A Nonlinear Programming Approach to Biaffine Matrix Inequality Problems in Multiobjective and Structured Controls

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Abstract: In this paper, a new nonlinear programming approach is suggested to solve biaffine matrix inequality (BMI) problems in multiobjective and structured controls. It is shown that these BMI problems are reduced to nonlinear minimization problems. An algorithm that is easily implemented with existing convex optimization codes is presented for the nonlinear minimization problem. The efficiency of the proposed algorithm is illustrated by numerical examples.

Keywords: Biaffine matrix inequality, multiobjective control, nonlinear programming, structured control.

1. INTRODUCTON

Multiobjective control problems have received considerable attention because of their practical importance, as noted in [3, 13, 18, 24]. Structured control problems, in which the structure of the control is specified *a priori*, have also been investigated by many researchers [9, 15, 17, 23]. It is known that multiobjective and structured control problems cannot be represented by linear matrix inequality (LMI) problems. In most cases they are represented by biaffine matrix inequality (BMI) problems. BMI problems are nonconvex and most of them are known to be NP-hard [4]. To date, there have been many approaches for solving these problems.

In [3, 18, 24], common Lyapunov functions were used to obtain multiobjective controls. In [8, 29], iterative LMI (ILMI) methods were proposed for multiobjective controls. In [13], a finite dimensional Q-parameterization was used to compute multiobjective H_2/H_∞ control, in which very large LMI's appear. However, because these LMI methods were made under performance approximations or conservative assumptions, such methods may only produce approximate solutions or solutions for limited cases. Recently, multiple Lyapunov functions were introduced to reduce some conservatism in [31]. However, the method still has some difficulties in dealing with multiobjective control problems. Moreover, it is noted that these approaches cannot be applied to structured control problems such as H_2/H_∞ PID control [9].

Most multiobjective and structured controls can be obtained by BMI methods rather than LMI methods. In [12, 16, 30], branch and bound algorithms were proposed for the general BMI problem. In [19, 27], randomized algorithms were proposed for BMI problems in robust control. In [9], a genetic algorithm was proposed for H_2/H_∞ PID control. However, these global search algorithms for BMI problems may be inefficient for problems of large size.

Recently, nonlinear programming approaches have been proposed for solving BMI problems in robust control [1, 2] and fixed order control [11]. It is noted that these existing nonlinear programming approaches have been derived by the elimination lemma [6]. However, the elimination lemma cannot be utilized to solve multiobjective and structured control problems, which are described as follows:

Find a structured Θ *satisfying*

$$A_i + B_i\Theta C_i + C_i^T\Theta^T B_i^T < 0 \tag{1}$$

for all $i = 1, 2, \ldots, m$.

The existing nonlinear programming methods therefore cannot be used to obtain multiobjective and structured controls.

In this paper, a new nonlinear programming approach is suggested by introducing a condition that is equivalent to (1). Using the derived condition, it is shown that BMI problems in multiobjective and structured controls can be reduced to nonlinear minimization problems. An explicit algorithm for

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solving the nonlinear minimization problem is also proposed. The algorithm is easily implemented using convex optimization codes [25]. Numerical examples show that the proposed algorithm is computationally efficient.

This paper is organized as follows. In Section 2, a nonlinear minimization problem for multiobjective and structured controls is presented. An algorithm for solving the nonlinear minimization problem is presented in Section 3. In Section 4, numerical examples are given. Finally, conclusions are presented in Section 5.

2. MULTIOBJECTIVE AND STRUCTURED CONTROL PROBLEMS

Consider a plant

$$\begin{aligned} \dot{x} &= Ax + B_w w + Bu, \\ z &= C_z x + D_{zw} w + D_z u, \\ y &= C_y x + D_w w, \end{aligned} \tag{2}$$

and a fixed order control

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y, \\ u &= C_c x_c + D_c y, \end{aligned} \tag{3}$$

where $x \in \mathbf{R}^{n_x}$ and $x_c \in \mathbf{R}^{n_c}$ are the states of the plant and the control, $u \in \mathbf{R}^{n_u}$ is the control input, $w \in \mathbf{R}^{n_w}$ is the exogenous input, $y \in \mathbf{R}^{n_y}$ is the measured output, and $z \in \mathbf{R}^{n_z}$ is the controlled output. Define a system matrix Θ_c of the control by

$$\Theta_c := \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}. \tag{4}$$

Then the closed-loop system is given by the state equation

$$\begin{aligned} \dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w, \\ z &= C_{cl} x_{cl} + D_{cl} w, \end{aligned} \tag{5}$$

where the system matrix Θ_{cl} of the closed-loop system is given by

$$\begin{aligned} \Theta_{cl} &= \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \\ &= \begin{bmatrix} A & 0 & B_w \\ 0 & 0 & 0 \\ C_z & 0 & D_{zw} \end{bmatrix} + \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \\ 0 & D_z \end{bmatrix} \Theta_c \begin{bmatrix} 0 & I_{n_c} & 0 \\ C_y & 0 & D_w \end{bmatrix}, \end{aligned} \tag{6}$$

which shows that Θ_{cl} is an affine transform of Θ_c. Most linear controls can be obtained by solving BMI problems on Θ_c and some matrix variables [24]. For example, consider the following H_∞ control problem.

Let T_{wz} be the closed-loop transfer function from w to z. Then, the H_∞ norm constraint, $\| T_{wz} \|_\infty < \gamma$, is equivalent to the existence of a symmetric matrix $P_\infty \in \mathbf{R}^{(n_x+n_c)\times(n_x+n_c)}$ that satisfies

$$\begin{bmatrix} A_{cl}^T P_\infty + P_\infty A_{cl} & P_\infty B_{cl} & C_{cl}^T \\ B_{cl}^T P_\infty & -\gamma I_{n_w} & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I_{n_z} \end{bmatrix} < 0, \tag{7}$$

$$P_\infty > 0. \tag{8}$$

The inequality (7) can be denoted by

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma I_{n_w} & 0 \\ 0 & 0 & -\gamma I_{n_z} \end{bmatrix} \\ &+ \begin{bmatrix} P_\infty & 0 \\ 0 & 0 \\ 0 & I_{n_z} \end{bmatrix} \Theta_{cl} \begin{bmatrix} I_{n_x+n_c} & 0 & 0 \\ 0 & I_{n_w} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} I_{n_x+n_c} & 0 \\ 0 & I_{n_w} \\ 0 & 0 \end{bmatrix} \Theta_{cl}^T \begin{bmatrix} P_\infty & 0 & 0 \\ 0 & 0 & I_{n_z} \end{bmatrix} < 0, \end{aligned} \tag{9}$$

and by substituting (6) into (9), we obtain a BMI on Θ_c and P_∞ as follows:

$$\mathbf{A}_\infty(P_\infty) + \mathbf{B}_\infty(P_\infty)\Theta_c \mathbf{C}_\infty + \mathbf{C}_\infty^T \Theta_c^T \mathbf{B}_\infty^T(P_\infty) < 0, \tag{10}$$

where the matrices $\mathbf{A}_\infty(P_\infty)$ and $\mathbf{B}_\infty(P_\infty)$ are affine transforms of the matrix P_∞, and $\mathbf{C}_\infty$ is a constant matrix obtained from (6) and (9).

Hence, the H_∞ control problem is to find Θ_c and P_∞ that satisfy (8) and a BMI (10). It is noted that as in [24], (10) can be reduced to an LMI if $n_c = n_x$. However, if $n_c < n_x$, then (10) cannot be reduced to an LMI.

Some control objectives such as H_2 performance require an equality constraint $D_{cl} = 0$ or

$$D_{zw} + D_z D_c D_w = 0. \tag{11}$$

The equality constraint (11) can be eliminated using a structured control as follows. If $D_z D_z^+ D_{zw} D_w^+ D_w = D_{zw}$, then a solution of (11) exists and all solutions of (11) can be represented by

$$D_c = -D_z^+ D_{zw} D_w^+ + \mathcal{D}_c - D_z^+ D_z \mathcal{D}_c D_w D_w^+ = -D_z^+ D_{zw} D_w^+ + \begin{bmatrix} I_{n_u} & -D_z^+ D_z \end{bmatrix} \begin{bmatrix} \mathcal{D}_c & 0 \\ 0 & \mathcal{D}_c \end{bmatrix} \begin{bmatrix} I_{n_y} \\ D_w D_w^+ \end{bmatrix}, \quad (12)$$

where D_z^+ and D_w^+ are pseudo inverse of D_z and D_w, respectively and Θ_c is an arbitrary matrix [5]. Hence, the control Θ_c that satisfies (11) can be denoted by

$$\Theta_c = \begin{bmatrix} 0 & 0 \\ 0 & -D_z^+ D_{zw} D_w^+ \end{bmatrix} + \begin{bmatrix} I_{n_c} & I_{n_c} & 0 & 0 & 0 \\ 0 & 0 & I_{n_u} & I_{n_u} & -D_z^+ D_z \end{bmatrix} \Theta \begin{bmatrix} I_{n_c} & 0 \\ 0 & I_{n_y} \\ I_{n_c} & 0 \\ 0 & I_{n_y} \\ 0 & D_w D_w^+ \end{bmatrix}, \quad (13)$$

where Θ is defined by

$$\Theta := \mathbf{diag}(A_c, B_c, C_c, \mathcal{D}_c, \mathcal{D}_c) \quad (14)$$

By using the structured control (13) we can remove the equality constraint (11) from control objectives. For example, consider the following H_2 control problem.

The H_2 norm constraint, $\| T_{wz} \|_2^2 < \nu$, is equivalent to the existence of symmetric matrices $P_2 \in \mathbf{R}^{(n_x+n_c)\times(n_x+n_c)}$ and $Q \in \mathbf{R}^{n_z \times n_z}$ that satisfy

$$\begin{bmatrix} A_{cl}^T P_2 + P_2 A_{cl} & P_2 B_{cl} \\ B_{cl}^T P_2 & -I_{n_w} \end{bmatrix} < 0, \quad (15)$$

$$\begin{bmatrix} P_2 & C_{cl}^T \\ C_{cl} & Q \end{bmatrix} > 0, \quad (16)$$

$$\text{Trace}(Q) < \nu, \quad (17)$$

$$D_{cl} = 0. \quad (18)$$

Using Θ_{cl}, (15) and (16) can be rewritten by

$$\begin{bmatrix} 0 & 0 \\ 0 & -I_{n_w} \end{bmatrix} + \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix} \Theta_{cl} + \Theta_{cl}^T \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad (19)$$

$$\begin{bmatrix} P_2 & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_z} \end{bmatrix} \Theta_{cl} \begin{bmatrix} I_{n_x+n_c} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x+n_c} & 0 \\ 0 & 0 \end{bmatrix} \Theta_{cl}^T \begin{bmatrix} 0 & 0 \\ 0 & I_{n_z} \end{bmatrix} > 0. \quad (20)$$

All controls that satisfy the equality constraint (11) can be represented by (13). Hence, by substituting (13) into (6) and substituting (6) into (19) and (20), we have a BMI

$$\mathbf{A}_{21}(P_2) + \mathbf{B}_{21}(P_2)\Theta\mathbf{C}_{21} + \mathbf{C}_{21}^T \Theta^T \mathbf{B}_{21}^T(P_2) < 0 \quad (21)$$

on Θ and P_2, and an LMI

$$\mathbf{A}_{22}(P_2, Q) + \mathbf{B}_{22}(P_2)\Theta\mathbf{C}_{22} + \mathbf{C}_{22}^T \Theta^T \mathbf{B}_{22}^T(P_2) < 0 \quad (22)$$

on Θ, P_2, and Q, where the bold faced matrices in (21) and (22) can be easily obtained from (6), (13), (19) and (20). Hence, the H_2 control problem is to find Θ, P_2, and Q satisfying (17), (21), and (22).

In addition, most multiobjective control problems are BMI problems. For example, the mixed H_2/H_∞ control problem is to find Θ, P_∞, P_2, and Q that satisfy (17), (21), (22), (8), and a BMI that is obtained by substituting (13) into (7).

Structured control means that the matrices A_c, B_c, C_c, and D_c of the control (3) have some structures. The decentralized stabilization by static output feedback is a typical example of structured control. Consider a system

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{r} B_k u_k(t), \quad y_k(t) = C_k x(t), \ \forall k = 1, \ldots, r \quad (23)$$

and a set of controls

$$u_k(t) = \Theta_k y_k(t), \forall k = 1, \ldots, r, \quad (24)$$

where $\Theta_k \in \mathbf{R}^{p_k \times q_k}$ for all $k = 1, \ldots, r$ as in [23]. Denote the matrices B and C by

$$B = [B_1 \ \cdots \ B_r], \ C = [C_1^T \ \cdots \ C_r^T]^T. \quad (25)$$

Then the decentralized control (24) guarantees stability if and only if a positive definite matrix P exists that satisfies the inequality

$$(A + B \begin{bmatrix} \Theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Theta_2 \end{bmatrix} C)^T P + P(A + B \begin{bmatrix} \Theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Theta_2 \end{bmatrix} C) < 0. \quad (26)$$

Remark 1: (24) represents a static control. In the case of a dynamic control, we may also have a BMI that is similar to (26) with appropriate matrices.

Hence it is clear that the system matrix $\Theta_c \in \mathbf{R}^{p\times q}$ of the structured control can be represented by

$$\Theta_c = U + V\Theta W \quad (27)$$

with appropriate constant matrices U, V, W, and a matrix variable $\Theta \in \mathbf{\Theta}_{\mathcal{K}}$, where $\mathbf{\Theta}_{\mathcal{K}}$ is a set of block diagonal matrices as follows:

$$\mathbf{\Theta}_{\mathcal{K}} := \{\mathbf{diag}(\overbrace{\Theta_1,\ldots,\Theta_1}^{J_1},\overbrace{\Theta_2,\ldots,\Theta_2}^{J_2},\ldots,\overbrace{\Theta_r,\ldots,\Theta_r}^{J_r}) \mid \Theta_k \in \mathbf{R}^{p_k \times q_k}, \forall k = 1,\ldots,r\}. \quad (28)$$

In (28), J_k is a repeat number of the submatrix Θ_k for all k = 1, …, r, and hence $\sum_{k=1}^r J_k p_k = p$ and $\sum_{k=1}^r J_k q_k = q$. For given $\mathbf{\Theta}_{\mathcal{K}}$, define sets of matrices, $\mathbf{S}_{\mathcal{K}}$ and $\mathbf{T}_{\mathcal{K}}$ as follows:

$$\mathbf{S}_{\mathcal{K}} := \{\mathbf{diag}(\overbrace{S_1,\ldots,S_1}^{J_1},\overbrace{S_2,\ldots,S_2}^{J_2},\ldots,\overbrace{S_r,\ldots,S_r}^{J_r}) \mid S_k \in \mathbf{R}^{p_k \times p_k}, \forall k = 1,\ldots,r\}. \quad (29)$$

$$\mathbf{T}_{\mathcal{K}} := \{\mathbf{diag}(\overbrace{T_1,\ldots,T_1}^{J_1},\overbrace{T_2,\ldots,T_2}^{J_2},\ldots,\overbrace{T_r,\ldots,T_r}^{J_r}) \mid T_k \in \mathbf{R}^{q_k \times q_k}, \forall k = 1,\ldots,r\}. \quad (30)$$

The above examples show that multiobjective and structured control problems can be described as the following problem:

Problem 1: Find $\Theta \in \mathbf{\Theta}_{\mathcal{K}}$ and $P \in \mathcal{P}$ that satisfy the inequalities

$$A_i(P) + B_i(P)\Theta C_i + C_i^T \Theta^T B_i^T(P) < 0 \quad (31)$$

for all i = 1, …, m.

In Problem 1, $A_i(P)$ is a symmetric affine transform of P for all i = 1, …, m.. That is, there are constant matrices K_i and L_i and a symmetric matrix H_i such that

$$A_i(P) = H_i + K_i P L_i + L_i^T P K_i^T \quad (32)$$

for all i = 1, …, m. $B_i(P)$ is an affine transform of P, and can be represented by

$$B_i(P) = E_i + F_i P G_i \quad (33)$$

for all i = 1, …, m, where E_i, F_i, and G_i are some constant matrices. Θ and P are matrix variables to be determined. $\mathbf{\Theta}_{\mathcal{K}}$ is a set of matrices defined by (28), and $\mathcal{P}$ is a convex set of matrices.

Remark 2: In Problem 1, P may be nonsquare, asymmetric, or structured, for example $P = \mathbf{diag}(P_1,\ldots,P_s)$. That is, $\mathcal{P}$ may be a convex set of arbitrarily structured matrices.

It can be shown that Problem 1 is equivalent to the following problem in which there are no BMI constraints.

Problem 2: Find $N \in \mathbf{\Theta}_{\mathcal{K}}$, $M \in \mathbf{S}_{\mathcal{K}}$, $R \in \mathbf{T}_{\mathcal{K}}$, $Z \in \mathbf{T}_{\mathcal{K}}$, and $P \in \mathcal{P}$ that satisfy the inequalities

$$\begin{bmatrix} A_i(P) - C_i^T Z C_i & B_i(P) & C_i^T \\ B_i^T(P) & -M & -N \\ C_i & -N^T & -R \end{bmatrix} < 0 \quad (34)$$

for all i = 1, …, m and an equality

$$Z = (R - N^T M^{-1} N)^{-1}. \quad (35)$$

We will make use of the following lemmas to prove that Problem 2 is equivalent to Problem 1.

Lemma 1: [6] Let A, B, and C be given constant matrices. If a matrix Θ exists that satisfies the inequality

$$A + B\Theta C + C^T \Theta^T B^T < 0, \quad (36)$$

then a positive scalar σ exists such that

$$A - \sigma B B^T < 0, \; A - \sigma C^T C < 0. \quad (37)$$

Lemma 2: Assume that $\Theta \in \mathbf{\Theta}_{\mathcal{K}}$, $X \in \mathbf{S}_{\mathcal{K}}$, and $Z \in \mathbf{T}_{\mathcal{K}}$, and

$$\begin{bmatrix} M & N \\ N^T & R \end{bmatrix} = \begin{bmatrix} X & \Theta \\ \Theta^T & Z \end{bmatrix}^{-1}, \quad (38)$$

then we have $N \in \mathbf{\Theta}_{\mathcal{K}}$, $M \in \mathbf{S}_{\mathcal{K}}$, and $R \in \mathbf{T}_{\mathcal{K}}$.

Proof: The result is clear from the definitions of the sets. □

Theorem 1: A solution of Problem 1 exists if and only if a solution of Problem 2 exists. Furthermore, if a solution of Problem 2 exists, then $\Theta = -M^{-1}N(R - N^T M^{-1} N)^{-1}$ is a solution of Problem 1.

Proof: (⇒) Assume that Θ and P exist that satisfy Problem 1. By Lemma 1, a positive scalar σ exists that satisfies the inequalities

$$A_i(P) - \sigma C_i^T C_i < 0 \quad (39)$$

for all i = 1, …, m. Hence, we obtain

$$A_i(P) - C_i^T Z C_i < 0 \quad (40)$$

for all i = 1, …, m, where $Z \in \mathbf{T}_{\mathcal{K}}$ such that $Z \geq \sigma I_q$. A sufficiently small scalar $\varepsilon > 0$ exists that satisfies the inequalities

$$A_i(P)+B_i(P)\Theta C_i + C_i^T\Theta^T B_i^T(P)+\varepsilon B_i(P)B_i^T(P)<0 \tag{41}$$

for all $i = 1, \ldots, m$. Hence we obtain

$$A_i(P)+B_i(P)\Theta C_i + C_i^T\Theta^T B_i^T(P)+B_i(P)XB_i^T(P)<0 \tag{42}$$

for all $i = 1, \ldots, m$, where $X \in \mathbf{S}_{\mathcal{K}}$ such that $0 < X \le \varepsilon I_p$.

From (42), we have

$$A_i(P)-C_i^T ZC_i + [B_i(P) \quad C_i^T]\begin{bmatrix} X & \Theta \\ \Theta^T & Z\end{bmatrix}\begin{bmatrix} B_i^T(P) \\ C_i\end{bmatrix}<0 \tag{43}$$

for all $i = 1, \ldots, m$. Using a matrix $Z \in \mathbf{T}_{\mathcal{K}}$ that satisfies $Z \ge \sigma I_q$ and

$$\begin{bmatrix} X & \Theta \\ \Theta^T & Z\end{bmatrix} > 0, \tag{44}$$

we obtain

$$\begin{bmatrix} A_i(P)-C_i^T ZC_i & [B_i(P) \quad C_i^T] \\ \begin{bmatrix} B_i^T(P) \\ C_i\end{bmatrix} & -\begin{bmatrix} X & \Theta \\ \Theta^T & Z\end{bmatrix}^{-1}\end{bmatrix}<0 \tag{45}$$

for all $i = 1, \ldots, m$. Denote the inverse matrix in (45) by

$$\begin{bmatrix} M & N \\ N^T & R\end{bmatrix} = \begin{bmatrix} X & \Theta \\ \Theta^T & Z\end{bmatrix}^{-1}, \tag{46}$$

then we have $Z = (R - N^T M^{-1} N)^{-1}$, and by Lemma 2, $M \in \mathbf{S}_{\mathcal{K}}$, $N \in \mathbf{\Theta}_{\mathcal{K}}$, and $R \in \mathbf{T}_{\mathcal{K}}$. Hence, we have a solution to Problem 2.

($\Leftarrow$) Assume that the matrices *Z*, *M*, *N*, *R*, and *P* are solutions of Problem 2, then we have

$$A_i(P)+B_i(P)\Theta C_i + C_i^T\Theta^T B_i^T(P)+B_i(P)XB_i^T(P)<0 \tag{47}$$

for all $i = 1, \ldots, m$, where

$$\Theta := -M^{-1}N(R-N^TM^{-1}N)^{-1}, \tag{48}$$

$$X := (M - NR^{-1}N^T)^{-1}. \tag{49}$$

It is easy to show that $\Theta \in \mathbf{\Theta}_{\mathcal{K}}$ and $X > 0$. Hence Θ and *P* are solutions of Problem 1. □

From Theorem 1, we can obtain a solution of Problem 1 by solving Problem 2, which has LMI constraints (34) and a nonlinear matrix equality constraint (35). It is easy to show that (35) can be replaced with (see [11])

$$R - N^T M^{-1} N \ge Z^{-1} \tag{50}$$

and

$$\mathbf{Tr}(R - N^T M^{-1} N - Z^{-1}) = 0, \tag{51}$$

where (50) can be reduced to an LMI as follows:

Lemma 3: Assume that $M > 0$. The matrices *Z*, *M*, *N*, and *R* satisfy an LMI

$$\begin{bmatrix} Z & 0 & I_q \\ 0 & M & N \\ I_q & N^T & R\end{bmatrix} \ge 0 \tag{52}$$

if and only if $Z > 0$, $R > 0$, and

$$R - N^T M^{-1} N - Z^{-1} \ge 0. \tag{53}$$

Proof: See Appendix A. □

From Lemma 3, it is clear that the nonlinear matrix equality constraint (35) can be replaced with (51) and (52). Hence we obtain the following theorem:

Theorem 2: A solution of Problem 1 exists if and only if the minimum of the following problem is 0.

Problem 3:

$$\underset{M\in\mathbf{S}_{\mathcal{K}}, N\in\mathbf{\Theta}_{\mathcal{K}}, Z, R\in\mathbf{T}_{\mathcal{K}}, P\in\mathcal{P}}{Minimize} \quad \mathbf{Tr}(R - N^T M^{-1} N - Z^{-1})$$

subject to

$$\begin{bmatrix} A_i(P)-C_i^T ZC_i & B_i(P) & C_i^T \\ B_i^T(P) & -M & -N \\ C_i & -N^T & -R\end{bmatrix}<0 \tag{54}$$

for all $i = 1, \ldots, m$ and

$$\begin{bmatrix} Z & 0 & I_q \\ 0 & M & N \\ I_q & N^T & R\end{bmatrix} \ge 0. \tag{55}$$

□

Remark 3: (55) can be denoted by a set of LMI's

$$\begin{bmatrix} Z_k & 0 & I_{q_k} \\ 0 & M_k & N_k \\ I_{q_k} & N_k^T & R_k\end{bmatrix} \ge 0, \forall k = 1,\ldots,r, \tag{56}$$

where Z_k, M_k, N_k, and R_k are submatrices of the block diagonal matrices, *Z*, *M*, *N*, and *R*, respectively. (56) is preferred to (55) because (56) requires less computer memory.

If the minimum of Problem 3 is 0, then the solution of Problem 1 is given by

$$\Theta = -M^{-1}N(R-N^TM^{-1}N)^{-1}$$

or

$$\Theta_k = -M_k^{-1}N_k(R_k - N_k^T M_k^{-1} N_k)^{-1}, \forall k = 1, \ldots, r.$$

However, we obtain a solution of Problem 1 even if the object value of Problem 3 is not zero from the following theorem:

Theorem 3: If $M \in \mathbf{S}_{\mathcal{K}}$, $N \in \mathbf{\Theta}_{\mathcal{K}}$, $Z \in \mathbf{T}_{\mathcal{K}}$, $R \in \mathbf{T}_{\mathcal{K}}$, and $P \in \mathcal{P}$ exist that satisfy (54), (55), and

$$C_i^T(Z-(R-N^TM^{-1}N)^{-1})C_i \le B_i(P)XB_i^T(P) \tag{57}$$

for all $i = 1, \ldots, m$, where $X = (M - NR^{-1}N^T)^{-1}$, then $\Theta = -M^{-1}N(R - N^T M^{-1}N)^{-1}$ is a solution of Problem 1.

Proof: From (54), we have

$$A_i(P) - C_i^T Z C_i + [B_i(P) \quad C_i^T]\begin{bmatrix} X & \Theta \\ \Theta^T & (R - N^T M^{-1}N)^{-1}\end{bmatrix}\begin{bmatrix} B_i^T(P) \\ C_i \end{bmatrix} < 0 \tag{58}$$

for all $i = 1, \ldots, m$. Hence from (57), we have (31). □

3. OPTIMIZATION ALGORITHM

It is easy to see that the objective function of Problem 3 is concave. Hence there is no algorithm that always guarantees a global minimum of Problem 3 in feasible time. To obtain the local minima of such concave minimization problems, a linearization method is used, such as in [1, 11, 20]. At a given point $(\mathcal{Z}, \mathcal{M}, \mathcal{N}, \mathcal{R})$, a linear approximation of Problem 3 is given by

Problem 4:

$$\underset{M \in \mathbf{S}_{\mathcal{K}}, N \in \mathbf{\Theta}_{\mathcal{K}}, Z, R \in \mathbf{T}_{\mathcal{K}}, P \in \mathcal{P}}{Minimize} \quad \mathbf{Tr}(F_1(Z, M, N, R))$$

subject to (54) and (55), where

$$F_1(Z, M, N, R) = R - 2N^T \mathcal{M}^{-1}\mathcal{N} + \mathcal{N}^T \mathcal{M}^{-1} M \mathcal{M}^{-1}\mathcal{N} + \mathcal{Z}^{-1} Z \mathcal{Z}^{-1}.$$

An algorithm for obtaining a local minimum of Problem 3 is given as follows:

Algorithm 1:

1. Find a random feasible solution $(\mathcal{Z}, \mathcal{M}, \mathcal{N}, \mathcal{R})$ of Problem 4.
2. Find a solution (Z, M, N, R) of Problem 4.
3. If the stopping criterion (57) is satisfied, then exit. If it reaches a stationary point, then go to Step 1. Otherwise, set $(\mathcal{Z}, \mathcal{M}, \mathcal{N}, \mathcal{R}) = (Z, M, N, R)$ and go to Step 2.

In Step 1, a random initial feasible solution can be obtained as follows:

- Select a random matrix $(\mathcal{Z}, \mathcal{M}, \mathcal{N}, \mathcal{R})$ and a large bound **M**.
- Solve the following problem:

$$\underset{M \in \mathbf{S}_{\mathcal{K}}, N \in \mathbf{\Theta}_{\mathcal{K}}, Z, R \in \mathbf{T}_{\mathcal{K}}, P \in \mathcal{P}}{Minimize} \quad \mathbf{Tr}(F_2(Z, M, N, R))$$

subject to (54), (55), and

$$\mathbf{Tr}(\mathbf{diag}(Z, M, R, P)) \le \mathbf{M}, \tag{59}$$

where

$$F_1(Z, M, N, R) = \mathcal{Z}^T Z + \mathcal{M}^T M + \mathcal{N}^T N + \mathcal{R}^T R.$$

- Set $(\mathcal{Z}, \mathcal{M}, \mathcal{N}, \mathcal{R}) = (Z, M, N, R)$.

In (59), a large bound **M** is introduced to prevent unbounded solutions. Step 1 and Step 2 are LMI problems that can be solved by semidefinite programming algorithms [7, 21, 25, 26].

There can be multiple local minima in BMI problems. In Algorithm 1, one or several local minima can be searched by using random initial feasible solutions This random search method does not guarantee that each local minimum has a different value. However, the numerical experiments will show that the proposed local search method is effective.

4. NUMERICAL EXPERIMENTS

In this section, the efficiency of the proposed method is illustrated by numerical examples: mixed H_2/H_∞ control, mixed H_2/H_∞ PID control, simultaneous stabilization by decentralized static output feedback, and simultaneous stabilization by static output feedback. To solve LMI problems in Algorithm 1, we used the semidefinite programming code **SP** [25] and Matlab on a Sparc Workstation. The **SP** parameters for absolute and relative convergence were both set to 10^{-8}.

5.1. Mixed H_2/H_∞ control

This example is taken from [24]. Consider a three-state unstable plant with equations

$$\dot{x} = \begin{pmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} w + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u, \qquad y = x_2 + 2w, \tag{60}$$

and performance outputs

$$z_\infty = \begin{pmatrix} x_1 \\ u \end{pmatrix}, \quad z_2 = \begin{pmatrix} x_2 \\ x_3 \\ u \end{pmatrix}. \tag{61}$$

In [24], a mixed control was obtained by solving the following problem

Minimize $\| T_{wz_2} \|_2$ *subject to* $\| T_{wz_\infty} \|_\infty < 23.6$.

The optimal value of the above problem is 7.748. The control obtained in [24] guarantees an H_2 performance of 8.07. To use the method proposed in this paper, we solved a modified problem

Find a control that satisfies $\| T_{wz_2} \|_2 < 8.0$ *and* $\| T_{wz_\infty} \|_\infty < 23.6$.

Because of the H_2 performance objective, D_c of the controller must be 0. Hence the problem is to find a structured control

$$\Theta_c = \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_{n_c} & I_{n_c} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} A_c & 0 & 0 \\ 0 & B_c & 0 \\ 0 & 0 & C_c \end{bmatrix} \begin{bmatrix} I_{n_c} & 0 \\ 0 & I_{n_y} \\ I_{n_c} & 0 \end{bmatrix} \quad (62)$$

that achieves the specified performances.

By using Algorithm 1, we obtained a controller

$$\dot{x}_c = \begin{pmatrix} -6.7262 & 5.7486 & 32.9562 \\ 31.6809 & -78.2219 & 3.1284 \\ 30.6526 & -72.2637 & -10.0598 \end{pmatrix} x_c + \begin{pmatrix} 33.75 \\ 55.169 \\ 23.826 \end{pmatrix} y,$$

$$u = (-0.0303 \quad -0.901 \quad 0.3320). \quad (63)$$

The control has an H_2 performance of 7.9029 and an H_∞ performance of 23.31. Reduced order H_2/H_∞ controls also can be obtained by the proposed method. For example, a second order control

$$\dot{x}_c = \begin{pmatrix} -2.6649 & -0.3836 \\ -1.9013 & -5.2169 \end{pmatrix} x_c + \begin{pmatrix} -9.5812 \\ -14.8499 \end{pmatrix} y, \quad (64)$$
$$u = (-0.2440 \quad 0.9762)$$

was obtained using Algorithm 1. The control has an H_2 performance of 8.81 and an H_∞ performance of 23.3.

5.2. Mixed H_2/H_∞ PID control

This example is taken from [9] in which a genetic algorithm for the mixed H_2/H_∞ PID control was proposed. Consider a plant

$$P(s) = \frac{0.8}{s(0.5s+1)}, \quad (65)$$

a weighting function $W(s) = 1/(s+1)$, and PID control

$$C(s) = k_1 + k_2/s + k_3 s. \quad (66)$$

Then the weighted transfer function is given by

$$T_{dy}(s) = \frac{W(s)}{1 + P(s)C(s)}$$
$$= \frac{s^3 + 2s^2}{(s+1)(0.5s^3 + (1+0.8k_3)s^2 + 0.8k_1 s + 0.8k_2)}. \quad (67)$$

The state space realization of the transfer function is given by

$$A_{cl} = \begin{pmatrix} -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} -1.6 \\ 0 \\ 0 \\ 0 \end{pmatrix} (k_1 \quad k_2 \quad k_3) \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{cl} = (1 \quad 0 \quad 0 \quad 0)^T, \quad C_{cl} = (1 \quad 2 \quad 0 \quad 0), \quad D_{cl} = 0 \quad (68)$$

The problem is to find the best H_2 control that guarantees the H_∞ performance of 0.1. Using the genetic algorithm in [9], PID control parameters $k_1 = 29.9884$, $k_2 = 0.1845$, and $k_3 = 30.0000$ were obtained. These parameters guarantee the H_2 performance of 0.1015 and the H_∞ performance of 0.0238.

Using Algorithm 1 of this paper, the following problem was solved.

Find a control that satisfies $\| T_{wy} \|_2 < 0.1$ *and* $\| T_{wy} \|_\infty < 0.1$

We obtained PID parameters $k_1 = 11.450$, $k_2 = 17.3243$, and $k_3 = 97.1921$ that guarantee the H_2 performance of 0.0576 and the H_∞ performance of 0.0439. In contrast to the genetic algorithm, the parameter domain is not required. Hence, we obtained better control.

5.3. Simultaneous stabilization by decentralized static output feedback

Consider the problem of decentralized and simultaneous stabilization by static output feedback as follows:

Find a structured control Θ *and* P_i *that satisfy the inequalities*

$$(A_i + B_i \Theta C_i)^T P_i + P_i (A_i + B_i \Theta C_i) < 0, \quad P_i > \alpha I, \quad (69)$$

for all $i = 1, \ldots, m$.

In the above problem, $A_i \in \mathbf{R}^{n \times n}$, $B_i \in \mathbf{R}^{n \times p}$, $C_i \in \mathbf{R}^{q \times n}$ and α is a positive scalar. This problem is known to be NP-hard [4].

As in [22], we randomly generated $\tilde{A}_1$, B_1, and C_1 so that $\tilde{A}_1$ was stable. Θ was randomly generated so that $(\tilde{A}_1 - B_1 \Theta C_1)$ was unstable and we took $A_1 = (\tilde{A}_1 - B_1 \Theta C_1)$. A_i's, B_i's, and C_i's were randomly generated so that A_i was unstable and $(A_i + B_i \Theta C_i)$ was stable for all i = 2, …, m. All random matrices were generated using the function `rand` of Matlab so that all elements in the matrices were between −1 and 1. We set $\alpha = 10$ for all problems.

We generated one hundred problems for each tuple

Table 1. Distribution of outer iteration numbers for decentralized stabilization.

(n,p,q,m)\Iter.	=1	≤100	>1000	Average
(3,2,2,1)	81%	95%	0%	12.65
(3,2,2,2)	61%	88%	1%	39.19
(3,2,2,3)	50%	87%	4%	43.61
(5,3,3,1)	69%	85%	1%	18.00

(n, p, q, m) in Table 1. For each problem we allowed up to one thousand outer iterations. The averages of the outer iteration numbers were computed excluding the cases that failed to obtain a solution.

We considered some cases:

(i) $n = 5, p = 3, q = 3, m = 1$, and

$$\Theta = \begin{bmatrix} \theta_1 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \\ 0 & 0 & \theta_4 \end{bmatrix}. \quad (70)$$

(ii) $n = 5, p = 2, q = 2, m = 1, 2, 3$, and

$$\Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}. \quad (71)$$

Table 1 shows that most problems could be solved in feasible time.

5.4. Simultaneous stabilization by static output feedback

Consider a set of systems

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t), \end{aligned} \quad (72)$$

and static output feedback controls $u_i(t) = \Theta y_i(t)$ for all $i = 1, \ldots, m$. The simultaneous stabilizer can be obtained by solving the following problem:

Find a control Θ and P_i that satisfy the inequalities

$$\begin{aligned} (A_i + B_i\Theta C_i)^T P_i + P_i(A_i + B_i\Theta C_i) < 0, \\ P_i > \alpha I, \end{aligned} \quad (73)$$

for all $i = 1, \ldots, m$.

In the above problem, $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^{n\times p}$, $C_i \in \mathbf{R}^{q\times n}$ and α is a positive scalar. This problem is known to be NP-hard [4].

As in Section 5.3, we generated A_i's, B_i's, and C_i's for all $i = 1, 2, \ldots, m$, and we set $\alpha = 100$ for all problems. We generated one thousand problems for each tuple (n, p, q, m) in Table 2, Table 3, and Table 4. For each problem we tried up to one thousand outer iterations. The averages of the outer iteration numbers were computed excluding the cases that we failed to obtain a solution within one thousand outer iterations.

Table 2. Distribution of outer iteration numbers when $m = 1$.

(n,p,q)\Iter.	=1	≤100	>1000	Average
(3,1,1)	87.8%	98.5%	0.0%	5.3
(3,2,1)	93.8%	99.7%	0.0%	3.3
(3,2,2)	96.7%	99.5%	0.0%	2.6
(5,1,1)	82.8%	98.3%	0.0%	7.5
(5,2,1)	81.7%	98.3%	0.2%	7.2
(6,1,1)	79.1%	97.5%	0.0%	10.4

Table 3. Distribution of outer iteration numbers when $m = 3$.

(n,p,q)\Iter.	=1	≤100	>1000	Average
(3,1,1)	68.6%	97.7%	0.0%	12.3
(3,2,1)	53.8%	94.8%	1.8%	15.9
(3,2,2)	48.7%	90.9%	2.2%	33.1
(5,1,1)	60.9%	96.4%	0.4%	15.5
(5,2,1)	32.5%	95.0%	0.8%	22.8
(6,1,1)	53.1%	95.1%	0.4%	17.7

Table 4. Distribution of outer iteration numbers when $n = 3, p = 1, q = 1$, and $1 \leq m \leq 5$.

m\Iter.	=1	≤100	>1000	Average
1	87.8%	98.5%	0%	5.3
2	77.0%	96.6%	0%	12.5
3	68.6%	97.7%	0%	12.3
4	64.6%	95.9%	0%	15.9
5	61.0%	97.5%	0%	15.4

In the special case $m = 1$, several methods exist for obtaining a static control in [10, 11, 14]. Their performances are compared in [22]. The results in Table 2 are comparable with those in [14, 22].

The results for the case $m > 1$ are shown in Table 3 and Table 4. The averages of outer iterations were increased and there were more problems where the number of outer iterations exceeded 1000 than the case for $m = 1$. However it can be seen that most problems were solved in feasible time using Algorithm 1.

Consider the following two plants

$$P_1(s) = \frac{a}{(s+2)^2}, \quad P_2(s) = \frac{1}{(s+2)(s-1)}, \quad (74)$$

which are taken from [8] where an iterative LMI (ILMI) method was proposed for simultaneous stabi-

Table 5. Outer iteration numbers and control gains for stabilization the plants $P_1(s)$ and $P_2(s)$.

a	Algorithm 1		ILMI method	
	Iter.	Gains	Iter.	Gains
5	18	−2.0718	3	−4.1981
0.5	1	−2.9735	3	−4.1988
-1	1	−2.7413	3	−2.9769
-1.5	19	−2.0027	13	−2.3312
-1.9	113	−2.0064	759	−2.0638

lization by static output feedback. A static output feedback control exists if $a > -2$. Table 5 shows the number of iterations and resultant controls for each value of a. It can be seen that the outer iteration numbers of Algorithm 1 are comparable with those of the ILMI method. Moreover, in case of $a = -1.9$, Algorithm 1 is superior to the ILMI method.

5. CONCLUSIONS

In this paper, a nonlinear minimization problem is proposed to obtain a solution of the BMI problem that arises in multiobjective and structured controls. An explicit algorithm for solving the proposed nonlinear minimization problem is also presented using a linearization method. The proposed algorithm is easily implemented using efficient convex optimization codes. Numerical experiments show that the proposed nonlinear programming approach is more efficient than other existing approaches for multiobjective and structured control problems. The proposed nonlinear programming approach can be applied to all BMI problems in multiobjective and structured controls, such as simultaneous stabilization by static output feedback, mixed H_2/H_∞ control, and simultaneous stabilization by decentralized output feedback.

APPENDIX A

PROOF OF LEMMA 3

The following lemmas will be used in the proof of Lemma 3.

Lemma 4: [6] If X and Y are symmetric matrices and satisfy the inequality

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \qquad \text{(A,1)}$$

then $X > 0$, $Y > 0$, and $X - Y^{-1} \geq 0$.

Lemma 5: [28] The inequality

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \geq 0 \qquad \text{(A,2)}$$

is satisfied if and only if $X_{11} \geq 0$ and $X_{22} - X_{12}^T X_{11}^+ X_{12} \geq 0$ where X_{11}^+ is the pseudo inverse of X_{11}.

Proof: From (52), we have

$$\begin{bmatrix} Z & I_q & 0 \\ I_q & R & N^T \\ 0 & N & M \end{bmatrix} \geq 0. \qquad \text{(A,3)}$$

Hence, we also have

$$\begin{bmatrix} Z & I_q \\ I_q & R \end{bmatrix} \geq 0. \qquad \text{(A,4)}$$

From Lemma 4, we have $Z > 0$ and $R > 0$. By applying Lemma 5 to (52), we also have

$$\begin{bmatrix} M & N \\ N^T & R \end{bmatrix} - \begin{bmatrix} 0 \\ I_q \end{bmatrix} Z^{-1} \begin{bmatrix} 0 & I_q \end{bmatrix} = \begin{bmatrix} M & N \\ N^T & R - Z^{-1} \end{bmatrix} \geq 0. \qquad \text{(A,5)}$$

By applying Lemma 5 to (A.5), we have $R - N^T M^{-1} N \geq Z^{-1} > 0$. □

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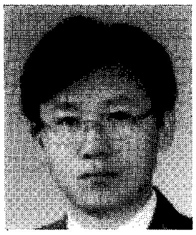

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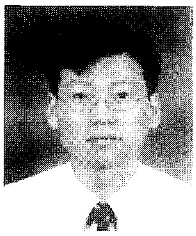

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