Journal of the Korean Society of Marine Engineers, Vol.27, No.4, pp.526~534, 2003(ISSN 1226-9549)

A path planning of free flying object and its application to the control of gymnastic robot

Taek-Kun Nam† · Myung-Ok So*

(Manuscript : Received JUN 3, 2003 ; Revised JUN 19, 2003)

Key words : Path planning, Nonholonomic system, Bang-bang control, Gymnastic robot

Abstract

Motions of animals and gymnasts in the air as well as free flying space robots without thruster are subject to nonholonomic constraints generated by the law of conservation of angular momentum. The interest in nonholonomic control problems is motivated by the fact that such systems can not stabilized to its equilibrium points by the smooth control input. The purpose of this paper is to derive analytical posture control laws for free flying objects in the air. We propose a control method using bang-bang control for trajectory planning of a 3 link mechanical system with initial angular momentum. We reduce the DOF (degrees of freedom) of control object in the first control phase and determine the control inputs to steer the reduced order system from its initial position to its desired position. Computer simulation for a motion planning of an athlete approximated by 3 link is presented to illustrate the effectiveness of the proposed control scheme.

1. Introduction

The nonholonomic system can be derived from the constraints that are not integrable in mechanical system. Various motions of an athlete such as platform diving, horizontal bar, horse vault, floor exercises are subject to nonholonomic constraints generated by the law of conservation of angular momentum. In particular, athlete's performances also include rotation in midair called a somersault or a flip. The initial angular momentum that the athletes acquire in the pre-flight phase remains constant after he or she takes off the ground, but it is transferred to their rotation axis- in other words, the gymnasts will automatically begin to rotate about their center of gravity. Therefore, the initial

† Corresponding Author(Korea Electrotechnology Research Institute), E-mail:tknam@keri.re.kr, Tel:055)280-1541

* Korea Maritime University, E-mail:smo@hanara.kmaritime.ac.kr

angular momentum plays an important role in posture control of an athletes in midair. We plan to develop a planar gymnastic robot and to accomplish the motion of an athlete in the flight phase. To realize the motion of gymnast, we need to derive a control law to achieve its goal. The purpose of this paper is to design a configuration control law for a free flying gymnast with an initial angular momentum.

There are related works of trajectory planning for nonholonomic systems with initial angular momentum[1]~[3]. Sampei et al. [2] shown that error of the system becomes locally controllable when the reference trajectory of the body angle is given by a certain first order function of time and proposed a linear feedback control method to stabilize the closed loop system. However, there is no guarantee that the error converges to zero when the control terminal time is finite. Kamon et al[3]. formulated the configuration control of a 3 dimensions somersault as a path planning problem and derived a minimum energy trajectory by numerical optimization method. Godhavn et al[1]. has proposed motion planning for a planar diver using reaching control and manifold control based on numerical computation. In this scheme the solution is not unique since the control input is generated by a random process. An analytical solution for motion planning of 2 DOF free flying object with drift by time optimal control was derived in Mita[6].

In this paper, we propose a configuration control law for 3 DOF free flying objects with initial angular momentum using bang-bang control [7] which has n-1 switching; n is the number of general coordinates. The proposed method reduce the DOF in finite time t_1, $0 < t_1 < T$(*final time*), and plan the trajectory from its initial states to desired position for the reduced order system. The computer simulation for a motion planning of an athlete approximated by 3 links is presented to show the effectiveness of proposed control algorithm.

2. Control problem

2.1 Control object

We deal with the configuration control problems of a 3 link planar gymnastic robot as shown in Fig.1.

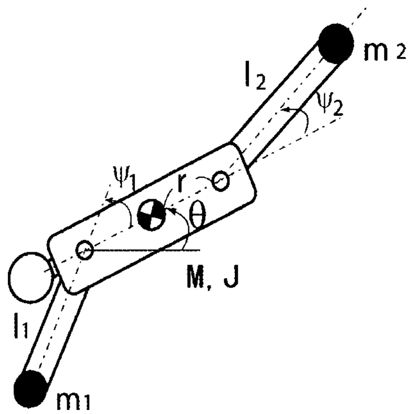

Fig. 1 Planar gymnastic robot composed of revolute joints.

The robot is composed of an arm and a leg of lengths l_1, l_2, weights m_1 and m_2 and moments of inertia J_1, J_2. M and J denote the body of weight and the moment of inertia, respectively. The limbs are attached to the body via revolute joints with distance r from the center of CG (center of gravity) of the body. The configuration of the robot can be described by ϕ_1, ϕ_2, θ. ϕ_1 reveals the relative angle between the body and the arm, and ϕ_2 is the relative angle between the body and the leg, and θ implies the absolute angle of the body with respect to horizontal.

2.1 Control problem

Suppose that the robot has a nonzero constant angular momentum P_0 which is provided by contact with the floor. Such an angular momentum remains constant after the robot takes off the ground as an initial angular momentum. The initial angular momentum is given as (see Appendix. I)

$$P_0 = (J_a + \frac{a_0}{m_0})\dot{\theta} - (J_1 + \frac{b_0}{m_0})\dot{\phi}_1 - (J_2 + \frac{c_0}{m_0})\dot{\phi}_2 \quad (1)$$

where J_a, J_1, J_2 are functions of ϕ_1, ϕ_2. Then the law of conservation of angular momentum can be derived as

$$\begin{aligned}\dot{\theta} &= \frac{m_0 P_0}{m_0 J_a + a_0} + \frac{J_1 + b_0}{m_0 J_a + a_0}\dot{\phi}_1 + \frac{J_2 + c_0}{m_0 J_a + a_0}\dot{\phi}_2 \\ &= \gamma_1(\phi_1, \phi_2) + \gamma_2(\phi_1, \phi_2)\dot{\phi}_1 + \gamma_3(\phi_1, \phi_2)\dot{\phi}_2\end{aligned} \quad (2)$$

Defining the generalized coordinates as $x = (\phi_1, \phi_2, \theta)^T$ and the control inputs as $u_1 = \dot{\phi}_1, u_2 = \dot{\phi}_2$, then we have

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ \gamma_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \gamma_2 & \gamma_3 \end{bmatrix} u \quad (3)$$

The control problem is to derive a control input which can drive the system from its initial state x_0 to its desired state x_r at fixed final time T. Considering the error $(q = x - x_r)$ between the current state value and the desired value, we have

$$\begin{aligned}\dot{q} &= \begin{bmatrix} 0 \\ 0 \\ \alpha_1(q_1, q_2) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha_2(q_1, q_2) & \alpha_3(q_1, q_2) \end{bmatrix} u \\ &= f(q) + G(q)u\end{aligned} \quad (4)$$

where $\alpha_1, \alpha_2, \alpha_3$ are functions of q (see Appendix. II). We can see that the control problem changed to derivation of control input to stabilize error variables q. Godhavn [1] generalized the STLC(Small time locally controllability) for an affine nonlinear control system with drift. Applying the control input $u = \lambda v, \lambda > 0$ to eq. (4), and scaling the time with $\tau = \lambda t$, then we have

$$\frac{dq}{d\tau} = \frac{1}{\lambda} f(q) + G(q)v \quad (5)$$

Since the drift term can be compensated by a large control input, i.e. $f/\lambda \cong 0$, the system will be equivalent to a driftless system and the controllability is guaranteed. This STLC property is guaranteed for 3 DOF nonholonomic systems that have 2 control inputs, but not generally guaranteed for a 2 DOF nonholonomic system with one control input.

3. Trajectory control by reducing DOF

We will design a control laws that steer the system eq. (4) from its initial configuration to the origin. As the first control phase, we consider the control input that reduces the order of system. If an initial state ($q_2(0) > q_1(0)$) lies in Reg. 1 ($q_2(0) > |q_1(0)|$) as shown in Fig.2, we will determine the control input

$$u(t) = (u_m - u_m)^T \quad (6)$$

where u_m denotes the maximum control input.
Applying control input (6) to eq. (4), trajectories of q_1, q_2 become

$$\begin{aligned} q_1(t) &= u_m t + q_1(0) \\ q_2(t) &= -u_m t + q_2(0) \end{aligned} \quad (7)$$

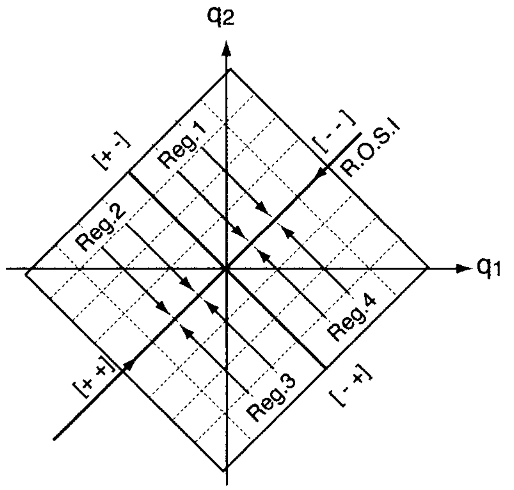

Fig. 2 Control input to reduce the order of system

Here, we define the reducing order surface(ROS I)

$$ROS\ I = \{q_1, q_2 \in R \mid q_1(t) = q_2(t)\} \quad (8)$$

The initial states in Reg.1 or Reg.2 will move to the ROS I by the control input. When the states q_1 and q_2 arrived at the ROS I, $q_1(t)$ becomes equal to $q_2(t)$, then the system is reduced to second order system. We can also consider control input

$$u(t) = (-u_m \quad u_m)^T \quad (9)$$

for an initial states in Reg.3 ($-q_2(0) \le |q_1(0)|$) or Reg.4 ($q_1(0) > q_2(0)$). Applying control input, eq. (9) to eq. (4), we have

$$q_1(t) = -u_m t + q_1(0), \quad q_2(t) = u_m t + q_2(0) \quad (10)$$

We see that the state variables q_1, q_2 move to ROS I and the system becomes to second order system. We will define t_1, the time when the states satisfy $q_1(t) = q_2(t)$. Since the DOF of system is reduced, we just consider the problem as how to move states q_1, q_3 from its initial states of reduced order system to the origin. The states and control inputs of reduced order system, i.e. the state variables on the ROS I ($t > t_1$) are defined as

$$\begin{aligned} q_1 &= q_2 = \eta, \\ u_1 &= u_2 = u \end{aligned} \quad (11)$$

Now the system equations of the reduced order system can be described as

$$\begin{aligned} \dot{\eta} &= u \\ \dot{q}_3 &= \alpha_1(q) + (\alpha_2(q) + \alpha_3(q))u \\ &= \alpha_1(\eta) + \beta_1(\eta)u \end{aligned} \quad (12)$$

In order to steer the states η, q_3 to the origin, we apply bang-bang control input. We have to analyze the trajectories of system, eq. (12) in order to apply a bang-bang input, e.g. $u = u_m$ or $u = -u_m$. Let us control input be

$$u = -u_m \quad (13)$$

Substituting eq. (13) into eq. (12), then we have

$$\begin{aligned} \dot{\eta} &= -u_m \\ \dot{q}_3 &= \alpha_1(\eta) - \beta_1(\eta)u_m \end{aligned} \quad (14)$$

The body angle q_3 which is the solution of eq. (14) is given as

$$q_3(t)=h_1(\eta)+C_1 \quad (15)$$

where $h_1(\eta)=-\frac{1}{u_m}\int_0^{\eta}(\alpha_1(p)-\beta_1(p)u_m)dp$

and C_1 denotes integral constant. The second equation of (14) can be derived from differentiation of eq. (15). Substituting eq. (4) into eq. (15), then we have [8]

$$h_1(\eta)=\int_0^{\eta}\frac{W_1/2\sin p+W_2/2\cos p+W_4}{W_1\sin p+W_2\cos p+W_3}dp$$
$$=\frac{\eta}{2}+\frac{2W_4-W_3}{\sqrt{W_3^2-W_1^2-W_2^2}}\tan^{-1}\left(\frac{(W_3-W_2)\tan(\eta/2)+W_1}{\sqrt{W_3^2-W_1^2-W_2^2}}\right) \quad (16)$$

where

$$W_1=-2(k_6\sin\phi_{1r}+k_7\sin\phi_{2r})$$
$$W_2=2(k_6\cos\phi_{1r}+k_7\cos\phi_{2r})$$
$$W_3=m_0J_a+k_1+k_2+k_3+2k_8\cos(\phi_{1r}-\phi_{2r})$$
$$W_4=\frac{-m_0P_0}{u_m}+J_1+J_2+k_4+k_5+2k_8\cos(\phi_{1r}-\phi_{2r})$$

The function $\tan^{-1}(*)$ in eq. (16) has discontinuity at $\eta=\pm\pi$. Therefore, we need to modify the calculation of $\tan^{-1}(*)$as

$$k\pi+\tan^{-1}\left(\frac{(W_3-W_2)\tan(\eta/2-k\pi)+W_1}{\sqrt{W_3^2-W_1^2-W_2^2}}\right) \quad (17)$$

The trajectory of $h_1(\eta)$ is shown in Fig. 3. Next, we consider control input $u=u_m$, then we have

$$\begin{aligned}\dot{\eta}&=u_m\\ \dot{q}_3&=\alpha_1(\eta)+\beta_1(\eta)u_m\end{aligned} \quad (18)$$

from (12). The solution of q_3 is

$$q_3(t)=h_2(\eta)+C_2 \quad (19)$$

where

$$h_2(\eta)=\frac{1}{u_m}\int_0^{\eta}(\alpha_1(p)+\beta_1(p)u_m)dp$$
$$=\frac{\eta}{2}+\frac{2W_5-W_3}{\sqrt{W_3^2-W_1^2-W_2^2}}\tan^{-1}\left(\frac{(W_3-W_2)\tan(\eta/2)+W_1}{\sqrt{W_3^2-W_1^2-W_2^2}}\right) \quad (20)$$

where $W_5=m_0P_0/u_m+J_1+J_2+k_4+k_5+2k_8\cdot\cos(\phi_{1r}-\phi_{2r})$

The trajectory $h_2(\eta)$ after compensating the discontinuity is shown in Fig. 3.

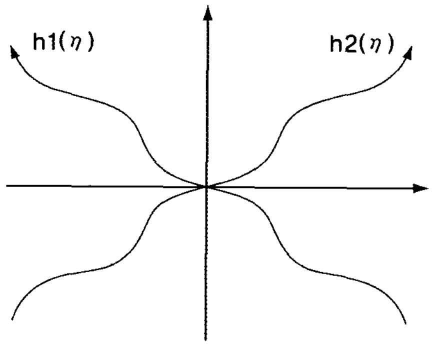

Fig. 3 Trajectories of $h_1(\eta)$ and $h_2(\eta)$.

The trajectories of q_3 calculated from eq. (15), eq. (19) are shown in Fig. 4. The trajectories I leading to left and trajectories II leading to right are obtained by changing C_1 in (15) and C_2 in (19), respectively.

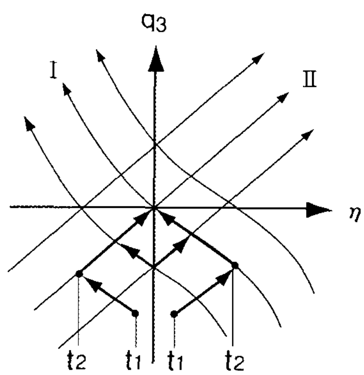

Fig. 4 Trajectory of q_3 to η.

From fig. 4, we can determine the suitable trajectory from initial states to the origin. Suppose that initial states $\eta(t_1), q_3(t_1)$ were located in the fourth quadrant and under the invariant manifold I, it becomes $\eta(t_1) \geq 0$. In this case, we apply control input as $u = u_m$ until the trajectory meets path which leads to left. Then the trajectory leading to right will intersect at $t = t_2$ with other path which moves from the fourth quadrant to the origin. The final approach is to switch the control input as $u(t) = -u_m$, $t_2 \leq t \leq T$. Therefore, we can steer the states of reduced system from its initial position to the origin. Repeating this procedure, initial states $\eta(t_1), q_3(t_1)$ in the third quadrant and under the invariant manifold II can be steered to the origin by the control inputs $u(t) = u_m, t_1 \leq t \leq t_2$, $u(t) = -u_m, t_2 \leq t \leq T$. When the initial states $\eta(t_1), q_3(t_1)$ are not exist under the invariant manifold *I* or *II*, we introduce new initial states as $\eta(t_1), q_3(t_1) - 2\pi$. It gives the solution for the robot body rotated 2π.

Now we can compile configuration control for the gymnastic robot from its initial states to the origin. In case an initial states located in

$$\begin{aligned} &Reg.1\ q_2(0) > |q_1(0)|, q_3(0) < 0 \\ &Reg.2\ q_1(0) < |q_2(0)|, q_3(0) < 0 \end{aligned} \tag{21}$$

then the control inputs become

$$u(t) = \begin{cases} (u_m \ \ -u_m)^T, (t_0 \leq t \leq t_1) \\ (-u_m \ \ -u_m)^T, (t_1 \leq t \leq t_2) \\ (u_m \ \ u_m)^T, (t_2 \leq t \leq T) \end{cases} \tag{22}$$

For the initial states in

$$\begin{aligned} &Reg.3\ q_2(0) < |q_1(0)|, q_3(0) < 0 \\ &Reg.4\ q_1(0) > |q_2(0)|, q_3(0) < 0 \end{aligned} \tag{23}$$

The control inputs become

$$u(t) = \begin{cases} (-u_m \ \ u_m)^T, (t_0 \leq t \leq t_1) \\ (-u_m \ \ -u_m)^T, (t_1 \leq t \leq t_2) \\ (u_m \ \ u_m)^T, (t_2 \leq t \leq T) \end{cases} \tag{24}$$

Consequently, the gymnastic robot represented by eq. (3) can be controlled from its initial position to its desired one by two times switching control inputs eq. (22), eq. (24).

4. Switching time calculation

In this section we will calculate switching time to steer all states to origin with 2 times switching. Let us assume the initial states lie in Reg. 2. From eq. (4) and eq. (22) we can get

$$q_1(t) = u_m t + q_1(0), \quad q_2(t) = -u_m t + q_2(0) \tag{25}$$

Since the states in ROS I satisfy $q_1(t_1) = q_2(t_1)$, we have

$$t_1 = \frac{q_2(0) - q_1(0)}{2u_m}, \quad q_1(0) > q_2(0) \tag{26}$$

Applying the second control input $u = (-u_m \ \ -u_m)^T$, $t_1 < t \leq t_2$, we get

$$q_1(t_2) = -u_m(t_2 - t_1) + q_1(t_1) \tag{27}$$

From eq. (27), we obtain

$$t_2 = \frac{u_m t_1 + q_1(t_1) - q_1(t_2)}{u_m} \tag{28}$$

Finally, applying $u = (u_m \ \ u_m)^T$, $t_2 < t \leq T$, we have

$$q_1(T) = u_m(T - t_2) + q_1(t_2) \tag{29}$$

using $q_1(T) = 0$, we have final time

$$T = \frac{u_m t_2 - q_1(t_2)}{u_m} \tag{30}$$

However we need to obtain $q_1(t_2)$ before calculating the switching time t_2 and final time T. In order to get information for $q_1(t_2)$, consider the state trajectory $q_3(t)$, $t_1 \le t \le T$. From eq. (15), eq. (19) we get

$$\begin{aligned} q_3(t) &= h_1(\eta) + C_1, \quad t_1 \le t \le t_2 \\ q_3(t) &= h_2(\eta) + C_2, \quad t_2 \le t \le T \end{aligned} \tag{31}$$

We can calculate an integral constant

$$C_1 = q_3(t_1) - h_1(\eta(t_1)) \tag{32}$$

from $q_3(t) = h_1(\eta) + C_1$, $t = t_1$. Considering $q_3(T) = h_2(\eta(T)) = 0$ and (32), we have

$$\begin{aligned} q_3(t) &= h_1(\eta) + q_3(t_1) - h_1(\eta(t_1)), \quad t_1 \le t \le t_2 \\ q_3(t) &= h_2(\eta), \quad t_2 \le t \le T \end{aligned} \tag{33}$$

Both the trajectories of eq. (33) should be equal at $t = t_2$, i.e.

$$h_1(\eta(t_2)) - h_2(\eta(t_2)) = -q_3(t_1) + h_1(\eta(t_1)) \tag{34}$$

We used quad8 function in MATLAB to calculate $q_3(t_1)$ which satisfies eq. (34). Finally, we can derive $\eta(t_2)$, *i.e.* $q_1(t_2)$

$$\begin{aligned} \eta(t_2) = & -2W_1 + 2\tan^{-1} \\ & (C_3 \tan(-q_3(t_1) + h_1(\eta(t_1)))) \end{aligned} \tag{35}$$

where $C_3 = \dfrac{W_3^2 - W_1^2 - W_2^2}{2(W_3 - W_2)(W_4 - W_5)}$.

Using the information of $q_1(t_2)$, we can obtain switching time t_2, final time T from eq. (28), eq. (30), respectively.

5. Application to planar diving

We applied proposed control scheme to the configuration control of planar diving. Simulation results for a 3 link gymnastic robot with parameters

$P_0 = 70[kgm^2/s], M = 40[kg], m_1 = 10[kg], m_2 = 15[kg]$
$l_1 = 0.6[m], l_2 = 0.9[m], r = 0.15[m]$ with $u_m = 4[/s]$

are shown in Fig. 5.

The initial and desired state values were $x_0 = (-0.5\ 0\ -0.5\pi)^T$ and $x_r = (0\ 0\ 2.5\pi)^T$, i.e. $q_0 = (-0.5\ 0\ -3\pi)^T$ and $q_r = (0\ 0\ 0)^T$, respectively. We can see that the control performance will be one and half rotation in midair. The switching time were determined as $t_1 = 0.0625[s]$, $t_2 = 0.9628[s]$, $T = 1.9256[s]$. Fig. 5(a) and Fig. 5(b) depict time evolution of the states and the control inputs, respectively.

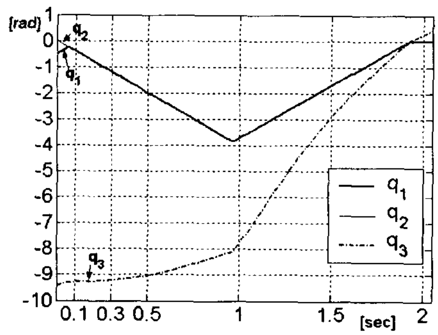

Fig. 5 (a) Time evolution of states in planar diving.

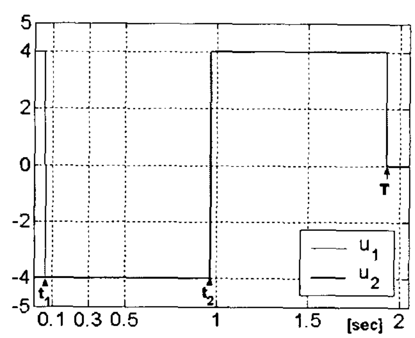

Fig. 5 (b) Time evolution of control inputs in planar diving.

We can observe that all state variables q_1, q_2, q_3 were zero at the final time T from fig. 5(a), i.e. the control purpose was achieved by the proposed control scheme. The animation of the simulation result is given in Fig. 5 (c).

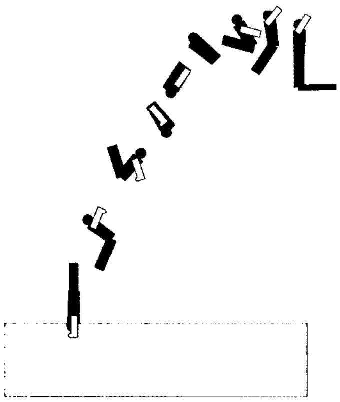

Fig. 5 (c) Animation of the simulation result.

6. Conclusions

We addressed control problem of the 3 DOF free flying objects from its initial configuration to the desired configuration using bang-bang control inputs. We reduced the DOF of original systems in the first control phase and determined the control inputs to steer the reduced order systems to the desired configuration by bang-bang control. The computer simulation for a motion planning of the planar diver approximated by 3 links was carried out to verify the effectiveness of the proposed control scheme. From the simulation results, this bang-bang control input which has switching time information is useful for real experiment because of its simplicity.

References

[1] J. M. Godhavn, A. Balluchi, L. S. Crawford, S. S. Sastry, "Steering of a class of nonholonomic systems with drift terms", Automatica, vol. 35, pp. 837-847, 1999.

[2] M. Sampei, H. Kiyota, "Control of Underactuated Flying Objects", Proc. of Titech COE Workshop, pp. 266-271, 1999.

[3] M. Kamon, K. Yoshida, "3 Dimensional attitude control methods for free-flying dynamc system with initial angular momentum", Journal of RSJ, pp. 223-231,1998.

[4] H. Nakamura, T. Mita, "Posture control of a free flying robot with initial angular momentum", Proc. SICE DST Symposium, pp. 255-260, 1999.

[5] Zexiang Li, J. F. Canny, "Nonholonomic motion planning", Kluwer Academic Publishers.

[6] T. Mita, S.H. Hyon, and T.K. Nam, "Analytical time optimal control solution for free flying objects with drift term", Proc. of IEEE CDC, pp. 91-94, 2000.

[7] A.P. Sage, C.C. White, "Optimum systems control", Prentice Hall.

[8] S. Moriguchi, K. Udagawa, S. Ichimatsu, "Mathematical Formulal", Iwanami Pub Co., 1987.

[9] Y. Chen et al., "A Proof the structure of the minimum time control law of robotic manipulators using a Hamilton formulation", IEEE Trans. on Robotics and Automation, vol. 6, no. 3, pp. 388-393, 1990.

[10] D. E. Rosenthal and M. A. Sherman, "High performance multibody simulations via symbolic equation manipulation and kane's method", Journal of Astronautical Science, vol. 34, no. 3, pp. 223-239, 1986.
[11] P. J. Brancazio, "Sport science-Physical laws and optimum performance", Simon and Schuster Inc, 1984.

Appendix. I

The angular momentum of 3 link flying object is

$$P_0 = (J_a + \frac{a}{m_0})\dot{\theta} - (J_1 + \frac{b}{m_0})\dot{\phi}_1 - (J_2 + \frac{c}{m_0})\dot{\phi}_2 \qquad \text{(A.1)}$$

where

$$\begin{aligned} a_0 = {} & m_1 M(l_1^2 + r_2^2) + m_1 m_2 (l_1^2 + l_2^2 + 4r^2) \\ & + m_2 M(l_2^2 + r^2) + 2m_1 l_1 r(M + 2m_2)\cos\phi_1 \\ & + 2m_2 l_2 r(M + 2m_1)\cos\phi_2 + 2m_1 m_2 l_1 l_2 \\ & \cos(\phi_1 - \phi_2) := k_1 + k_2 + k_3 + 2k_6 \cos\phi_1 \\ & + 2k_7 \cos\phi_2 + 2k_8 \cos(\phi_1 - \phi_2) \end{aligned}$$

$$\begin{aligned} b_0 = {} & m_1(M + m_2) l_1^2 + m_1 l_1 r(M + 2m_2)\cos\phi_1 \\ & + m_1 m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \\ & := k_4 + k_6 \cos\phi_1 + k_8 \cos(\phi_1 - \phi_2) \end{aligned}$$

$$\begin{aligned} c_0 = {} & m_2(M + m_1) l_2^2 + m_2 l_2 r(M + 2m_1)\cos\phi_2 \\ & + m_1 m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \\ & := k_5 + k_7 \cos\phi_2 + k_8 \cos(\phi_1 - \phi_2) \end{aligned}$$

$$m_0 = m_1 + m_2 + M, \quad J_a = J_1 + J_2 + J$$

Appendix. II

$$\alpha_1(q) = \frac{m_0 P_0}{m_0 J_a + a(q)}, \quad \alpha_2(q) = \frac{J_1 + b(q)}{m_0 J_a + a(q)}, \quad \alpha_3(q) = \frac{J_2 + c(q)}{m_0 J_a + a(q)}, \qquad \text{(A.2)}$$

where

$$\begin{aligned} a(q) = {} & k_1 + k_2 + k_3 + 2k_6(\cos(\phi_{1r})\cos(q_1) \\ & - \sin(\phi_{1r})\sin(q_1)) + 2k_7(\cos(\phi_{2r})\cos(q_2) \\ & - \sin(\phi_{2r})\sin(q_2)) + 2k_8(\cos(\phi_{1r} - \phi_{2r}) \\ & \cos(q_1 - q_2) - \sin(\phi_{1r} - \phi_{2r})\sin(q_1 - q_2)) \end{aligned}$$

$$\begin{aligned} b(q) = {} & k_4 + k_6(\cos(\phi_{1r})\cos(q_1) - \sin(\phi_{1r})\sin(q_1)) \\ & + k_8(\cos(\phi_{1r} - \phi_{2r})\cos(q_1 - q_2) \\ & - \sin(\phi_{1r} - \phi_{2r})\sin(q_1 - q_2)) \end{aligned}$$

$$\begin{aligned} c(q) = {} & k_5 + k_7(\cos(\phi_{2r})\cos(q_2) - \sin(\phi_{2r}) \\ & \sin(q_2)) + k_8(\cos(\phi_{1r} - \phi_{2r}) \\ & \cos(q_1 - q_2) - \sin(\phi_{1r} - \phi_{2r})\sin(q_1 - q_2)) \end{aligned}$$