1156 *KSME International Journal, Vol. 17 No. 8, pp. 1156~1163, 2003*

In-Plane Free Vibration Analysis of Curved Timoshenko Beams by the Pseudospectral Method

Jinhee Lee*
Department of Mechano-Informatics, Hongik University,
Chochiwon, Yeonki-kun, Choongnam 339-701, Korea

The pseudospectral method is applied to the analysis of in-plane free vibration of circularly curved Timoshenko beams. The analysis is based on the Chebyshev polynomials and the basis functions are chosen to satisfy the boundary conditions. Natural frequencies are calculated for curved beams of rectangular and circular cross sections under hinged-hinged, clamped-clamped and hinged-clamped end conditions and the results are compared with those by transfer matrix method. The present method gives good accuracy with only a limited number of collocation points.

Key Words : Eigenvalue Analysis, Curved Timoshenko Beam, In-Plane Mode, Pseudospectral Method

Nomenclature

A	: Cross sectional area of the beam
B_k, C_k, D_k	: Basis functions
b_k, c_k, d_k	: Pseudospectral coefficients
E	: Young's modulus
G	: Shear modulus
I	: Second moment of area
M, N, Q	: Stress resultants
R	: Radius of curvature of the curved beam
s_l	: Slenderness ratio
T_n	: Chebyshev polynomial of the first kind
U, u	: Axial displacement
V, w	: Transverse displacement
κ	: Shear coefficient
ρ	: Density of the beam
Θ	: Angle of the curved beam
Ψ, ψ	: Bending rotation
ω	: Natural frequency in [rad/sec]

* **E-mail** : jinhlee@wow.hongik.ac.kr
TEL : +82-41-860-2589; **FAX** : +82-41-862-2664
Department of Mechano-Informatics, Hongik University, Chochiwon, Yeonki-kun, Choongnam 339-701, Korea. (Manuscript **Received** February 10, 2003; **Revised** April 7, 2003)

1. Introduction

Curved beams are frequently used in many practical applications. Because of their importance the free vibration analysis of curved beams has been extensively studied and new methods have been proposed as can be found in the review articles (Markus and Nanasi, 1981 ; Laura and Maurizi, 1987 ; Chidamparam and Leissa, 1993). Although there exists a vast amount of research on the free vibration analysis of curved beams, most of the work has been done on the basis of the Bernoulli-Euler beam theory. Real beams may have appreciable thickness where the shear deformation and the rotary inertia are not negligible as assumed in the classical beam theory. As a result the Timoshenko beam model has gained more popularity.

Free vibration analysis of curved beams based on the Timoshenko theory has been carried out using various methods such as the transfer matrix method (Bickford and Strom, 1975 ; Irie et al., 1982 ; Irie et al., 1983 ; Yildirim, 1997), the dynamic stiffness method (Issa et al., 1987 ; Howson et al., 1995 ; Tseng et al., 1997 ; Howson and Jemah, 1999), the differential quadrature method

(Kang et al., 1995) and the finite element method (Davis et al., 1972; Prathap and Babu, 1986; Heppler, 1992; Lee and Sin, 1994; Yang and Sin, 1995). In this study a free vibration analysis for the in-plane mode of curved Timoshenko beams using the pseudospectral method is presented. The pseudospectral method can be considered as a spectral method that performs a collocation process, which can be made as spatially accurate as desired through exponential rate of convergence with mesh refinement.

The pseudospectral method, however, remains largely unnoticed by the structural analysis community and the application of the pseudospectral method to the vibration analysis is scarce. The pseudospectral method was applied to the free vibration analyses of axisymmetric Mindlin plate (Soni and Amba-Rao, 1975) and axisymmetric annular Mindlin plate (Gupta and Lal, 1985). The collocation method along with the power series representation was also used in the eigenvalue analysis of rectangular Mindlin plates (Mikami and Yoshimura, 1984). Recently, the pseudospectral method was applied to the eigenvalue problems of straight Timoshenko beams and axisymmetric Mindlin plates (Lee and Schultz) and rectangular Mindlin plates (Lee, 2003).

2. Pseudospectral Formulations

Fig. 1 depicts the geometric configuration of the title problem and the dependent variables. The slenderness ratio s_l of the curved beam is defined by

$$s_l=\sqrt{AR^2/I} \tag{1}$$

Fig. 2 shows the schematics of stress resultants. The equations of motion for the in-plane modes are given as follows

$$\begin{aligned} \frac{1}{R}\frac{\partial N}{\partial \theta}-\frac{Q}{R}&=\rho A\frac{\partial^2 U}{\partial t^2} \\ \frac{1}{R}\frac{\partial Q}{\partial \theta}+\frac{N}{R}&=\rho A\frac{\partial^2 V}{\partial t^2} \\ \frac{1}{R}\frac{\partial M}{\partial \theta}-\frac{Q}{R}&=\rho I\frac{\partial^2 \Psi}{\partial t^2} \end{aligned} \tag{2}$$

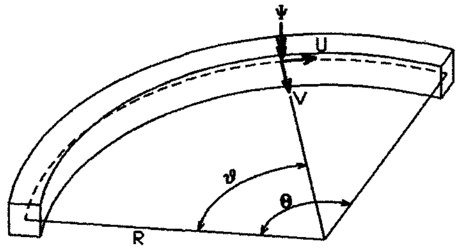

Fig. 1 Geometry of curved beam and generalized displacements

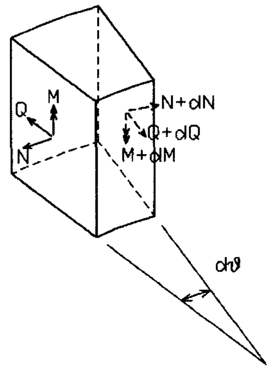

Fig. 2 Stress resultants acting on an infinitesimal element

The stress resultants M, N and Q in (2) are defined by

$$\begin{aligned} M&=\frac{EI}{R}\frac{\partial \Psi}{\partial \theta} \\ N&=\frac{EA}{R}\left(\frac{\partial U}{\partial \theta}-V\right) \\ Q&=\kappa AG\left(\frac{1}{R}\frac{\partial V}{\partial \theta}+\frac{U}{R}-\Psi\right) \end{aligned} \tag{3}$$

Assuming the simple harmonic motions in time

$$\begin{aligned} U(\theta, t)&=u(\theta)\cos\omega t \\ V(\theta, t)&=v(\theta)\cos\omega t \\ \Psi(\theta, t)&=\psi(\theta)\cos\omega t \end{aligned} \tag{4}$$

the substitution of (3) into (2) yields

$$\begin{aligned} &\frac{EA}{R^2}\frac{d^2u}{d\theta^2}-\frac{\kappa AG}{R^2}u-\frac{EA+\kappa AG}{R^2}\frac{dv}{d\theta}+\frac{\kappa AG}{R}\psi=-\omega^2\rho Au \\ &\frac{EA+\kappa AG}{R^2}\frac{du}{d\theta}+\frac{\kappa AG}{R^2}\frac{d^2v}{d\theta^2}-\frac{EA}{R^2}v-\frac{\kappa AG}{R}\frac{d\psi}{d\theta}=-\omega^2\rho Av \\ &\frac{\kappa AG}{R}u+\frac{\kappa AG}{R}\frac{dv}{d\theta}+\frac{EI}{R^2}\frac{d^2\psi}{d\theta^2}-\kappa AG\psi=-\omega^2\rho I\psi \end{aligned} \tag{5}$$

When the range of the independent variable is given by $(0 \leq \theta \leq \Theta)$ it is convenient to use the normalized variable

$$\xi=\frac{2\theta-\Theta}{\Theta}\in[-1,1] \tag{6}$$

and (5) can be rewritten as

$$\begin{aligned}&\frac{4EA}{R^2\Theta^2}u''-\frac{\kappa AG}{R^2}u-\frac{2(EA+\kappa AG)}{R^2\Theta}v'+\frac{\kappa AG}{R}\psi=-\omega^2\rho Au\\&\frac{2(EA+\kappa AG)}{R^2\Theta}u'+\frac{4\kappa AG}{R^2\Theta^2}v''-\frac{EA}{R^2}v-\frac{2\kappa AG}{R\Theta}\psi'=-\omega^2\rho Av\\&\frac{\kappa AG}{R}u+\frac{2\kappa AG}{R\Theta}v'+\frac{4EI_z}{R^2\Theta^2}\psi''-\kappa AG\psi=-\omega^2\rho I\psi\end{aligned} \tag{7}$$

where $'$ stands for the differentiation with respect to ξ. The series expansions of the exact solutions $u(\xi)$, $v(\xi)$ and $\psi(\xi)$ have infinite numbers of terms. In this study, however, the dependent variables are approximated by the K-th partial sums as follows:

$$\begin{aligned}u(\xi)&\approx\tilde{u}(\xi)=\sum_{k=1}^{K}b_kB_k(\xi)\\v(\xi)&\approx\tilde{v}(\xi)=\sum_{k=1}^{K}c_kC_k(\xi)\\\psi(\xi)&\approx\tilde{\psi}(\xi)=\sum_{k=1}^{K}d_kD_k(\xi)\end{aligned} \tag{8}$$

The end conditions considered in this study are clamped-clamped, hinged-hinged, and clamped-hinged boundary conditions. The boundary conditions for the in-plane mode are given by

$$\begin{cases}\text{hinged support}: u=0,\ v=0,\ M=0\\\text{clamped support}: u=0,\ v=0,\ \psi=0\end{cases} \tag{9}$$

The basis functions

$$\begin{aligned}B_{2n-1}(\xi)&=C_{2n-1}(\xi)=T_{2n}(\xi)-T_0(\xi)\\B_{2n}(\xi)&=C_{2n}(\xi)=T_{2n+1}(\xi)-T_1(\xi)\\&(n=1,2,\cdots)\end{aligned} \tag{10}$$

satisfy the boundary conditions $u=0$ and $v=0$ at $\xi=\pm1$. The basis function $D_k(\xi)$ is required to satisfy either $\psi=0$ or $\psi'=0$ at the ends, and it is assumed

$$\begin{aligned}D_{2n-1}(\xi)&=T_{2n}(\xi)-T_0(\xi)+a_1\xi^2+a_2\xi\\D_{2n}(\xi)&=T_{2n+1}(\xi)-T_1(\xi)+a_3\xi^2+a_4\xi\\&(n=1,2,\cdots)\end{aligned} \tag{11}$$

The calculation of constants a_1, a_2, a_3 and a_4 that satisfy each of clamped-clamped ($\psi=0$ at $\xi=\pm1$), hinged-hinged ($\psi'=0$ at $\xi=\pm1$), and clamped-hinged ($\psi=0$ at $\xi=-1$ and $\psi'=0$ at $\xi=1$) boundary conditions is given in APPENDIX. By substituting (8) into (7) and by setting the residuals equal to zero at the collocation points

$$\xi_i=-\cos\frac{(2i-1)\pi}{2K},\ (i=1,\cdots,K) \tag{12}$$

the pseudospectral algebraic system of equations for the in-plane mode is given by

$$\begin{aligned}&\sum_{k=1}^{K}\left[b_k\left\{\frac{4EA}{R^2\Theta^2}B_k''(\xi_i)-\frac{\kappa AG}{R^2}B_k(\xi_i)\right\}-c_k\frac{2(EA+\kappa AG)}{R^2\Theta}C_k'(\xi_i)\right.\\&\left.+d_k\frac{\kappa AG}{R}D_k(\xi_i)\right]=-\omega^2\sum_{k=1}^{K}\rho Ab_kB_k(\xi_i)\\&\sum_{k=1}^{K}\left[b_k\frac{2(EA+\kappa AG)}{R^2\Theta}B_k'(\xi_i)+c_k\left\{\frac{4\kappa AG}{R^2\Theta^2}C_k''(\xi_i)-\frac{EA}{R^2}C_k(\xi_i)\right\}\right.\\&\left.-d_k\frac{2\kappa AG}{R\Theta}D_k'(\xi_i)\right]=-\omega^2\sum_{k=1}^{K}\rho Ac_kC_k(\xi_i)\\&\sum_{k=1}^{K}\left[b_k\frac{\kappa AG}{R}B_k(\xi_i)+c_k\frac{2\kappa AG}{R\Theta}C_k'(\xi_i)\right.\\&\left.+d_k\left\{\frac{4EI_z}{R^2\Theta^2}D_k''(\xi_i)-\kappa AGD_k(\xi_i)\right\}\right]=-\omega^2\sum_{k=1}^{K}\rho I_zd_kD_k(\xi_i)\\&\qquad(i=1,\cdots,K)\end{aligned} \tag{13}$$

The total number of pseudospectral coefficients $b_1, \cdots, b_K, c_1, \cdots c_K, d_1, \cdots, d_K$ is $3K$, which matches the total number of equations in (13), and (13) is solved for the eigenvalues of the in-plane modes.

3. Numerical Examples

A preliminary test is run to check the convergence of the pseudospectral method applied to the in-plane free vibration analysis of curved Timoshenko beams. The eigenvalues of circularly curved beam of circular cross section with clamped-clamped boundary condition for the slenderness ratio $s_l=100$ are computed for different collocation number K, and the computed results are listed in Table 1. This shows the rapid convergence nature of the pseudospectral method such that the convergence of lowest 4 eigenvalues

Table 1 Convergence test of nondimensinalized frequency parameters λ_i ($s_t=100$, $\Theta=120$, circular cross section, clamped-clamped boundary condition)

Vibration mode	$K=7$	$K=10$	$K=15$	$K=20$	$K=25$	$K=30$
1	11.910	11.790	11.790	11.790	11.790	11.790
2	27.207	23.255	23.249	23.249	23.249	23.249
3	69.963	42.869	42.367	42.367	42.367	42.367
4	90.996	65.903	61.425	61.424	61.424	61.424
5	141.99	99.288	89.930	89.872	89.872	89.872
6	179.82	107.20	94.081	94.074	94.074	94.074
7	316.22	176.96	124.55	124.20	124.20	124.20
8	460.45	288.85	155.69	150.94	150.94	150.94
9	641.06	329.22	182.56	179.06	179.06	179.06
10	734.30	379.20	205.49	193.30	193.18	193.18

Table 2 Nondimensionalized frequency parameter λ_i of circularly curved Timoshenko beam (clamped-clamped boundary condition)

s_t		Rectangular cross section			Circular cross section		
		$\Theta=60°$	$\Theta=120°$	$\Theta=180°$	$\Theta=60°$	$\Theta=120°$	$\Theta=180°$
10	1	15.256	8.2796	3.6163	15.384	8.3526	3.6367
	2	24.251	8.7386	6.3265	24.546	8.7515	6.3571
	3	32.777	17.042	10.565	32.902	17.070	10.598
	4	42.655	17.175	10.923	43.344	17.354	10.976
	5	59.012	25.323	15.267	60.015	25.631	15.342
20	1	23.713	10.574	4.1494	23.772	10.612	4.1570
	2	38.646	15.156	8.5123	38.988	15.182	8.5354
	3	62.919	24.629	15.399	62.958	24.716	15.455
	4	69.877	30.450	17.891	70.676	30.553	17.913
	5	102.08	38.935	25.410	103.41	39.076	25.513
50	1	44.734	11.615	4.3443	44.746	11.623	4.3456
	2	50.177	22.091	9.4524	50.298	22.115	9.4576
	3	99.753	40.655	17.464	100.10	40.721	17.477
	4	144.71	45.162	26.202	145.11	45.179	26.229
	5	165.29	64.563	37.792	165.67	64.683	37.844
100	1	52.779	11.788	4.3743	52.815	11.790	4.3746
	2	75.973	23.242	9.6014	76.004	23.249	9.6027
	3	117.81	42.349	17.805	117.87	42.367	17.809
	4	170.79	61.389	27.207	171.07	61.424	27.215
	5	255.14	89.800	39.294	255.69	89.872	39.308
				Irie et al.			
20	1	23.70	12.57*	4.143	23.75	10.61	4.151
	2	38.73	15.17	8.519	39.05	15.19	8.542
	3	62.35	24.63	15.40	62.38	24.72	15.46
	4	69.97	30.38	17.90	70.71	30.47	17.91
100	1	52.78	11.79	4.374	52.82	11.79	4.374
	2	75.98	23.24	9.602	76.01	23.25	9.603
	3	117.8	42.35	17.81	117.9	42.37	18.81
	4	170.8	61.39	27.21	171.1	61.43	27.22

Table 3 Nondimensionalized frequency parameter λ_i of circularly curved Timoshenko beam (hinged-hinged boundary condition)

s_l		Rectangular cross section			Circular cross section		
		Θ=60°	Θ=120°	Θ=180°	Θ=60°	Θ=120°	Θ=180°
10	1	11.529	5.7874	2.0726	11.548	5.8117	2.0773
	2	21.109	8.6120	5.6055	21.288	8.6209	5.6267
	3	32.471	14.530	9.8142	32.527	14.622	9.8229
	4	40.578	17.036	10.299	41.114	17.062	10.350
	5	57.480	23.319	14.372	58.809	23.521	14.398
20	1	19.558	6.5803	2.2132	19.564	6.5895	2.2146
	2	28.570	14.382	6.5336	28.691	14.409	6.5425
	3	60.074	20.932	12.801	60.510	20.953	12.830
	4	62.729	27.922	17.853	62.750	28.023	17.874
	5	94.486	36.542	22.149	95.409	36.575	22.198
50	1	32.620	6.8676	2.2579	32.650	6.8693	2.2582
	2	44.197	17.033	6.8574	44.205	17.042	6.8591
	3	76.981	32.708	13.771	77.116	32.738	13.777
	4	126.32	44.985	22.224	126.69	45.006	22.238
	5	158.72	55.287	32.830	158.76	55.339	32.860
100	1	33.365	6.9118	2.2645	33.373	6.9122	2.2646
	2	68.985	17.381	6.9067	69.013	17.384	6.9071
	3	101.50	33.500	13.925	101.51	33.508	13.927
	4	137.44	52.436	22.672	137.56	52.456	22.675
	5	214.73	77.336	33.650	215.02	77.377	33.658
200	1	33.560	6.9230	2.2662	33.562	6.9231	2.2662
	2	73.994	17.468	6.9191	74.004	17.468	6.9192
	3	140.51	33.705	13.965	140.55	33.707	13.965
	4	183.64	53.236	22.783	183.65	53.241	22.784
	5	230.02	78.396	33.859	230.01	78.406	33.861

to 5 digits is achieved for $K=15$, and lowest 9 eigenvalues to 5 digits for $K=20$. Poisson's ratio ν is 0.3 and shear correction factor κ for the circular and the rectangular cross section are 0.89 and 0.85, respectively, throughout the paper. The numbers given in Tables 1~4 are the non-dimensionalized frequency parameters λ_i defined as

$$\lambda_i=\sqrt{\rho A R^4 \omega_i^2 / EI} \tag{14}$$

Eigenvalues are computed with $K=30$ for various slenderness ratios and curved beam angles Θ under clamped-clamped, hinged-hinged and clamped-hinged boundary conditions, and lowest 5 eigenvalues for each boundary condition are listed in Tables 2~4. The eigenvalues computed by the transfer matrix method (Irie et al., 1983) are also given for the purpose of comparison in Table 2. The eigenvalues for $s_l=100$ in Table 2 show excellent agreement with those by the transfer matrix method, where it was pointed out that 12.57* among Irie et al.'s results must be a typo for 10.57 (Kang et al., 1995).

4. Conclusions

The Chebyshev pseudospectral method is applied to the analysis of in-plane free vibration of curved Timoshenko beams. The pseudospectral formulation is straightforward and efficient for writing a code for computation. Numerical examples are provided for circularly curved beams of rectangular and circular cross sections under clamped-clamped, hinged-hinged and clamped-hinged boundary conditions for various slenderness ratios and curved beam angles. The results

Table 4 Nondimensionalized frequency parameter λ_i of circularly curved Timoshenko beam (clamped-hinged boundary condition)

s_l		Rectangular cross section			Circular cross section		
		$\Theta=60°$	$\Theta=120°$	$\Theta=180°$	$\Theta=60°$	$\Theta=120°$	$\Theta=180°$
10	1	13.161	6.9137	2.8228	13.224	6.9561	2.8342
	2	22.933	8.7247	6.0392	23.183	8.7388	6.0670
	3	32.600	15.872	9.9903	32.684	16.003	10.012
	4	41.594	17.042	10.809	42.211	17.071	10.859
	5	58.783	24.357	14.719	59.803	24.614	14.766
20	1	20.931	8.4650	3.1287	20.952	8.4860	3.1326
	2	33.926	14.924	7.5533	34.152	14.952	7.5689
	3	62.637	22.519	14.071	62.680	22.568	14.113
	4	65.263	29.532	17.884	65.863	29.644	17.903
	5	98.425	37.407	23.711	99.557	37.481	23.786
50	1	39.927	9.0789	3.2329	39.981	9.0830	3.2336
	2	44.985	19.601	8.1313	45.006	19.617	8.1344
	3	88.185	36.495	15.561	88.415	36.541	15.570
	4	137.02	45.126	24.222	137.48	45.145	24.242
	5	160.46	59.692	35.259	160.58	59.776	35.300
100	1	42.333	9.1764	3.2486	42.351	9.1775	3.2488
	2	73.727	20.260	8.2209	73.760	20.264	8.2217
	3	107.58	37.763	15.800	107.62	37.776	15.802
	4	153.98	56.993	24.903	154.17	57.020	24.909
	5	234.65	83.291	36.410	235.06	83.346	36.421
200	1	42.789	9.2012	3.2526	42.794	9.2015	3.2526
	2	84.711	20.421	8.2436	84.728	20.422	8.2438
	3	157.71	38.081	15.861	157.76	38.084	15.861
	4	183.74	58.327	25.069	183.75	58.334	25.071
	5	250.83	84.875	36.698	250.94	84.890	36.701

under the clamped-clamped boundary condition are compared with the solutions by the transfer matrix method and it is shown that they are in excellent agreement. The title problem demonstrates the rapid convergence and accuracy as well as the conceptual simplicity of the pseudospectral method.

References

Bickford, W. B. and Strom, B. T., 1975, "Vibration of Plane Curved Beams," *Journal of Sound and Vibration*, Vol. 39, No. 2, pp. 135~146.

Chidamparam, P. and Leissa, A. W., 1993, "Vibrations of Planar Curved Beams, Rings, and Arches," *Applied Mechanics Reviews*, Vol. 46, No. 9, pp. 467~483.

Davis, R., Henshell, R. D. and Warburton, G. B., 1972, "Constant Curvature Beam Finite Elements for In-plane Vibration," *Journal of Sound and Vibration*, Vol. 25, No. 4, pp. 561~576.

Gupta, U. S. and Lal, R., 1985, "Axisymmetric Vibrations of Ploar Orthotropic Mindlin Annular Plates of Variable Thickness," *Journal of Sound and Vibration*, Vol. 98, No. 4, pp. 565~573.

Heppler, G. R., 1992, "An Element for Studying the Vibration of Unrestrained Curved Timoshenko Beams," *Journal of Sound and Vibration*, Vol. 158, No. 3, pp. 387~404.

Howson, W. P. and Jemah, A. K., 1999, "Exact Out-of-plane Natural Frequencies of Curved Timoshenko Beam," *Journal of Engineering Mechanics*, Vol. 125, No. 1, pp. 19~25.

Howson, W. P., Jemah, A. K. and Zhou, J. Q., 1995, "Exact Natural Frequencies for Out-of-plane Motion of Plane Structures Composed of Curved Beam Members," *Computers & Structures*, Vol. 55, No. 6, pp. 989~995.

Irie, T., Yamada, G. and Tanaka, K., 1982, "Natural Frequencies of Out-of-plane Vibration of Arcs," *Journal of Applied Mechanics, Transactions of ASME*, Vol. 49, No. 4, pp. 910~913.

Irie, T., Yamada, G. and Tanaka, K., 1983, "Natural Frequencies of In-plane Vibration of Arcs," *Journal of Applied Mechanics, Transactions of ASME*, Vol. 50, No. 2, pp. 449~452.

Issa, M. S., Wang, T. M. and Hsiao, B. T., 1987, "Extensional Vibrations of Continuous Circular Curved Beams with Rotary Inertia and Shear Deformation I: Free Vibration," *Journal of Sound and Vibration*, Vol. 114, No. 2, pp. 297~308.

Kang, K., Bert, C. W. and Striz, A. G., 1995, "Vibration Analysis of Shear Deformable Circular Arches by the Differential Quadrature Method," *Journal of Sound and Vibration*, Vol. 183, No. 2, pp. 353~360.

Laura, P. A. A. and Maurizi, M. J., 1987, "Recent Research on Vibrations of Arch-type Structures," *The Shock and Vibration Digest*, Vol. 19, No. 1, pp. 6~9.

Lee, P. -G. and Sin, H. -C., 1994, "Locking Free Curved Beam Element Based on Curvature," *International Journal for Numerical Methods in Engineering*, Vol. 37, No. 6, pp. 989~1007.

Lee, J., 2003, "Eigenvalue Analysis of Rectangular Mindlin Plates by Chebyshev Pseudospectral Method," *KSME International Journal*, Vol. 17, No. 3, pp. 370~379.

Lee, J. and Schultz, W. W., "Eigenvalue Analysis of Timoshenko Beams and Axisymmetric Mindlin Plates by Pseudospectral Method," *to be published by Journal of Sound and Vibration.*

Markus, S. and Nanasi, T., 1981, "Vibration of Curved Beams," *The Shock and Vibration Digest*, Vol. 13, No. 4, pp. 3~14.

Mikami, T. and Yoshimura, J., 1984, "Application of the Collocation Method to Vibration Analysis of Rectangular Plates," *Computers & Structures*, Vol. 18, No. 3, pp. 425~432.

Prathap, G. and Babu, C. R., 1986, "An Isoparametric Quadratic Thick Curved Beam Element," *International Journal for Numerical Methods in Engineering*, Vol. 23, No. 9, pp. 1583~1600.

Soni, S. R. and Amba-Rao, C. L., 1975, "On Radially Symmetric Vibrations of Orthotropic Non-uniform Disks Including Shear Deformation," *Journal of Sound and Vibration*, Vol. 42, No. 1, pp. 57~63.

Tseng, Y. -P., Huang, C. S. and Lin, C. -J., 1997, "Dynamic Stiffness Analysis for In-plane Vibrations of Arches with Variable Curvature," *Journal of Sound and Vibration*, Vol. 207, No. 1, pp. 15~31.

Yildrim, V., 1997, "A Computer Program for the free Vibration Analysis of Elastic Arcs," *Computers & Structures*, Vol. 62, No. 3, pp. 475~485.

Yang, S. -Y. and Sin, H. -C., 1995, "Curvature Base Beam Element for the Analysis of Timoshenko and Shear-Deformable Beam," *Journal of Sound and Vibration*, Vol. 187, No. 4, pp. 569~584.

Appendix

Calculation of Constants for Basis Function $D_k(\xi)$

1. The clamped-clamped boundary condition for the Timoshenko beam is given by

$$u=0,\ v=0,\ \psi=0 \text{ at } \xi=\pm 1 \tag{A1}$$

$u=0$ and $v=0$ at $\xi=\pm 1$ are satisfied by the condition given in Eq. (10), and the remaining condition $\psi=0$ at $\xi=\pm 1$ can be satisfied simply by choosing

$$\begin{aligned} D_{2n-1}(\xi) &= T_{2n}(\xi) - T_0(\xi) \\ D_{2n}(\xi) &= T_{2n+1}(\xi) - T_1(\xi) \\ &(n=1,\ 2,\ \cdots) \end{aligned} \tag{A2}$$

which makes $a_1=a_2=a_3=a_4=0$.

2. The hinged-hinged boundary condition is

$$\begin{cases} u=0,\ v=0,\ M=0 \text{ at } \xi=-1 \\ u=0,\ v=0,\ M=0 \text{ at } \xi=1 \end{cases} \tag{A3}$$

$u=0$ and $v=0$ at $\xi=\pm 1$ are satisfied by the condition given in Eq. (10), and the remaining condition is

$$M\,|_{\xi=\pm1}=\frac{EI}{R}\frac{d\psi}{d\theta}\bigg|_{\xi=\pm1}=\frac{2EI}{R\Theta}\frac{d\psi}{d\xi}\bigg|_{\xi=\pm1}=0 \quad \text{(A4)}$$

Using the relationship (8), it is worthwhile to note that

$$\frac{dD_k}{d\xi}\bigg|_{\xi=\pm1}=0 \quad (k=1,\,2,\,\cdots,\,K) \qquad \text{(A5)}$$

is a sufficient condition for the zero-moment condition (A4). Having the differentiation of the odd numbered terms of $D_k(\xi)$ with respect to ξ equal to zero makes

$$\frac{dD_{2n-1}}{d\xi}\bigg|_{\xi=\pm1}=\left(\frac{dT_{2n}}{d\xi}+2a_1\xi+a_2\right)\bigg|_{\xi=\pm1}=0 \quad \text{(A6)}$$
$$(n=1,\,2,\,\cdots)$$

Eq. (A6) is rewritten as

$$\begin{cases} -4n^2-2a_1+a_2=0 \text{ at } \xi=-1 \\ 4n^2+2a_1+a_2=0 \text{ at } \xi=1 \end{cases} \qquad \text{(A7)}$$

and we have

$$a_1=-2n^2,\ a_2=0 \qquad \text{(A8)}$$

The differentiation of the even numbered terms with respect to ξ makes

$$\frac{dD_{2n}}{d\xi}\bigg|_{\xi=\pm1}=\left(\frac{dT_{2n+1}}{d\xi}-1+2a_3\xi+a_4\right)\bigg|_{\xi=\pm1}=0 \quad \text{(A9)}$$

Eq. (A9) is also rewritten as

$$\begin{cases} (2n+1)^2-1-2a_3+a_4=0 \text{ at } \xi=-1 \\ (2n+1)^2-1+2a_3+a_4=0 \text{ at } \xi=1 \end{cases} \qquad \text{(A10)}$$

from which the constants and a_3 are a_4 found to be

$$a_3=0,\ a_4=-4n(n+1) \qquad \text{(A11)}$$

3. The clamped-hinged boundary condition is given by

$$\begin{cases} u=0,\ v=0,\ \psi=0 \text{ at } \xi=-1 \\ u=0,\ v=0,\ M=0 \text{ at } \xi=1 \end{cases} \qquad \text{(A12)}$$

$u=0$ and $v=0$ at $\xi=\pm1$ are satisfied by the condition given in Eq. (10), and the remaining condition is satisfied by the introduction

$$\begin{cases} D_k=0 \quad \text{at } \xi=-1 \\ \dfrac{dD_k}{d\xi}=0 \text{ at } \xi=1 \end{cases} \qquad \text{(A13)}$$

Using the relationships of Eq. (11), the condition for the odd numbered terms is given by

$$\begin{cases} D_{2n-1}|_{\xi=-1}=(T_{2n}-T_0+a_1\xi^2+a_2\xi)|_{\xi=-1}=a_1-a_2=0 \\ \dfrac{dD_{2n-1}}{d\xi}\bigg|_{\xi=1}=\left(\dfrac{dT_{2n}}{d\xi}+2a_1\xi+a_2\right)\bigg|_{\xi=1}=4n^2+2a_1+a_2=0 \end{cases} \qquad \text{(A14)}$$

from which we have

$$a_1=a_2=-\frac{4n^2}{3} \qquad \text{(A15)}$$

For the even numbered terms

$$\begin{cases} D_{2n}|_{\xi=-1}=(T_{2n+1}-T_1+a_3\xi^2+a_4\xi)|_{\xi=-1}=a_3-a_4=0 \\ \dfrac{dD_{2n}}{d\xi}\bigg|_{\xi=1}=\left(\dfrac{dT_{2n+1}}{d\xi}-1+2a_3\xi+a_4\right)\bigg|_{\xi=1}=(2n+1)^2-1+2a_3+a_4=0 \end{cases} \qquad \text{(A16)}$$

from which we have

$$a_3=a_4=-\frac{4n(n+1)}{3} \qquad \text{(A17)}$$