

CONFORMAL DEFORMATION ON A SEMI-RIEMANNIAN MANIFOLD (II)

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ABSTRACT. In this paper, when N is a compact Riemannian manifold, we considered the positive time solution to equation $\square_g u(t, x) - c_n u(t, x) + c_n u(t, x)^{(n+3)/(n-1)}$ on $M = (-\infty, +\infty) \times_f N$, where $c_n = (n-1)/4n$ and \square_g is the d'Alembertian for a Lorentzian warped manifold.

1. INTRODUCTION

In a recent study, Leung [5, 6] has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of a positive solution to equation

$$(1.1) \quad \Delta_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}},$$

where Δ_{g_0} is the Laplacian operator for an n -dimensional Riemannian manifold (N, g_0) and $d_n = \frac{n-2}{4(n-1)}$. Equation (1.1) is derived from the conformal deformation of Riemannian metric (*cf.* Aviles & McOwen [1]; Kazdan & Warner [4]).

Similarly, let (N, g_0) be a compact Riemannian n -dimensional manifold with constant scalar curvature. We consider the $(n+1)$ -dimensional Lorentzian warped manifold $M = (-\infty, \infty) \times_f N$ with the metric $g = -dt^2 + f(t)^2 g_0$, where f is a positive function on $[a, \infty)$. Let $u(t, x)$ be a positive smooth function on M and let g have a constant scalar curvature equal to $+1$.

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If the conformal metric $g_c = u(t, x)^{4/(n-1)}g$ also has constant scalar curvature equal to $+1$, then $u(t, x)$ satisfies equation

$$(1.2) \quad \square_g u(t, x) - c_n u(t, x) + c_n u(t, x)^{\frac{n+3}{n-1}} = 0,$$

where $c_n = \frac{n-1}{4n}$ and \square_g is the d'Alembertian for a Lorentzian warped manifold $M = (-\infty, \infty) \times_f N$.

In this paper, we study the uniqueness of positive solution to equation (1.2). Leung [5, 6] considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in Ehrlich, Jung & Kim [3], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature. Indeed, in Ehrlich, Jung & Kim [3], authors proved that when the fiber manifold has the constant scalar curvature, then there is no obstruction of the existence of Lorentzian warped metric with constant scalar equal to $+1$. So we may assume that the Lorentzian warped product metric g has the constant scalar equal to $+1$.

2. MAIN RESULTS

In this section, we let (N, g_0) be a compact Riemannian n -dimensional manifold with $n \geq 3$ and without boundary. Then Beem, Ehrlich & Powell [2, Theorem 5.4] implies the following proposition.

Proposition 1. *Let $M = (-\infty, \infty) \times_f N$ have a Lorentzian warped product metric*

$$g = -dt^2 + f(t)^2 g_0.$$

Then the d'Alembertian \square_g is given by

$$\square_g = -\frac{\partial^2}{\partial t^2} - \frac{nf'(t)}{f(t)} \frac{\partial}{\partial t} + \frac{1}{f(t)^2} \Delta_x,$$

where Δ_x is the Laplacian on fiber manifold N .

By Proposition 1, Equation (1.2) is changed into the following equation

$$(2.1) \quad \frac{\partial^2 u(t, x)}{\partial t^2} + \frac{nf'(t)}{f(t)} \frac{\partial u(t, x)}{\partial t} - \frac{1}{f(t)^2} \Delta_x u(t, x) + c_n u(t, x) - c_n u(t, x)^{\frac{n+3}{n-1}} = 0.$$

A positive solution to Equation (1.2) or Equation (2.1) is said to be *nonspacelike* (*timelike* or *null*) *complete* if the conformal metric $g_c = u^{4/(n-1)}g$ is a nonspacelike (timelike or null) complete Lorentzian metric on M .

In this section, we discuss whether the nonspacelike complete positive solution of Equation (2.1) is unique. By Powell [8, Lemma 2 and Theorem 5] and Beem, Ehrlich & Powell [2, Theorem 4.1], we have the following proposition.

Proposition 2. *Let $M = (-\infty, \infty) \times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2g_0$. Then all future directed timelike (resp. null) geodesics are future complete if and only if for some $t_0 \in (-\infty, \infty)$,*

$$\int_{t_0}^{\infty} \frac{f(t)}{\sqrt{1+f(t)^2}} dt = \infty \quad (\text{resp. } \int_{t_0}^{\infty} f(t) dt = \infty).$$

Similarly, all past directed timelike (resp. null) geodesics are past complete if and only if for some $t_0 \in (-\infty, \infty)$,

$$\int_{-\infty}^{t_0} \frac{f(t)}{\sqrt{1+f(t)^2}} dt = \infty \quad (\text{resp. } \int_{-\infty}^{t_0} f(t) dt = \infty).$$

If $u(t, x)$ is a positive function with only time- variable t , then Equation (2.1) becomes

$$(2.2) \quad u''(t) + \frac{nf'(t)}{f(t)}u'(t) = c_n \left(u^{\frac{n+3}{n-1}}(t) - u(t) \right).$$

Lemma 3. *Let $u(t)$ be a solution of Equation (2.2) and $u(a) = 1$ for some $a \in (-\infty, \infty)$. We have four cases:*

i) *If there exists $t_1 > a$ such that $u(t_1) \geq 1$ and $u'(t_1) > 0$, then*

$$u'(t) > 0 \text{ for all } t > t_1.$$

ii) *If there exists $t_2 < a$ such that $u(t_2) \geq 1$ and $u'(t_2) < 0$, then*

$$u'(t) < 0 \text{ for all } t < t_2.$$

iii) *If there is a point t_3 such that $u(t_3) < 1$ and $u'(t_3) < 0$, then*

$$u'(t) < 0 \text{ for all } t > t_3.$$

iv) *If there exists $t_4 < a$ such that $u(t_4) < 1$ and $u'(t_4) > 0$, then*

$$u'(t) > 0 \text{ for all } t < t_4.$$

Proof. For case i), suppose not. Then there exists a point $s_1 > a$ such that $u(s_1) > 1$, $u'(s_1) = 0$ and $u''(s_1) \leq 0$, but Equation (2.2) shows that this is not possible. For the other cases, they are similar with the case i). \square

The proof of following theorem is similar with that of Leung [7, Theorem 4.9].

Theorem 4. *Let $u(t)$ be a positive solution of Equation (2.2). Assume that there exist positive constants t_0 and C_0 such that*

$$\left| \frac{f'(t)}{f(t)} \right| \leq C_0 \text{ for all } |t| > t_0.$$

Then $u(t)$ is bounded from above.

Proof. From Equation (2.2) we have

$$(2.3) \quad \frac{(f^n u')'}{f^n} = c_n \left(u^{\frac{n+3}{n-1}} - u \right).$$

Let $\chi \in C_0^\infty((-\infty, \infty))$ be a cut-off function. Multiplying both sides of Equation (2.3) by $\chi^{n+1}u$ and then using integration by parts we obtain

$$(2.4) \quad c_n \int_{-\infty}^{\infty} \chi^{n+1} u^2 dt - \int_{-\infty}^{\infty} (f^n u') \left(\frac{\chi^{n+1} u}{f^n} \right)' dt = c_n \int_{-\infty}^{\infty} \chi^{n+1} u^{\frac{2n+2}{n-1}} dt.$$

We have

$$-(f^n u') \left(\frac{\chi^{n+1} u}{f^n} \right)' = -(n+1) \chi^n u \chi' u' - \chi^{n+1} |u'|^2 + n \chi^{n+1} u u' \frac{f'}{f}.$$

Applying the Cauchy inequality we get

$$\begin{aligned} -(n+1) \chi^n u \chi' u' &= -2 \left(\frac{n+1}{\sqrt{2}} \chi^{\frac{n+1}{2}-1} u \chi' \right) \left(\frac{1}{\sqrt{2}} \chi^{\frac{n+1}{2}} u' \right) \\ &\leq \frac{(n+1)^2}{2} \chi^{n-1} u^2 |\chi'|^2 + \frac{1}{2} \chi^{n+1} |u'|^2 \end{aligned}$$

and

$$\begin{aligned} n \chi^{n+1} u u' \frac{f'}{f} &= 2 \left(\frac{n}{\sqrt{2}} \chi^{\frac{n+1}{2}} u \frac{f'}{f} \right) \left(\frac{1}{\sqrt{2}} \chi^{\frac{n+1}{2}} u' \right) \\ &\leq \frac{n^2}{2} \chi^{n+1} \left(\frac{f'}{f} \right)^2 u^2 + \frac{1}{2} \chi^{n+1} |u'|^2 \end{aligned}$$

Together with Equation (2.4), we obtain

$$\begin{aligned} c_n \int_{-\infty}^{\infty} \chi^{n+1} u^2 dt + \frac{n^2}{2} \int_{-\infty}^{\infty} \left(\frac{f'}{f} \right)^2 \chi^{n+1} u^2 dt + \frac{(n+1)^2}{2} \int_{-\infty}^{\infty} \chi^{n-1} u^2 |\chi'|^2 dt \\ \geq c_n \int_{-\infty}^{\infty} \chi^{n+1} u^{\frac{2n+2}{n-1}} dt. \end{aligned}$$

Applying Young's inequality and using the bound $\left|\frac{f'}{f}\right| \leq C_0$, we have

$$(2.5) \quad c_n \int_{-\infty}^{\infty} \chi^{n+1} u^{\frac{2n+2}{n-1}} dt \leq C' \int_{-\infty}^{\infty} (|\chi'|^{n+1} + \chi^{n+1}) dt,$$

where C' is a positive constant. Let $\chi \geq 0$ on $(-\infty, \infty)$,

$$\chi \equiv \begin{cases} 0 & \text{on } (-\infty, -r-3] \cup [r+3, \infty), \\ 1 & \text{on } [-r-2, -r-1] \cup [r+1, r+2] \end{cases}$$

with $r > t_0$, and $|\chi'| \leq \frac{1}{2}$. From Equation (2.5) we have

$$\int_{-r-2}^{-r-1} u^{\frac{2n+2}{n-1}} dt + \int_{r+1}^{r+2} u^{\frac{2n+2}{n-1}} dt \leq C''$$

for all $r > t_0$, where C'' is a constant independent on r . Therefore u is bounded from above. \square

Theorem 5. *Let (M, g) be a complete Lorentzian manifold with scalar curvature equal to 1. Let u be a positive smooth solution to Equation (1.2) on (M, g) with $g = -dt^2 + f^2(t)dx^2$ such that the conformal metric $g_c = u^{4/(n-1)}g$ is a (future and past) complete Lorentzian metric with scalar curvature equal to 1. Assume that $\lim_{t \rightarrow \pm\infty} f(t) = \infty$ and there exist positive constants t_0 and C_0 such that*

$$\left| \frac{f'(t)}{f(t)} \right| \leq C_0 \text{ for all } |t| > t_0.$$

If $u(t, x)$ is a positive function with only time-variable t , then $u(t) \equiv 1$ on M .

Proof. If $u = u(t)$ is a solution of Equation (2.2), then, by Theorem 4, $u(t)$ is bounded from above. Suppose that there exists a point $t_1 \in (-\infty, \infty)$ such that $u(t_1) > 1$. Then, by Omori-Yau maximum principle (cf. Ratto, Rigoli & Setti [9]), there exists a sequence $\{t_k\}$ such that

$$\lim_{k \rightarrow \infty} u(t_k) = \sup_{t \in (-\infty, \infty)} u(t), \quad |u'(t_k)| \leq \frac{1}{k} \text{ and } u''(t_k) \leq \frac{1}{k}.$$

Since $\sup_{t \in (-\infty, \infty)} u(t) > 1$, there exist a number $\varepsilon > 0$ and K such that

$$c_n \left(u(t_k)^{\frac{n+3}{n-1}} - u(t_k) \right) > \varepsilon \text{ for all } k > K.$$

This is a contradiction to Equation (2.2), so $u(t) \leq 1$.

Suppose that there exists a point $t_2 \in (-\infty, \infty)$ such that $u(t_2) < 1$. Assume that $u(t) \geq c$ for all $|t| > t_0$, where $c \in (0, 1)$ is a constant. Then we can find a sequence t'_k and a positive constant $\delta > 0$ such that $u(t'_k) > 1 - \delta$ for all k , $\lim_{k \rightarrow \infty} u'(t_k) = 0$ and $u''(t_k) \geq 0$. But this contradicts Equation (2.2). Therefore $\lim_{t \rightarrow \pm\infty} u(t) = 0$

and we must have $u'(t_3) < 0$ and $u'(t_4) > 0$ for positive large t_3 and negative large t_4 . The proof of Lemma 3 implies that we have $u'(t) < 0$ for all $t > t_3$ and $u'(t) > 0$ for all $t < t_4$. We consider two cases: Future case and past case.

i) (Future case) There exist positive constants $t' > t_3$ and $C > 0$ such that for $t \geq t'$ we have

$$(f^n u')'(t) = c_n f^n(t) \left(u^{\frac{n+3}{n-1}}(t) - u(t) \right) \leq -C f^n(t) u(t).$$

Integrating from t' to $t > t'$ we have

$$f^n(t) u'(t) \leq f^n(t') u'(t') - C \int_{t'}^t f^n(s) u(s) ds \leq -C u(t) \int_{t'}^t f^n(s) ds,$$

as $u' \leq 0$. Therefore

$$(2.6) \quad \frac{u'(t)}{u(t)} \leq -C \frac{\int_{t'}^t f^n(s) ds}{f^n(t)}.$$

Using the bound $\frac{f'}{f} < C_0$ we have $(f^n)' \leq C_0 n f^n$. An integration gives

$$f^n(t) - f^n(t') \leq C_0 n \int_{t'}^t f^n(s) ds.$$

As $\lim_{t \rightarrow \infty} f(t) = \infty$, if t is large we have $\frac{1}{2} f^n(t) \leq C_0 \int_{t'}^t f^n(s) ds$, that is,

$$(2.7) \quad \frac{\int_{t'}^t f^n(s) ds}{f^n(t)} \geq c'$$

for all t large and for some positive constant c' . Equations (2.6) and (2.7) give $u(t) \leq \tilde{C} e^{-ct}$ for all t large enough, where \tilde{C} is a positive constant. Thus the conformal metric $g_c = u^{4/(n-1)} g$ cannot be future complete. It is a contradiction.

ii) (Past case) There exist positive constants $t'' < t_4$ and $C' > 0$ such that for $t \leq t''$ we have

$$(f^n u')'(t) = c_n f^n(t) \left(u^{\frac{n+3}{n-1}}(t) - u(t) \right) \leq -C' f^n(t) u(t).$$

Integrating from $t (< t'')$ to t'' we have

$$-f^n(t) u'(t) \leq -f^n(t'') u'(t'') - C' \int_t^{t''} f^n(s) u(s) ds \leq -C' u(t) \int_t^{t''} f^n(s) ds,$$

as $u' \geq 0$. Therefore

$$(2.8) \quad \frac{u'(t)}{u(t)} \geq C' \frac{\int_t^{t''} f^n(s) ds}{f^n(t)}.$$

Using the bound $\frac{f'}{f} > -C_0$ we have $(f^n)' \geq -C_0 n f^n$. Integrating from t ($< t''$) to t'' gives

$$f^n(t'') - f^n(t) \geq -C_0 n \int_t^{t''} f^n(s) ds.$$

As $\lim_{t \rightarrow -\infty} f(t) = \infty$, if t is negative large, we have $-\frac{1}{2} f^n(t) \geq -C_0 \int_t^{t''} f^n(s) ds$, that is,

$$(2.9) \quad \frac{\int_t^{t''} f^n(s) ds}{f^n(t)} \leq c_1$$

for all t negative large and for some positive constant c_1 . Equations (2.8) and (2.9) give $u(t) \leq C_2 e^{c_1 t}$ for all t negative large enough, where C_2 is a positive constant. Thus the conformal metric $g_c = u^{4/(n-1)} g$ cannot be past complete. It is a contradiction.

Hence $u \equiv 1$ on $(-\infty, \infty)$. □

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