^{-t^2/(2+\frac{2}{3}C_0 t/s_n)}. \tag{2.3}$$

Hence the result follows by (2.2) and (2.3).

Proof of Lemma 2. To this end, for any arbitrary but fixed $c \in R$, the function $g(x) = e^{cx}$ is convex. Therefore, for each $i = 1, \cdots, n$,

$$Ee^{cX_i} \le \frac{b_i - \mu_i}{b_i - a_i} e^{ca_i} + \frac{\mu_i - a_i}{b_i - a_i} e^{cb_i},$$

and

$$Ee^{c(X_i - \mu_i)} \le e^{-c(\mu_i - a_i)} e^{ca_i} \cdot e\left\{ ln\left[\frac{b_i - \mu_i}{b_i - a_i} e^{ca_i} + \frac{\mu_i - a_i}{b_i - a_i} e^{cb_i} \right] \right\}$$

$$= e[-kp + \ln(1 - p + pe^{-k})] = e^{L(k)}.$$

Let $k = c(b_i - a_i)$, $p = \frac{\mu_i - a_i}{b_i - a_i}$, then $1 - p = \frac{b_i - \mu_i}{b_i - a_i}$ and $L(k) = -kp + \ln(1 - p + pc^k)$. It follows that $L(0) = L'(0) = 0$ and $L''(k) = u(1-u)$, $0 \le u \le \frac{(1-p)e^{-k}}{(1-p)e^k + p} \le 1$, so that $L''(u) \le 1/4$. Expanding $L(k)$ according to Taylor's formula up to terms involving the second derivative and using the above results, we obtain that $L(k) \le \frac{k^2}{8} \le c^2(b_i - a_i)^2/8$. Therefore

$$Ee^{c(X_i - \mu_i)} \le e^{c^2(b_i - a_i)^2/8}, \quad i = 1, \cdots, n.$$

By Markov's inequality, for $c > 0$,

$$\begin{aligned} P((\overline{X} - \mu) \ge t) &\le e^{-cnt} E(e^{c\sum_{i=1}^{n}(X_i - \mu_i)}) \\ &= e^{-cnt} E(e^{c\sum_{i=1}^{n-1}(X_i - \mu_i)} e^{c(X_n - \mu_n)}) \\ &= e^{-cnt} Cov(e^{c\sum_{i=1}^{n-1}(X_i - \mu_i)}, e^{c(X_n - \mu_n)}) + e^{-cnt} Ee^{c\sum_{i=1}^{n-1}(X_i - \mu_i)} Ee^{c(X_n - \mu_n)} \\ &=: I_1 + I_2 \end{aligned}$$

Again, we will show that I_1 is nonpositive.

$$\begin{aligned} e^{cnt} I_1 = \int\int [& P(c\sum_{i=1}^{n-1}(X_i - \mu_i) > \log x,\ c(X_n - \mu_n) > \log y] \\ & - P(c\sum_{i=1}^{n-1}(X_i - \mu_i) > \log x) P(c(X_n - \mu_n) > \log y) dF(x, y), \end{aligned}$$

where $F(x, y)$ is the joint distribution function of $\sum_{i=1}^{n-1}(X_i - \mu_i)$ and $(X_n - \mu_n)$.

Since $\sum_{i=1}^{n-1}(X_i - \mu_i)$ and $(X_n - \mu_n)$ are *NQD*, it follows that for all $x, y > 0$,

$$\begin{aligned} & P(c\sum_{i=1}^{n-1}(X_i - \mu_i) > \log x,\ c(X_n - \mu_n) > \log y) \\ & \qquad - P(c\sum_{i=1}^{n-1}(X_i - \mu_i) > \log x) \cdot P(c(X_n - \mu_n) > \log y) \le 0. \end{aligned}$$

So that I_1 is nonpositive. Thus

$$P((\overline{X}-\mu)\geq t)\leq I_2 = e^{-cnt}Ee^{c\sum_{i=1}^{n-1}(X_i-\mu_i)}Ee^{c(X_n-\mu_r)}.$$

A repetition of this argument leads to the inequality

$$\begin{aligned}P((\overline{X}-\mu)\geq t) &\leq e^{-cnt}\Pi_{i=1}^{n}Ee^{c(X_i-\mu_i)}\\ &\leq e^{\frac{c^2}{8}\sum_{i=1}^{n}(b_i-a_i)^2-cnt}\end{aligned}$$

Taking $c=4nt/\sum_{i=1}^{n}(b_i-a_i)^2$, we obtain that

$$P((\overline{X}-\mu)\geq t) \leq e^{-2n^2t^2/\sum(b_i-a_i)^2}.$$

Since the same inequality, clearly, holds when $(\overline{X}-\mu)$ is replaced by $-(\overline{X}-\mu)$,

$$P(|\overline{X}-\mu|\geq t)\leq 2e^{-2n^2t^2/\sum_{i=1}^{n}(b_i-a_i)^2}.$$

Proof of Theorem 1. For $\varepsilon>0$ and $j=1,\cdots,n$,

$P(\hat{\xi}_{pn}>\xi_p+\varepsilon)=P(\hat{F}_n(\xi_p+\varepsilon)<p)=P(\sum_{j=1}^{n}(U_{jn}-EU_{jn}>n[(1-p)-EU_{1n}])$, where $U_{jn}=1-K((\xi_p+\varepsilon-X_j)/h_n)$. Note that for all sufficiently large n, $1-p-E(U_{1n})\to F(\xi_p+\varepsilon)-F(\xi_p)\geq\delta(\varepsilon)$ and $1-p-E(U_{1n})>\delta(\varepsilon)/2$. Furthermore, the random variables U_{jn}, $j=1,\cdots,n$, are $LNQD$, since X_j, $j=1,\cdots,n$, are $LNQD$, and U_{jn} is a nondecreasing function of X_j, it follows by Remark 2 that for all sufficiently large n,

$$P(\hat{\xi}_{pn}>\xi_p+\varepsilon)\leq e^{-n\delta^2(\varepsilon)/8}.$$

In a similar method, and for all sufficiently large n, it follows that

$$P(\hat{\xi}_{pn}<\xi_p-\varepsilon)\leq e^{-n\delta^2(\varepsilon)/8}.$$

References

[1] Bennett, George (1962). Probability inequalities for the sum of independent random variables. J. Amer. Statist. Assoc. 57 33-45.

[2] Bilingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

[3] Birkel, T. (1998). Moment bounds for associated sequences. Ann. Probab. 16, 1184-1193.

[4] Esary, J.D., Proschan, F., Walkup, D.W. (1967). Association of random variables, with application. Ann. Math. Statist. 38, 1466-1474.

[5] Hoeffeing, Wassily (1963). Probability inequalities for sums of bounded random variables. J. Amer. Statist. Assoc. 58 13-30.

[6] Ioannides, D. A. and Roussas, G.G.(1999), Exponential inequality for associated random variables. Statistics & Pro. Let. 42. 423-431.

[7] Joag-Dev, K. (1983). Independence via uncorrelatedness under certain dependence structures. Ann. Probab. 17, 362-371.

[8] Joag-Dev, K. And Proschan, F. (1983). Nagative association of random variables, with applications. Ann. Statist. 11 286-295.

[9] Lehmann, E. (1966). Some concepts of dependence. Ann. Math. Statist. 37, 1137-1153.

[10] Newman. C.M. (1980). Normal fluctuations and the FKG inequalities. Commun. Math. Phys. 74. 119-128.

[11] Newman. C.M. (1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. In: Tong, Y.L.(Ed.), Inequalities in Statistics and Probability. IMS Lecture Notes-Monograph Series. vol. 5, pp. 127-140, Hayward, CA.

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