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FRENET EQUATIONS OF NULL CURVES

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ABSTRACT. The purpose of this paper is to study the geometry of null curves in a 6-dimensional semi-Riemannian manifold $\mathbf{M}_q$ of index q, since the general n-dimensional cases are too complicated. We show that it is possible to construct three types of Frenet equations of null curves in $\mathbf{M}_q$, supported by one example. We find each types of Frenet equations invariant under any causal change. And we discuss some properties of null curves in $\mathbf{M}_q$.

1. INTRODUCTION

Theory of space curves of a Riemannian manifold is fully developed and its local and global geometry is well-known. But its counter part of the curve theory of a semi-Riemannian manifold is relatively new and in a developing stage. In case of semi-Riemannian manifolds, there are three categories of curves, namely, spacelike, timelike and null, depending on their causal character. We know from O'Neill [10], that the study of timelike curves has many similarities with the spacelike curves. However, since the induced metric of a null curve is degenerate, this case is much more complicated and also different from the non-degenerate case.

Duggal & Bejancu [4, Chapter 3] published their work on *"general theory of null curves in Lorentz manifolds"*. They constructed a Frenet frame and proved the fundamental existence and uniqueness theorem for this class of null curves. Their study was restricted to Lorentz manifold, since for the general semi-Riemannian manifolds of index greater than one, they have shown (by an example) that their Frenet frame is not invariant with respect to causal change of any of its generating vector fields.

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The objective of this paper is to study on null curves in a 6-dimensional semi-Riemannian manifold $\mathbf{M}_q$ of index q, since the general n-dimensional cases are too complicated. And we guess their Frenet equations from the 6-dimensional cases. We show that it is possible to construct three types of Frenet frames suitable for $\mathbf{M}_q$ and cite one example, each invariant under any causal change. This is then followed by constructing general Frenet equations (called *compound Frenet equations*) which include all the possible forms of the three types. And we study some properties of null curves in $\mathbf{M}_q$.

2. TRANSVERSAL VECTOR BUNDLES

Let $\mathbf{M}_q$ be a real 6-dimensional semi-Riemannian manifold of constant index q $(1 \leq q \leq 3)$ and C be a smooth null curve in $\mathbf{M}_q$ locally given by

$$x^i = x^i(t), \qquad t \in \mathbf{I} \subset \mathbb{R}, \quad i \in \{0, 1, \dots, 5\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Since C is a null curve, the tangent vector field $\frac{d}{dt} = \lambda$ on $\mathcal{U}$ satisfies $g(\lambda, \lambda) = 0$. Denote by TC the tangent bundle of C and $TC^{\perp}$ the TC-perp. Clearly, $TC^{\perp}$ is a vector bundle over C of rank 5 and TC is a vector subbundle of $TC^{\perp}$ of rank 1 (*cf.* Duggal & Bejancu [4] and O'Neill [10]). This implies that $TC^{\perp}$ is not complementary to TC in $T\mathbf{M}_q|_C$. Thus we must find complementary vector bundle to TC in $T\mathbf{M}_q$ which will play the role of the normal bundle $TC^{\perp}$ consistent with the classical non-degenerate theory. A few researchers have done research on this matter dealing with only specified problems (*cf.* Bonnor [2], Duggal [3], Graves [6], Ikawa [8]). Duggal & Bejancu [4] developed a general mathematical theory to deal with the null case, which we brief as follows:

Suppose $S(TC^{\perp})$ denotes the complementary vector subbundle to TC in $TC^{\perp}$, *i. e.*, we have

$$TC^{\perp} = TC \perp S(TC^{\perp}),$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^{\perp})$ is a non-degenerate 4-dimensional vector subbundle of $T\mathbf{M}_q$. We call $S(TC^{\perp})$ a *screen vector bundle* of C, which being non-degenerate, we have

$$T\mathbf{M}_q|_C = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp}, \tag{1}$$

where $S(TC^{\perp})^{\perp}$ is a 2-dimensional complementary orthogonal vector subbundle to $S(TC^{\perp})$ in $T\mathbf{M}_q|_C$.

Throughout this paper we denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle.

Theorem 2.1 (Duggal & Bejancu [4]). *Let C be a null curve on a semi-Riemannian manifold $\mathbf{M}_q$ and $S(TC^{\perp})$ a screen vector bundle of C. Then there exists a unique vector bundle $\mathrm{ntr}(C)$ over C of rank 1, such that on each coordinate neighborhood $\mathcal{U} \subset C$ there is a unique section $N \in \Gamma(\mathrm{ntr}(C)|_{\mathcal{U}})$ satisfying*

$$g(\lambda, N) = 1, \quad g(N, N) = g(N, X) = 0 \tag{2}$$

for every $X \in \Gamma(S(TC^{\perp})|_{\mathcal{U}})$.

We call $\mathrm{ntr}(C)$ the *null transversal bundle* of C with respect to $S(TC^{\perp})$. Next consider the vector bundle

$$\mathrm{tr}(C) = \mathrm{ntr}(C) \perp S(TC^{\perp}),$$

which according to (1) and (2) is complementary but not orthogonal to TC in $T\mathbf{M}_q|_C$. More precisely, we have

$$T\mathbf{M}_q|_C = TC \oplus \mathrm{tr}(C) = (TC \oplus \mathrm{ntr}(C)) \perp S(TC^{\perp}). \tag{3}$$

We call $\mathrm{tr}(C)$ the *transversal vector bundle* of C with respect to $S(TC^{\perp})$. The vector field N in Theorem 2.1 is called the *null transversal vector field* of C with respect to λ. As $\{\lambda, N\}$ is a null basis of $\Gamma((TC \oplus \mathrm{ntr}(C))|_{\mathcal{U}})$ satisfying (2), we obtain

Proposition 2.1 (Duggal & Bejancu [4]). *Let C be a null curve on a semi-Riemannian manifold $\mathbf{M}_q$. Then any screen vector bundle of C is semi-Riemannian of index $q-1$.*

3. Frenet equations of Type 1

Let C be a null curve on $\mathbf{M}_3$ and N be the null transversal vector field of C. Denote ∇ the Levi-Civita connection on $\mathbf{M}_3$. In this section we study a class of null curves C whose Frenet frame is made up of two null vector fields λ and N, two timelike and two spacelike vector fields. We denote the Frenet equations of this particular class of C by Type 1.

From $g(\lambda, \lambda) = 0$ and $g(\lambda, N) = 1$ we have

$$g(\nabla_\lambda \lambda, \lambda) = 0 \quad \text{and} \quad g(\nabla_\lambda \lambda, N) = -g(\lambda, \nabla_\lambda N) = h,$$

where h is a smooth function on $\mathcal{U}$. These relations and the equation (3) imply that

$$\nabla_\lambda \lambda = h\lambda + S_1,$$

where $S_1 \in S(TC^\perp)$. Thus S_1 is everywhere perpendicular to both λ and N. Since $S(TC^\perp)$ is a semi-Riemannian vector bundle of rank 4 and index 2, in general there are three cases (timelike, spacelike and null) by the causality of the vector field S_1.

In this section we assume that S_1 is non-null. Based on this restriction, we define the first curvature function κ_1 by $\kappa_1 = \sigma_1\varepsilon_1$, where

$$\sigma_1 = \|S_1\|$$

and $\varepsilon_1 = 1$ or -1 according as S_1 is spacelike or timelike, *i. e.*, ε_1 is the signature of S_1. Now we set $U_1 = S_1/\sigma_1$ so that U_1 is a unit vector field along C which is everywhere perpendicular to λ and N. Thus using above, we have

$$\nabla_\lambda \lambda = h\lambda + \kappa_1\varepsilon_1 U_1.$$

Now, from $g(\nabla_\lambda N, \lambda) = -h$, $g(\nabla_\lambda N, N) = 0$ and $g(\nabla_\lambda N, U_1) = \kappa_2$, where κ_2 denotes the second curvature function. We have

$$\nabla_\lambda N = -hN + \kappa_2\varepsilon_1 U_1 + S_2,$$

where S_2 is a vector field on $S(TC^\perp)$. Thus S_3 is perpendicular to λ, N and U_1. We assume that S_2 is also non-null. Define the third curvature function κ_3 by $\kappa_3 = \sigma_2\varepsilon_2$, where

$$\sigma_2 = \|T_2\|$$

and ε_2 is the signature of S_2. Set $U_2 = S_2/\sigma_2$ so that U_2 is also a unit vector field along C and is everywhere parallel to S_2. Thus we have

$$\nabla_\lambda N = -hN + \kappa_2\varepsilon_1 U_1 + \kappa_3\varepsilon_2 U_2.$$

Repeating above process we have an orthonormal basis $\{U_1, U_2, U_3, U_4\}$ of $S(TC^\perp)$ which is made up of two timelikes and two spacelikes. Setting

$$W_i = \varepsilon_i U_i, \qquad i \in \{1, 2, 3, 4\},$$

we obtain the following equations

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3, \\ \varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\ \varepsilon_3 \nabla_\lambda W_3 &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\ \varepsilon_4 \nabla_\lambda W_4 &= -\kappa_7 W_2 - \kappa_8 W_3, \end{cases} \tag{4}$$

where h and $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ are smooth functions on $\mathcal{U}$, $\{W_1, W_2, W_3, W_4\}$ is a certain orthonormal basis of $\Gamma(S(TC^\perp)|_{\mathcal{U}})$ and $\varepsilon_i = g(W_i, W_i)$ is the signature of each W_i, such that $\varepsilon_i = +1$ or -1. We call

$$F_1 = \{\lambda, N, W_1, \ldots, W_4\} \tag{5}$$

a *Frenet frame of Type 1 on* $\mathbf{M}_3$ along C with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (4) are called its *Frenet equations of Type 1.* Finally, the functions $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ are called *curvature functions* of C with respect to the frame F_1.

Remark 1. Since the screen bundle is semi-Riemannian of index 2, this implies that two of W_i's are timelikes and another two are spacelikes. We know that the choice of different timelikes W_i generates different Frenet equations of the same type.

4. Frenet equations of Type 2

In this section we study a class of null curves C whose Frenet frame is generated by a pseudo-orthonormal basis consisting of the two null vector fields λ and N, additional two null vector fields L_i and L_{i+1} such that $g(L_i, L_{i+1}) = 1$, one timelike vector field U_j and one spacelike vector field U_k, $\{j, k\} \neq \{i, i+1\}$. If we set

$$U_i = \frac{1}{\sqrt{2}}(L_i - L_{i+1}), \quad \text{and} \quad U_{i+1} = \frac{1}{\sqrt{2}}(L_i + L_{i+1}) \tag{6}$$

then U_i and U_{i+1} are timelike and spacelike vector fields, respectively, and $F = \{\lambda, N, U_1, \ldots, U_4\}$ is a Frenet frame of C, but have Frenet equations of another type. We denote the Frenet equations of this particular class of C by Type 2. There are three choices for $\{L_i, L_{i+1}\}$: $\{L_1, L_2\}$, $\{L_2, L_3\}$ and $\{L_3, L_4\}$.

To choose $\{L_1, L_2\}$, we let the vector field $\nabla_\lambda \lambda - h\lambda$ be null and define the curvature function K_1 by

$$\nabla_\lambda \lambda = h\lambda + K_1 L_1,$$

where $L_1 \in \Gamma(S(TC^\perp))$. Thus L_1 is a null vector field along C which is everywhere perpendicular to λ and N. Since $S(TC^\perp)$ is semi-Riemannian vector bundle of rank 4, we can take a vector field V along C such that $g(L_1, V) \neq 0$, otherwise $S(TC^\perp)$ is degenerate. Set

$$L_2 = \frac{1}{g(L_1, V)} \left\{ V - \frac{g(V, V)}{g(L_1, V)} L_1 \right\},$$

then $g(L_1, L_2) = 1$ along C. Set this case so that the equation (6) holds for $i = 1$. Therefore U_1 and U_2 are perpendicular to λ and N and we have

$$\nabla_\lambda \lambda = h\lambda + \kappa_1 \varepsilon_1 U_1 + \tau_1 \varepsilon_2 U_2$$

where $\kappa_1 = -\tau_1 = -\frac{K_1}{\sqrt{2}}$ and ε_i is the signature of each U_i. Also,

$$g(\nabla_\lambda N, \lambda) = -h, \quad g(\nabla_\lambda N, N) = 0, \quad g(\nabla_\lambda N, U_1) = \kappa_2, \quad g(\nabla_\lambda N, U_2) = \kappa_3$$

implies that

$$\nabla_\lambda N = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + S_3,$$

where S_3 is a vector field perpendicular to λ, N, U_1 and U_2. Since $S(TC^\perp)$ is a semi-Riemannian vector bundle of index 2, there are three cases by the causality of the vector field S_3. In this section we assume that S_3 is non-null. Now define a torsion function τ_3 by $\tau_3 = \sigma_3 \varepsilon_3$ where $\sigma_3 = \|S_3\|$ and ε_3 is the signature of S_3. Set $U_3 = S_3/\sigma_3$, then U_3 is a unit non-null vector field along C which is also perpendicular to λ, N, U_1 and U_2. Thus we obtain

$$\nabla_\lambda N = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + \tau_3 \varepsilon_3 U_3. \tag{7}$$

Also from the following results

$$\begin{aligned} g(\nabla_\lambda U_1, \lambda) &= -\kappa_1, \\ g(\nabla_\lambda U_1, N) &= -\kappa_2, \\ g(\nabla_\lambda U_1, U_1) &= 0, \\ g(\nabla_\lambda U_1, U_2) &= \kappa_4, \\ g(\nabla_\lambda U_1, U_3) &= \kappa_5, \end{aligned}$$

we obtain

$$\nabla_\lambda U_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 \varepsilon_2 U_2 + \kappa_5 \varepsilon_3 U_3 + S_4$$

where S_4 is a non-null vector field perpendicular to λ, N, U_1, U_2 and U_3. Now we define another torsion function τ_5 by $\tau_5 = \sigma_4 \varepsilon_4$ where $\sigma_4 = \|S_4\|$. Set $U_4 = S_4/\sigma_4$, then U_4 is also a unit non-null vector field along C which is everywhere perpendicular to λ, N, U_1, U_2 and U_3. Thus we obtain

$$\nabla_\lambda U_1 = -\kappa_2 \lambda - \kappa_1 N + \kappa_4 \varepsilon_2 U_2 + \kappa_5 \varepsilon_3 U_3 + \tau_5 \varepsilon_4 U_4. \tag{8}$$

In a similar way we get

$$\begin{cases} \nabla_\lambda U_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 \varepsilon_1 U_1 + \kappa_6 \varepsilon_3 U_3 + \kappa_7 \varepsilon_4 U_4, \\ \nabla_\lambda U_3 &= -\tau_3 \lambda - \kappa_5 \varepsilon_1 U_1 - \kappa_6 \varepsilon_2 U_2 + \kappa_8 \varepsilon_4 U_4, \\ \nabla_\lambda U_4 &= -\tau_5 \varepsilon_1 U_1 - \kappa_7 \varepsilon_2 U_2 - \kappa_8 \varepsilon_3 U_3, \end{cases} \tag{9}$$

where

$$\kappa_6 = g(\nabla_\lambda U_2, U_3), \quad \kappa_7 = g(\nabla_\lambda U_2, U_4), \quad \kappa_8 = g(\nabla_\lambda U_3, U_4).$$

Setting

$$W_i = \varepsilon_i U_i, \quad i \in \{1, 2, 3, 4\}$$

we have the following equations

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1 + \tau_1 W_2, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4, \\ \varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\ \varepsilon_3 \nabla_\lambda W_3 &= -\tau_3 \lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\ \varepsilon_4 \nabla_\lambda W_4 &= -\tau_5 W_1 - \kappa_7 W_2 - \kappa_8 W_3. \end{cases} \tag{10}$$

In the above case, we call

$$F_2^{(1)} = \{\lambda, N, W_1, W_2, W_3, W_4\} \tag{11}$$

a *Frenet frame of Type 2* on $\mathbf{M}_3$ along C with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (10) its *Frenet equations of Type 2*. The functions $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ and $\{\tau_1, \tau_3, \tau_5\}$ are called the *curvature functions* and the *torsion functions* of C with respect to the frame $F_2^{(1)}$.

On the other hand, using $\{L_1, L_2\}$ such that $g(L_1, L_2) = 1$ and

$$g(\nabla_\lambda N, \lambda) = -h, \quad g(\nabla_\lambda N, L_1) = K_3, \quad g(\nabla_\lambda N, L_2) = K_2,$$

we can write

$$\nabla_\lambda N = -hN + K_2 L_1 + K_3 L_2 + Q_3$$

where Q_3 is perpendicular to λ, N, L_1 and L_2. Also

$$\begin{aligned}&\nabla_\lambda N + hN - K_2 L_1 - K_3 L_2\\&= \nabla_\lambda N + hN - \frac{K_2}{\sqrt{2}}(U_2 + U_1) - \frac{K_3}{\sqrt{2}}(U_2 - U_1)\\&= \nabla_\lambda N + hN + \left(\frac{K_3 - K_2}{\sqrt{2}}\right) U_1 - \left(\frac{K_3 + K_2}{\sqrt{2}}\right) U_2\end{aligned}$$

and

$$\kappa_2 = g(\nabla_\lambda N, U_1) = \frac{1}{\sqrt{2}}\{g(\nabla_\lambda N, L_1 - L_2)\} = \frac{K_3 - K_2}{\sqrt{2}},$$
$$\kappa_3 = g(\nabla_\lambda N, U_2) = \frac{1}{\sqrt{2}}\{g(\nabla_\lambda N, L_1 + L_2)\} = \frac{K_2 + K_3}{\sqrt{2}}.$$

Using above results and the equation (7) we conclude that $Q_3 = \tau_3 \varepsilon_3 U_3$. Therefore

$$\nabla_\lambda N = -hN + K_2 L_1 + K_3 L_2 + \tau_3 \varepsilon_3 U_3. \tag{12}$$

In a similar way we obtain

$$\nabla_\lambda L_1 = -K_3 \lambda + K_4 L_1 + K_5 \varepsilon_3 U_3 + \overline{K}_6 \overline{\varepsilon}_4 \overline{U}_4, \tag{13}$$
$$\nabla_\lambda L_2 = -K_2 \lambda - K_1 N - K_4 L_2 + K_7 \varepsilon_3 U_3 + \overline{K}_8 \overline{\varepsilon}_4 \overline{U}_4, \tag{14}$$

where $\overline{U}_4$ is a unit non-null vector field perpendicular to λ, N, L_1, L_2 and U_3, and the smooth functions K_i $(i = 4, 5, 7)$ and $\overline{K}_j$ $(j = 6, 8)$ are defined by

$$\begin{aligned}K_4 &= g(\nabla_\lambda L_1, L_2) = -g(L_1, \nabla_\lambda L_2),\\K_5 &= g(\nabla_\lambda L_1, U_3) = -g(L_1, \nabla_\lambda U_3),\\\overline{K}_6 &= g(\nabla_\lambda L_1, \overline{U}_4) = -g(L_1, \nabla_\lambda \overline{U}_4),\\K_7 &= g(\nabla_\lambda L_2, U_3) = -g(L_2, \nabla_\lambda U_3),\\\overline{K}_8 &= g(\nabla_\lambda L_2, \overline{U}_4) = -g(L_2, \nabla_\lambda \overline{U}_4).\end{aligned}$$

Next, by the transformations (6) for $i = 1$ we have

$$\nabla_\lambda U_1 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_1 - \nabla_\lambda L_2), \quad \nabla_\lambda U_2 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_1 + \nabla_\lambda L_2).$$

Using (13) and (14) in above equations and the following results

$$\begin{aligned}\kappa_4 &= g(\nabla_\lambda U_1, U_2) = g(\nabla_\lambda L_1, L_2) = K_4,\\\kappa_5 &= g(\nabla_\lambda U_1, U_3) = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_1 - \nabla_\lambda L_2, U_3) = \frac{1}{\sqrt{2}}(K_5 - K_7),\\\kappa_6 &= g(\nabla_\lambda U_2, U_3) = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_1 + \nabla_\lambda L_2, U_3) = \frac{1}{\sqrt{2}}(K_5 + K_7),\end{aligned}$$

we obtain

$$\overline{K}_6\overline{\varepsilon}_4\overline{U}_4 = K_6\varepsilon_4 U_4, \qquad \overline{K}_8\overline{\varepsilon}_4\overline{U}_4 = K_8\varepsilon_4 U_4,$$

where

$$K_6 = \frac{\kappa_7 + \tau_5}{\sqrt{2}}, \qquad K_8 = \frac{\kappa_7 - \tau_5}{\sqrt{2}}.$$

Thus (13) and (14) become

$$\nabla_\lambda L_1 = -K_3\lambda + K_4 L_1 + K_5\varepsilon_3 U_3 + K_6\varepsilon_4 U_4,$$
$$\nabla_\lambda L_2 = -K_2\lambda - K_1 N - K_4 L_2 + K_7\varepsilon_3 U_3 + K_8\varepsilon_4 U_4.$$

In a similar way we get

$$\nabla_\lambda U_3 = -\tau_3\lambda - K_7 L_1 - K_5 L_2 + \kappa_8\varepsilon_4 U_4,$$
$$\nabla_\lambda U_4 = -K_8 L_1 - K_6 L_2 - \kappa_8\varepsilon_3 U_3.$$

Setting

$$W_i = \varepsilon_i U_i, \quad i \in \{3,4\},$$

we get the following equations

$$\begin{cases} \nabla_\lambda\lambda &= h\lambda + K_1 L_1, \\ \nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + \tau_3 W_3, \\ \nabla_\lambda L_1 &= -K_3\lambda + \kappa_4 L_1 + K_5 W_3 + K_6 W_4, \\ \nabla_\lambda L_2 &= -K_2\lambda - K_1 N - \kappa_4 L_2 + K_7 W_3 + K_8 W_4, \\ \varepsilon_3\nabla_\lambda W_3 &= -\tau_3\lambda - K_7 L_1 - K_5 L_2 + \kappa_8 W_4, \\ \varepsilon_4\nabla_\lambda W_4 &= -K_8 L_1 - K_6 L_2 - \kappa_8 W_3, \end{cases} \tag{15}$$

where

$$K_1 = -\sqrt{2}\kappa_1 = \sqrt{2}\tau_1, \quad K_2 = \frac{\kappa_3 - \kappa_2}{\sqrt{2}}, \quad K_3 = \frac{\kappa_3 + \kappa_2}{\sqrt{2}},$$
$$K_5 = \frac{\kappa_6 + \kappa_5}{\sqrt{2}}, \quad K_6 = \frac{\kappa_7 + \tau_5}{\sqrt{2}}, \quad K_7 = \frac{\kappa_6 - \kappa_5}{\sqrt{2}}, \quad K_8 = \frac{\kappa_7 - \tau_5}{\sqrt{2}}.$$

In this case, we also call

$$F_2^{(1)} = \{\lambda, N, L_1, L_2, W_3, W_4\} \tag{16}$$

a *Frenet frame of Type 2* on $\mathbf{M}_3$ along C with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (15) its *Frenet equations of Type 2*.

In the next cases, to choose $\{L_2, L_3\}$ and $\{L_3, L_4\}$, we let the vector fields

$$\nabla_\lambda N + hN - k_2 U_1 \quad \text{and} \quad \nabla_\lambda U_1 + k_2\lambda + k_1 N - k_4 U_2$$

be null in turn, then using a procedure same as above for each such cases, we obtain the equations of the form (10) with the torsion functions $\{\tau_1 = 0, \tau_3 = -\kappa_3, \tau_5\}$, or equivalently,

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + K_3 L_2, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + K_4 L_2 + K_5 L_3 + \tau_5 W_4, \\ \nabla_\lambda L_2 &= -K_5 W_1 + \kappa_6 L_2 + K_8 W_4, \\ \nabla_\lambda L_3 &= -K_3 \lambda - K_4 W_1 - \kappa_6 L_3 + K_9 W_4, \\ \varepsilon_4 \nabla_\lambda W_4 &= -\tau_5 W_1 - K_9 L_2 - K_8 L_3, \end{cases} \tag{17}$$

where

$$\frac{K_3}{\sqrt{2}} = -\kappa_3, \quad \frac{1}{\sqrt{2}}(K_4 - K_5) = -\kappa_4, \quad \frac{1}{\sqrt{2}}(K_4 + K_5) = \kappa_5,$$
$$\frac{K_3}{\sqrt{2}} = \tau_3, \quad \frac{1}{\sqrt{2}}(K_8 - K_9) = \kappa_7, \quad \frac{1}{\sqrt{2}}(K_8 + K_9) = \kappa_8,$$

and $\{\tau_1 = 0, \tau_3 = 0, \tau_5 = -\kappa_5\}$, or equivalently,

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + K_5 L_3, \\ \varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \kappa_4 W_1 + K_6 L_3 + K_7 L_4, \\ \nabla_\lambda L_3 &= -K_7 W_2 + \kappa_8 L_3, \\ \nabla_\lambda L_4 &= -K_5 W_1 - K_6 W_2 - \kappa_8 L_4, \end{cases} \tag{18}$$

where

$$\frac{K_5}{\sqrt{2}} = -\kappa_5 = \tau_5, \qquad \frac{1}{\sqrt{2}}(K_6 - K_7) = \kappa_6, \qquad \frac{1}{\sqrt{2}}(K_6 + K_7) = \kappa_7,$$

respectively. In the cases $\{L_2, L_3\}$ and $\{L_3, L_4\}$, we also call

$$F_2^{(2)} = \{\lambda, N, W_1, L_2, L_3, W_4\} \quad \text{and} \quad F_2^{(3)} = \{\lambda, N, W_1, W_2, L_3, L_4\},$$

respectively, *Frenet frames of Type 2* on $\mathbf{M}_3$ along C with respect to a given screen vector bundle $S(TC^\perp)$, and the equations (17) and (18) their *Frenet equations of Type 2*, respectively.

Remark 2. We know that the Frenet equations (10) include all three different Frenet equations of Type 2. Hence we call the equations (10) the *general Frenet equations* *of Type 2* of the null curve C and $F_2 = \{\lambda, N, W_1, \ldots, W_4\}$ the *general Frenet frame of Type 2 on* $\mathbf{M}_3$ along the null curve C.

5. Frenet equations of Type 3

In this section we study a class of null curves C whose Frenet frame is generated by a pseudo-orthonormal basis consisting of the two null vector fields λ and N and additional four null vector fields L_1, L_2, L_3 and L_4 such that $g(L_i, L_{i+1}) = 1$, $i = 1, 3$. If we set also

$$U_i = \frac{L_i - L_{i+1}}{\sqrt{2}}, \qquad U_{i+1} = \frac{L_i + L_{i+1}}{\sqrt{2}}, \quad i = 1, 3. \tag{19}$$

then $\{U_1, U_3\}$ and $\{U_2, U_4\}$ are timelike and spacelike vector fields respectively, and $F = \{\lambda, N, U_1, \ldots, U_4\}$ is also a Frenet frame of C, but have Frenet equations of the other type. We denote the Frenet equations of this particular class of C by Type 3. There is only one choice for $\{L_1, L_2, L_3, L_4\}$.

In the section 4 for Type 2, there exist two null vector fields $\{L_1, L_2\}$, or equivalently, one timelike vector field and one spacelike vector field $\{U_1, U_2\}$ satisfying

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + K_1 L_1 = h\lambda + \kappa_1 \varepsilon_1 U_1 + \tau_1 \varepsilon_2 U_2, \\ \nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + S_3 = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + S_3. \end{cases} \tag{20}$$

In this section we let the vector field S_3 be null and define the curvature function T_3 by

$$\nabla_\lambda N = -hN + K_2 L_1 + K_3 L_2 + T_3 L_3 = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + T_3 L_3,$$

where L_3 is a null vector field along C perpendicular to $\lambda, N, L_1, L_2, U_1$ and U_2. Then there exists another null vector field L_4 along C such that $g(L_3, L_4) = 1$ and is everywhere perpendicular to $\lambda, N, L_1, L_2, U_1$ and U_2. Set this case so that the equation (19) hold for $i = 3$. Therefore U_3 and U_4 are perpendicular to $\lambda, N, L_1, L_2, U_1$ and U_2 and we have

$$\nabla_\lambda N = -hN + \kappa_2 \varepsilon_1 U_1 + \kappa_3 \varepsilon_2 U_2 + \tau_3 \varepsilon_3 U_3 + \tau_2 \varepsilon_4 U_4, \tag{21}$$

where $\tau_2 = -\tau_3 = \frac{T_3}{\sqrt{2}}$. Also from the following results

$$\begin{aligned} g(\nabla_\lambda U_1, \lambda) &= -\kappa_1, & g(\nabla_\lambda U_1, N) &= -\kappa_2, & g(\nabla_\lambda U_1, U_1) &= 0, \\ g(\nabla_\lambda U_1, U_2) &= \kappa_4, & g(\nabla_\lambda U_1, U_3) &= \kappa_5, & g(\nabla_\lambda U_1, U_4) &= \tau_5, \end{aligned}$$

we obtain

$$\nabla_\lambda U_1 = -\kappa_2\lambda - \kappa_1 N + \kappa_4\varepsilon_2 U_2 + \kappa_5\varepsilon_3 U_3 + \tau_5\varepsilon_4 U_4. \tag{22}$$

In a similar way we get

$$\begin{cases} \nabla_\lambda U_2 &= -\kappa_3\lambda - \tau_1 N - \kappa_4\varepsilon_1 U_1 + \kappa_6\varepsilon_3 U_3 + \kappa_7\varepsilon_4 U_4, \\ \nabla_\lambda U_3 &= -\tau_3\lambda - \kappa_5\varepsilon_1 U_1 - \kappa_6\varepsilon_2 U_2 + \kappa_8\varepsilon_4 U_4, \\ \nabla_\lambda U_4 &= -\tau_2\lambda - \tau_5\varepsilon_1 U_1 - \kappa_7\varepsilon_2 U_2 - \kappa_8\varepsilon_3 U_3. \end{cases} \tag{23}$$

Setting

$$W_i = \varepsilon_i U_i, \quad i \in \{1, 2, 3, 4\}$$

we have the following equations

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1 + \tau_1 W_2, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2\lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4, \\ \varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3\lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\ \varepsilon_3 \nabla_\lambda W_3 &= -\tau_3\lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \\ \varepsilon_4 \nabla_\lambda W_4 &= -\tau_2\lambda - \tau_5 W_1 - \kappa_7 W_2 - \kappa_8 W_3. \end{cases} \tag{24}$$

In this case, we call

$$F_3 = \{\lambda, N, W_1, W_2, W_3, W_4\} \tag{25}$$

a *Frenet frame of Type 3* on $\mathbf{M}_3$ along C with respect to a give screen vector bundle $S(TC^\perp)$ and the equations (24) its *Frenet equations of Type 3.* The functions $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ and $\{\tau_1, \tau_2, \tau_3, \tau_5\}$ are called the *curvature functions* and the *torsion functions* of C with respect to the frame F_3.

On the other hand, using the set of null vector fields $\{L_1, L_2, L_3, L_4\}$ such that $g(L_1, L_2) = 1$, $g(L_3, L_4) = 1$ and all other $g(L_i, L_j) = 0$, $i \leq j$, we have

$$\begin{cases} \nabla_\lambda \lambda &= h\lambda + K_1 L_1, \\ \nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + T_3 L_3, \\ \nabla_\lambda L_1 &= -K_3\lambda + K_4 L_1 + T_4 L_3 + T_5 L_4, \\ \nabla_\lambda L_2 &= -K_2\lambda - K_1 N - K_4 L_2 + T_6 L_3 + T_7 L_4, \\ \nabla_\lambda L_3 &= -T_7 L_1 - T_5 L_2 + T_8 L_3, \\ \nabla_\lambda L_4 &= -T_3\lambda - T_6 L_1 - T_8 L_4. \end{cases} \tag{26}$$

In this case, we also call

$$F_3 = \{\lambda, N, L_1, L_2, L_3, L_4\} \tag{27}$$

a *Frenet frame of Type 3* on **M** along C with respect to a given screen vector bundle $S(TC^{\perp})$ and the equations (26) its *Frenet equations of Type 3.* The functions $\{K_1, \ldots, K_4\}$ and $\{T_5, \ldots, T_8\}$ are called the *curvature functions* and the *torsion functions* of C with respect to the frame F_3.

Next, by the transformations (19) for $i = 3$, we have

$$\nabla_\lambda U_3 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_3 - \nabla_\lambda L_4), \qquad \nabla_\lambda U_4 = \frac{1}{\sqrt{2}}(\nabla_\lambda L_3 + \nabla_\lambda L_4).$$

Using the following relations

$$\begin{aligned}
&\frac{T_4 - T_5}{\sqrt{2}} = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_1, L_4 - L_3) = -g(\nabla_\lambda L_1, U_3) = -K_5,\\
&\frac{T_4 + T_5}{\sqrt{2}} = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_1, L_4 + L_3) = g(\nabla_\lambda L_1, U_4) \ = K_6,\\
&\frac{T_6 - T_7}{\sqrt{2}} = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_2, L_4 - L_3) = -g(\nabla_\lambda L_1, U_3) = -K_7,\\
&\frac{T_6 + T_7}{\sqrt{2}} = \frac{1}{\sqrt{2}} g(\nabla_\lambda L_2, L_4 + L_3) = g(\nabla_\lambda L_1, U_4) \ = K_8,\\
&T_8 = g(\nabla_\lambda L_3, L_4) = g(\nabla_\lambda U_3, U_4) = K_8 = \kappa_8,\\
&T_4 L_3 + T_5 L_4 = -K_5 U_3 + K_6 U_4,\\
&T_6 L_3 + T_7 L_4 = -K_7 U_3 + K_8 U_4,
\end{aligned}$$

we have

$$\begin{cases}
\nabla_\lambda \lambda &= h\lambda + K_1 L_1,\\
\nabla_\lambda N &= -hN + K_2 L_1 + K_3 L_2 + \tau_3 W_3 + \tau_2 W_4,\\
\nabla_\lambda L_1 &= -K_3 \lambda + \kappa_4 L_1 + K_5 W_3 + K_6 W_4,\\
\nabla_\lambda L_2 &= -K_2 \lambda - K_1 N - \kappa_4 L_2 + K_7 W_3 + K_8 W_4,\\
\varepsilon_3 \nabla_\lambda W_3 &= -\tau_3 \lambda - K_7 L_1 - K_5 L_2 + \kappa_8 W_4,\\
\varepsilon_4 \nabla_\lambda W_4 &= -\tau_2 - K_8 L_1 - K_6 L_2 - \kappa_8 W_3,
\end{cases} \tag{28}$$

where

$$\begin{aligned}
K_1 &= -\sqrt{2}\kappa_1, & K_2 &= \frac{\kappa_3 - \kappa_2}{\sqrt{2}}, & K_3 &= \frac{\kappa_3 + \kappa_2}{\sqrt{2}}, & K_4 &= \kappa_4,\\
K_5 &= \frac{\kappa_6 + \kappa_5}{\sqrt{2}}, & K_6 &= \frac{\kappa_7 + \tau_5}{\sqrt{2}}, & K_7 &= \frac{\kappa_6 - \kappa_5}{\sqrt{2}}, & K_8 &= \frac{\kappa_7 - \tau_5}{\sqrt{2}}.
\end{aligned}$$

In this case, we also call

$$F_3 = \{\lambda, N, L_1, L_2, W_3, W_4\} \tag{29}$$

a *Frenet frame of Type 3* on $\mathbf{M}_3$ along C with respect to a given screen vector bundle $S(TC^{\perp})$ and the equations (28) its *Frenet equations of Type 3.*

Remark 3. We know that the Frenet equations (24) include all of six different Frenet equations of Type 1 (in the case $\tau_1 = \tau_2 = \tau_3 = \tau_5 = 0$), three different Frenet equations of Type 2 (in the case $\tau_2 = 0$) and one Frenet equations of Type 3. By the same calculation, we find that $\mathbf{M}_2$ have Frenet equations of two types, named by Type 1 and Type 2, up to the signatures of W_i's and $\mathbf{M}_1$ have Frenet equations of only one type, named by Type 1, up to the signatures of W_i's. Hence we call the equations (24) the *compound Frenet equations* of the null curve C and $F = \{\lambda, N, W_1, \ldots, W_4\}$ the *compound Frenet frame* on $\mathbf{M}_q$ $(1 \le q \le 3)$ along C.

Remark 4. In general, let $(\mathbf{M}, g)$ be a real $(m+2)$-dimensional semi-Riemannian manifold of index q and C be a smooth null curve in $\mathbf{M}$. We know that C has k-type Frenet equations, namely Type 1, Type 2, ..., Type k, where $k = \min\{q, m+2-q\}$.

Example 1. Let $\mathbf{R}^6_3$ be a 6-dimensional semi-Riemannian space of index 3 with the semi-Riemannian metric

$$g(x,y) = -x^0y^0 - x^1y^1 - x^2y^2 + x^3y^3 + x^4y^4 + x^5y^5.$$

Suppose C is a null curve in $\mathbf{R}^6_3$ given by the equations

$$C : (A\cos t,\ A\sin t,\ B\sinh t,\ B\cosh t,\ At,\ Bt)$$

where $A, B, t \in R$ such that $(A, B) \neq (0,0)$. Then,

$$\lambda = (-A\sin t, A\cos t, B\cosh t, B\sinh t, A, B),$$
$$\nabla_\lambda \lambda = (-A\cos t, -A\sin t, B\sinh t, B\cosh t, 0, 0).$$

If we take a spacelike vector field V along C such that

$$V = \begin{cases} (0,0,0,0,1,0), & \text{if } A \neq 0, \\ (0,0,0,0,0,1), & \text{if } B \neq 0, \end{cases}$$

then $g(\lambda, V) = A$ or B and $g(V,V) = 1$. By the relation

$$N = \frac{1}{g(\lambda, V)}\left\{V - \frac{g(V,V)}{2g(\lambda,V)}\lambda\right\},$$

we obtain the following null transversal vector field

$$N = \begin{cases} \dfrac{1}{2A^2}(A\sin t, -A\cos t, -B\cosh t, -B\sinh t, A, -B), & \text{if } A \neq 0, \\ \dfrac{1}{2B^2}(A\sin t, -A\cos t, -B\cosh t, -B\sinh t, -A, B), & \text{if } B \neq 0. \end{cases}$$

We need to know the causal character of the vector field $H(t) = \nabla_\lambda \lambda - h\lambda$ along C. By direct calculations we obtain

$$g(H(t), H(t)) = B^2 - A^2.$$

Hence $H(t)$ is spacelike, timelike or lightlike according as $B^2 > A^2$, $B^2 < A^2$ or $B^2 = A^2$ respectively.

Choose $A = 1$ and $B = \sqrt{2}$, then $H(t)$ is spacelike and the curve

$$C : (\cos t,\ \sin t,\ \sqrt{2}\sinh t,\ \sqrt{2}\cosh t,\ t,\ \sqrt{2}t)$$

falls in the Type 1 with the Frenet frame $F = \{\lambda, N, W_1, W_2, W_3, W_4\}$ as follows

$$\begin{aligned}
\lambda &= (-\sin t, \cos t, \sqrt{2}\cosh t, \sqrt{2}\sinh t, 1, \sqrt{2}),\\
N &= \frac{1}{2}(\sin t, -\cos t, -\sqrt{2}\cosh t, -\sqrt{2}\sinh t, 1, -\sqrt{2}),\\
W_1 &= (-\cos t, -\sin t, \sqrt{2}\sinh t, \sqrt{2}\cosh t, 0, 0),\\
W_2 &= (-\sqrt{2}\cos t, -\sqrt{2}\sin t, \sinh t, \cosh t, 0, 0),\\
W_3 &= (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0, 0, 0, 1),\\
W_4 &= (\sqrt{2}\sin t, -\sqrt{2}\cos t, -\cosh t, -\sinh t, 0, -2).
\end{aligned}$$

The Frenet equations (4) and the curvature functions are given by

$$\begin{aligned}
\nabla_\lambda \lambda &= W_1, & \nabla_\lambda N &= -\frac{1}{2}W_1, & \nabla_\lambda W_1 &= \frac{1}{2}\lambda - N - \sqrt{2}W_3,\\
\nabla_\lambda W_2 &= -2W_3 - W_4, & \nabla_\lambda W_3 &= -\sqrt{2}W_1 + 2W_2, & \nabla_\lambda W_4 &= -W_2,
\end{aligned}$$

with

$$h = 0,\ \kappa_1 = 1,\ \kappa_2 = -\frac{1}{2},\ \kappa_3 = 0,\ \kappa_4 = 0,$$
$$\kappa_5 = -\sqrt{2},\ \kappa_6 = 2,\ \kappa_7 = 1,\ \kappa_8 = 0.$$

Next we set $A = \sqrt{2}$ and $B = 1$. The curve

$$C : (\sqrt{2}\cos t,\ \sqrt{2}\sin t,\ \sinh t,\ \cosh t,\ \sqrt{2}t,\ t)$$

also falls in Type 1 with the Frenet frame $F = \{\lambda, N, W_1, W_2, W_3, W_4\}$ as follows

$$\begin{aligned}
\lambda &= (-\sqrt{2}\sin t, \sqrt{2}\cos t, \cosh t, \sinh t, \sqrt{2}, 1),\\
N &= \frac{1}{2}(\sqrt{2}\sin t, -\sqrt{2}\cos t, -\cosh t, -\sinh t, -\sqrt{2},\ 1),
\end{aligned}$$

$$W_1 = (-\sqrt{2}\cos t, -\sqrt{2}\sin t, \sinh t, \cosh t,\ 0,\ 0),$$
$$W_2 = (-\cos t, -\sin t, \sqrt{2}\sinh t, \sqrt{2}\cosh t,\ 0,\ 0),$$
$$W_3 = (0, 0, \sqrt{2}\cosh t, \sqrt{2}\sinh t, 1, 0),$$
$$W_4 = (\sin t,\ -\cos t, -\sqrt{2}\cosh t, -\sqrt{2}\sinh t, -2,\ 0).$$

The Frenet equations (4) and the curvature functions are given by

$$\nabla_\lambda \lambda = W_1, \qquad \nabla_\lambda N = -\frac{1}{2}W_1, \qquad \nabla_\lambda W_1 = -\frac{1}{2}\lambda + N + \sqrt{2}W_3$$
$$\nabla_\lambda W_2 = 2W_3 + W_4, \quad \nabla_\lambda W_3 = -\sqrt{2}W_1 + 2W_2, \quad \nabla_\lambda W_4 = -W_2,$$

with

$$h = 0,\ \kappa_1 = 1,\ \kappa_2 = -\frac{1}{2},\ \kappa_3 = 0,\ \kappa_4 = 0,$$
$$\kappa_5 = -\sqrt{2},\ \kappa_6 = 2,\ \kappa_7 = 1,\ \kappa_8 = 0.$$

Finally we set $A = B = 1$, then $H(t)$ is lightlike. Therefore, the curve

$$C : (\cos t,\ \sin t,\ \sinh t,\ \cosh t,\ t,\ t)$$

falls in the Type 3 with the Frenet frame $F = \{\lambda, N, L_1, L_2, L_3, L_4\}$ as follows

$$\lambda = (-\sin t, \cos t, \cosh t, \sinh t, 1, 1),$$
$$N = \frac{1}{2}(\sin t, -\cos t, -\cosh t, -\sinh t, 1, -1),$$
$$L_1 = (-\cos t, -\sin t, \sinh t, \cosh t,\ 0,\ 0),$$
$$L_2 = \frac{1}{2}(\cos t, \sin t, \sinh t, \cosh t,\ 0,\ 0),$$
$$L_3 = (-\sin t, \cos t, 0, 0, 0, 1),$$
$$L_4 = (0, 0, \cosh t, \sinh t, 0, 1).$$

The Frenet equations (26) and the curvature and torsion functions are given by

$$\nabla_\lambda \lambda = L_1, \qquad \nabla_\lambda N = -\frac{1}{2}L_1,$$
$$\nabla_\lambda L_1 = -L_3 + L_4, \quad \nabla_\lambda L_2 = \frac{1}{2}\lambda - N - \frac{1}{2}(L_3 + L_4),$$
$$\nabla_\lambda L_3 = \frac{1}{2}L_1 - L_2, \quad \nabla_\lambda L_4 = \frac{1}{2}L_1,$$

with

$$h = 0, \quad K_1 = 1, \quad K_2 = -\frac{1}{2}, \quad K_3 = 0, \qquad K_4 = 0,$$
$$T_3 = 0, \quad T_4 = -1, \quad T_5 = 1, \quad T_6 = T_7 = -\frac{1}{2}, \quad T_8 = 0.$$

The Frenet equations (28) gives

$$\nabla_\lambda \lambda = L_1, \qquad \nabla_\lambda N = -\frac{1}{2}L_1, \quad \nabla_\lambda L_1 = \sqrt{2}W_3,$$
$$\nabla_\lambda L_2 = \frac{1}{2}\lambda - N - \frac{1}{\sqrt{2}}W_4, \quad \nabla_\lambda W_3 = \frac{1}{\sqrt{2}}L_2, \quad \nabla_\lambda W_4 = \frac{1}{\sqrt{2}}(L_1 - L_2).$$

Also the Frenet equations (24) gives

$$\begin{aligned}
\nabla_\lambda \lambda &= \frac{1}{\sqrt{2}}(W_2 - W_1),\\
\nabla_\lambda N &= \frac{1}{2\sqrt{2}}(W_1 - W_2)\\
\nabla_\lambda W_1 &= \frac{1}{2\sqrt{2}}\lambda - \frac{1}{\sqrt{2}}N - W_3 - \frac{1}{2}W_4,\\
\nabla_\lambda W_2 &= \frac{1}{2\sqrt{2}}\lambda - \frac{1}{\sqrt{2}}N + W_3 - \frac{1}{2}W_4,\\
\nabla_\lambda W_3 &= \frac{1}{2}(W_1 + W_2),\\
\nabla_\lambda W_4 &= -W_1.
\end{aligned}$$

6. Invariance of Frenet frames

In this section we show that each type of the Frenet frames always transform to the same type by the canonical parameter transformations of the coordinate neighborhood of C and the screen vector bundle. And we discuss some properties of null curves in $\mathbf{M}_q$.

First, with respect to a given screen vector bundle $S(TC^\perp)$, we consider two Frenet frames F and F^* along two neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively with non-null intersection. Then we have

$$\lambda^* = \frac{dt}{dt^*}\lambda, \quad N^* = \frac{dt^*}{dt}N, \tag{30}$$

$$W_\alpha^* = \sum_{\beta=1}^{4} A_\alpha^\beta W_\beta, \quad 1 \le \alpha \le 4, \tag{31}$$

where A_α^β are smooth functions on $\mathcal{U} \cap \mathcal{U}^*$ and the matrix $(A_\alpha^\beta(x))$ is an element of the semi-orthogonal group $O(q-1, 4-q+1)$ for any $x \in \mathcal{U} \cap \mathcal{U}^*$.

If we write the first and second equations of the compound Frenet equations (24) for both F and F^* and use (30) and (31), we obtain

$$\frac{d^2t}{dt^{*2}} + h\left(\frac{dt}{dt^*}\right)^2 = h^*\frac{dt}{dt^*}, \tag{32}$$

$$(\kappa_1^*, \tau_1^*, 0, 0)\left(A_\alpha^\beta(x)\right) = (\kappa_1, \tau_1, 0, 0)\left(\frac{dt}{dt^*}\right)^2, \tag{33}$$

$$(\kappa_2^*, \kappa_3^*, \tau_3^*, \tau_2^*)\left(A_\alpha^\beta(x)\right) = (\kappa_2, \kappa_3, \tau_3, \tau_2). \tag{34}$$

From these relations we have

Lemma 1. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$, and F and F^* be two Frenet frames of Type 1 on $\mathcal{U}$ and $\mathcal{U}^*$ with their respective curvature functions, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1\kappa_3 \neq 0$ on $\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$. Then at any point of $\mathcal{U} \cap \mathcal{U}^*$ we have*

$$\begin{cases} \kappa_1^* &= \kappa_1 A_1 \left(\dfrac{dt}{dt^*}\right)^2, \\ \kappa_2^* &= \kappa_2 A_1, \\ \kappa_3^* &= \kappa_3 A_2, \\ \kappa_\alpha^* &= \kappa_\alpha A_{\alpha-1}\dfrac{dt}{dt^*}, \end{cases} \tag{35}$$

where $4 \leq \alpha \leq 8$ and $A_i = \pm 1$.

Proof. From the relations (33) satisfying $\tau_1 = \tau_1^* = 0$, we have $k_1^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ and $A_1^2 = A_1^3 = A_1^4 = 0$. Since $(A_\alpha^\beta(x))$ is a semi-orthogonal matrix, we infer that $A_1^1 = A_1 = \pm 1$ and $A_2^1 = A_3^1 = A_4^1 = 0$. Then from the second equation of Type 1 with respect to F and F^*, and taking into account that $\kappa_3 \neq 0$, we obtain $k_3^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ which implies $A_2^3 = A_3^2 = A_2^4 = A_4^2 = 0$ and $A_2^2 = A_2 = \pm 1$. Repeating this process for all other equations we obtain all the relations in (35), which completes the proof. □

Proposition 6.1. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$, and F and F^* be two Frenet frames of Type 1 on $\mathcal{U}$ and $\mathcal{U}^*$ with their respective curvature functions, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1\kappa_3 \neq 0$ on $\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$. Then the second and third curvatures κ_2 and κ_3 are invariant to the coordinate transformations.*

Next, we let F and F^* be Frenet frames of Type 2 or 3 on $\mathcal{U}$ and $\mathcal{U}^*$ and assume that the orthonormal basis of a screen vector bundle $S(TC^{\perp})$ of C have the signature $(-,+,(\pm),+)$, where $(\pm)$ is $+$ or $-$, according to the index $q=2$ or $q=3$. From the equation (33) we have

$$A_1^1 - A_2^1 = A_2^2 - A_1^2, \qquad A_1^3 = A_2^3, \qquad A_1^4 = A_2^4, \tag{36}$$

because $\tau_1^* = -\kappa_1^*$ and $\tau_1 = -\kappa_1$. Using (31), (36) and $L_1 = \frac{1}{\sqrt{2}}(W_2 - W_1)$ due to (6) and (19), we obtain

$$L_1^* = (A_1^1 - A_2^1)L_1 \tag{37}$$

where $A_1^1 - A_2^1 \neq 0$, otherwise the matrix $(A_\alpha^\beta(x))$ is singular.

Since W_1^* is a timlike vector field and W_2^* is a spacelike vector fields, the first row $(A_1^1, \dots, A_1^4)$ of $(A_\alpha^\beta(x))$ is a timelike vector field and the second row $(A_2^1, \dots, A_2^4)$ is a spacelike vector field of $\mathbf{R}_{q-1}^4$ and these vectors are perpendicular to each other. Thus, we have

$$(A_1^1)^2 - (A_1^2)^2 - 1 = (A_2^1)^2 - (A_2^2)^2 + 1 = A_1^1 A_2^1 - A_1^2 A_2^2.$$

From this relation we have the following two relations

$$A_1^1 A_2^2 - A_1^2 A_2^1 = 1, \qquad (A_1^1 - A_2^1)(A_1^1 + A_1^2) = 1. \tag{38}$$

Using the relations

$$L_1 = \frac{1}{\sqrt{2}}(W_2 - W_1), \qquad L_2 = \frac{1}{\sqrt{2}}(W_2 + W_1),$$
$$W_1 = \frac{1}{\sqrt{2}}(L_2 - L_1), \qquad W_2 = \frac{1}{\sqrt{2}}(L_2 + L_1),$$

we have

$$W_\alpha^* = \frac{1}{\sqrt{2}}(A_\alpha^2 - A_\alpha^1)L_1 + \frac{1}{\sqrt{2}}(A_\alpha^2 + A_\alpha^1)L_2 + A_\alpha^3 W_3 + A_\alpha^4 W_4, \quad \alpha \in \{1,2\} \tag{39}$$

$$L_2^* = \frac{1}{2}(A_2^2 + A_1^2 - A_1^1 - A_2^1)L_1 + (A_1^1 + A_2^1)L_2 + \sqrt{2}A_1^3 W_3 \ + \ \sqrt{2}A_1^4 W_4. \tag{40}$$

The scalar product of L_1^* and W_α^*, L_2^* and L_2^*, and L_2^* and W_α^* provide

$$A_3^1 = -A_3^2,$$
$$A_4^1 = -A_4^2,$$
$$(A_2^2 + A_1^2 - A_1^1 - A_2^1)(A_1^1 + A_1^2 + A_2^1 + A_2^2) = 4\{(A_1^3)^2 - (A_1^4)^2\},$$
$$(A_\alpha^2 - A_\alpha^1)(A_1^1 + A_1^2 + A_2^1 + A_2^2) = 4\{A_1^3 A_\alpha^3 - A_1^4 A_\alpha^4\}, \quad \alpha \in \{3,4\},$$

respectively. Using (36) and (38), the last two equations reduce

$$\begin{aligned} A_2^2 + A_1^2 - A_1^1 - A_2^1 &= 2\{(A_1^3)^2 - (A_1^4)^2\}(A_1^1 - A_2^1), \\ A_\alpha^2 - A_\alpha^1 &= 2\{A_1^3 A_\alpha^3 - A_1^4 A_\alpha^4\}(A_1^1 - A_2^1), \quad \alpha \in \{3,4\}, \end{aligned}$$

respectively. From this two equations, we have

$$\begin{aligned} A_2^2 - A_1^1 &= \{(A_1^3)^2 - (A_1^4)^2\}(A_1^1 - A_2^1), \\ A_1^2 - A_2^1 &= 2\{(A_1^3)^2 - (A_1^4)^2\}(A_1^1 - A_2^1), \\ A_\alpha^2 &= \{A_1^3 A_\alpha^3 - A_1^4 A_\alpha^4\}(A_1^1 - A_2^1), \quad \alpha \in \{3,4\}. \end{aligned}$$

Thus if $A_1^3 = A_1^4 = 0$, then

$$A_2^3 = A_2^4 = A_3^1 = A_4^1 = A_3^2 = A_4^2 = 0 \text{ and } A_1^1 = A_2^2, \; A_2^1 = A_1^2.$$

Since the matrix $(A_\alpha^\beta(x))$ is an element of semi-orthogonal group $O(q-1, 4-q+1)$ and the signature of the orthonormal basis $\{W_1, W_2, W_3, W_4\}$ of the screen vector bundle $S(TC^\perp)$ is $(-,+,(\pm),+)$, the third and the fourth rows of $(A_\alpha^\beta(x))$ satisfy

$$(\pm)(A_3^3)^2 + (A_3^4)^2 = (\pm)1, \;\; (\pm)(A_4^3)^2 + (A_4^4)^2 = 1, \;\; (\pm)A_3^3 A_4^3 + A_3^4 A_4^4 = 0,$$

$$(\pm)(A_3^3)^2 + (A_4^3)^2 = (\pm)1, \;\; (\pm)(A_3^4)^2 + (A_4^4)^2 = 1, \;\; (\pm)A_3^3 A_3^4 + A_4^3 A_4^4 = 0,$$

This relations provide the following two relations:

$$A_3^4 = A_4^3, \quad A_3^3 = A_4^4 \qquad \text{or} \qquad A_3^4 = -A_4^3, \quad A_3^3 = -A_4^4$$

with the aid of the fact that $A_3^3 A_4^4 - A_3^4 A_4^3 = (\pm)1$.

Since the matrix $(A_\alpha^\beta(x))$ is a semi-orthogonal, this matrix is orthogonally diagonalizable by an orthogonal diagonalization P, *i. e.*,

$$({}^tPA(x)P) = \left(\begin{array}{cc|cc} \cosh\theta_1 & \sinh\theta_1 & 0 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ \hline 0 & 0 & \cosh\theta_2 & \sinh\theta_2 \\ 0 & 0 & (\pm)\sinh\theta_2 & (\pm)\cosh\theta_2 \end{array}\right).$$

Assume yhat, no loss generality, this matrix also denote

$$(A_\beta^\alpha(x)) = \left(\begin{array}{cc|cc} A_1^1 & A_1^2 & 0 & 0 \\ A_2^1 & A_2^2 & 0 & 0 \\ \hline 0 & 0 & A_3^3 & A_3^4 \\ 0 & 0 & A_4^3 & A_4^4 \end{array}\right).$$

The coordinate transformation (31) of this form is called *canonical.* For the canonical transformation, from the equations (39) and (40), we obtain

$$L_2^* = (A_1^1 + A_2^1)L_2,$$
$$W_\alpha^* = \sum_{\beta=3}^{4} A_\alpha^\beta W_\beta, \quad \alpha \in \{3,4\}.$$

Exchange the equations (15) for $i = 2,3$ and $i = 3,4$ we obtain the following general relations

$$L_i^* = (A_i^i - A_{i+1}^i)L_i, \quad L_{i+1}^* = (A_i^i + A_{i+1}^i)L_{i+1},$$
$$W_\alpha^* = \sum_{\beta=1}^{4} A_\alpha^\beta W_\beta, \quad \alpha \in \{1,2\}.$$

Thus we have

Proposition 6.2. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$ and F and F^* be two Frenet frames of Type 2 on $\mathcal{U} \cap \mathcal{U}^*$, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1 \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$. Then $S(TC^\perp)$ is an orthogonal direct sum of two invariant subspaces* $\mathrm{Span}\{L_i, L_{i+1}\} = \mathrm{Span}\{W_i, W_{i+1}\}$ *and* $\mathrm{Span}\{W_1, \hat{W}_i, \hat{W}_{i+1}, W_4\}$ *by the canonical transformation of coordinate neighborhoods of C, where overhat $(\hat{\ })$ denotes the deleted symbol for that term.*

Similarly, from the Frenet equations (26), we obtain the following general relations

$$L_i^* = (A_i^i - A_{i+1}^i)L_i, \quad L_{i+1}^* = (A_i^i + A_{i+1}^i)L_{i+1}, \quad i = 1,3.$$

Proposition 6.3. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$ and F and F^* be two Frenet frames of Type 3 on $\mathcal{U} \cap \mathcal{U}^*$, induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\kappa_1 \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$. Then $S(TC^\perp)$ is a orthogonal direct sum of two invariant subspaces* $\mathrm{Span}\{L_1, L_2\} = \mathrm{Span}\{W_1, W_2\}$ *and* $\mathrm{Span}\{L_3, L_4\} = \mathrm{Span}\{W_3, W_4\}$ *by the canonical transformation of coordinate neighborhoods of C.*

Proposition 6.4. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$, and F and F^* be two Frenet frames on $\mathcal{U}$ and $\mathcal{U}^*$ with curvature functions $\{\kappa_1, \kappa_2^*, \dots, \kappa_8\}$ and $\{\kappa_1^*, \kappa_2^*, \dots, \kappa_8^*\}$ and torsion functions $\{\tau_1, \tau_2, \tau_3, \tau_5\}$ and $\{\tau_1^*, \tau_2^*, \tau_3^*, \tau_5^*\}$ induced by the same screen vector bundle $S(TC^\perp)$ respectively. Suppose $\kappa_1\kappa_3 \neq 0$ on $\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$. Then the type of Frenet equations is invariant to the coordinate transformations.*

Proof. In the first case suppose $F^* = F_2^*$ or F_3^* and $F = F_1$. Then we have $\tau_1^* = -\kappa_1^*$ and $\tau_1 = 0$. This means from equation (33) that $A_1^2 = A_2^2$. Since W_1^* and W_2^* are the timlike and spacelike vector fields respectively, the first row $(A_1^1, A_1^2, 0, 0)$ of $(A_\alpha^\beta(x))$ is timelike vector field and the second row $(A_2^1, A_2^2, 0, 0)$ is a spacelike vector field of $\mathbf{R}_{q-1}^4$ and these vectors are perpendicular to each other. Thus, we have

$$(A_1^1)^2 - 1 = (A_2^1)^2 + 1 = A_1^1 A_2^1.$$

From this relation we have the contradictory relation $A_1^1 = A_2^1$. Hence this case is not possible.

Conversely, if $F^* = F_1^*$ and $F = F_2$ or F_3, then $\tau_1^* = 0$ and $\tau_1 = -\kappa_1$. From the equations (33) we have $A_1^1 = -A_1^2$, This means that the first row $(A_1^1, A_1^2, 0, 0)$ of the matrix $(A_\alpha^\beta(x))$ is an null vector, hence the vector field W_1^* is a null vector field, thus we conclude that this case is also not possible.

In the next cases suppose

$$F^* = F_3^*, F = F_2 \text{ and } F^* = F_2^*, F = F_3$$

respectively. From the equation (34), we have

$$\begin{cases} \kappa_2^* A_1^1 + \kappa_3^* A_2^1 = \kappa_2, \\ \kappa_2^* A_1^2 + \kappa_3^* A_2^2 = \kappa_3, \\ \tau_3^* A_3^3 + \tau_2^* A_4^3 = \tau_3, \\ \tau_3^* A_3^4 + \tau_2^* A_4^4 = \tau_2. \end{cases} \tag{41}$$

In case $F^* = F_3^*, F = F_2$, we have $A_3^4 = A_4^4$ and

$$(A_3^3)^2 - 1 = (A_4^3)^2 + 1 = A_3^3 A_4^3.$$

From this relation we have the contradictory relation $A_3^3 = A_4^3$. Hence this case is not possible. In another case $F^* = F_2^*, F = F_3$. From the equation (41) we conclude that this case is also not possible, which complete the proof. □

Using the Frenet equations of Type 2 and Type 3 in (15) and (26) and the method of Lemma 1, we have the following lemmas.

Lemma 2. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$, and F and F^* be two Frenet frames of Type 2 on $\mathcal{U}$ and $\mathcal{U}^*$ with respective curvature and torsion functions, induced by the same screen vector bundle $S(TC^\perp)$. If τ_3 is non-zero on*

$\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$, *then* $A_3^4 = A_4^3 = 0$ *and at any point of* $\mathcal{U} \cap \mathcal{U}^*$ *we have*

$$\begin{cases} K_1^* = K_1 D_1 \left(\frac{dt}{dt^*}\right)^2, & \tau_3^* = \tau_3 A_3, \\ K_2^* = K_2 D_1, & K_3^* = K_3 C_1, \\ K_5^* = K_5 C_2 \frac{dt}{dt^*}, & K_6^* = K_6 C_3 \frac{dt}{dt^*}, \\ K_7^* = K_7 D_2 \frac{dt}{dt^*}, & K_8^* = K_8 D_3 \frac{dt}{dt^*}, \\ \kappa_4^* = \kappa_4 \frac{dt}{dt^*} + D_1 \frac{dC_1}{dt^*}, & \kappa_8^* = \kappa_8 A_4 \frac{dt}{dt^*}, \end{cases} \tag{42}$$

where

$$C_1 = A_1^1 - A_2^1, \quad C_2 = C_1 A_3^3, \quad C_3 = C_1 A_4^4,$$
$$D_1 = A_1^1 + A_2^1, \quad D_2 = E_1 A_3^3, \quad D_3 = E_1 A_4^4,$$

$$C_i D_i = 1, \quad i \in \{1, 2, 3\},$$

$$A_3 = A_4 = \pm 1.$$

Lemma 3. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$, and F and F^* be two Frenet frames of Type 3 on $\mathcal{U}$ and $\mathcal{U}^*$ with respective curvature and torsion functions, induced by the same screen vector bundle $S(TC^\perp)$. Then we have*

$$\begin{cases} K_1^* = K_1 D_1 \left(\frac{dt}{dt^*}\right)^2, & K_2^* = K_2 D_1, \\ K_3^* = K_3 C_1, & K_4^* = K_4 \frac{dt}{dt^*} + D_1 \frac{dC_1}{dt^*}, \\ T_3^* = T_3 G_1, & T_4^* = T_4 G_3 \frac{dt}{dt^*}, \\ T_5^* = T_5 E_2 \frac{dt}{dt^*}, & T_6^* = T_6 G_2 \frac{dt}{dt^*}, \\ T_7^* = T_7 E_3 \frac{dt}{dt^*}, & T_8^* = T_8 \frac{dt}{dt^*} + G_1 \frac{dE_1}{dt^*}, \end{cases} \tag{43}$$

where

$$E_1 = A_3^3 - A_4^3, \quad E_2 = C_1 E_1, \quad E_3 = D_1 E_1,$$
$$G_1 = A_3^3 + A_4^3, \quad G_2 = D_1 G_1, \quad G_3 = C_1 G_1,$$

$$E_i G_i = 1, \quad i \in \{1, 2, 3\}.$$

Next, let $F = \{\lambda, N, W_1, \ldots, W_4\}$ and $\overline{F} = \{\overline{\lambda}, \overline{N}, \overline{W}_1, \ldots, \overline{W}_4\}$ be two Frenet frames with respect to $(t, S(TC^\perp), \mathcal{U})$ and $(\bar{t}, \overline{S}(TC^\perp), \overline{\mathcal{U}})$, respectively. Then the general transformations that relate elements of F and $\overline{F}$ on $\mathcal{U} \cap \mathcal{U}$, are

$$\begin{cases} \overline{\lambda} &= \dfrac{dt}{d\bar{t}}\lambda, \\ \overline{N} &= -\dfrac{1}{2}\dfrac{dt}{d\bar{t}}\displaystyle\sum_{\alpha=1}^{4}\varepsilon_\alpha (c_\alpha)^2\lambda + \dfrac{d\bar{t}}{dt}N + \displaystyle\sum_{\alpha=1}^{4} c_\alpha W_\alpha, \\ \overline{W}_\alpha &= \displaystyle\sum_{\beta=1}^{4} B_\alpha^\beta\left(W_\beta - \varepsilon_\beta \dfrac{dt}{d\bar{t}}c_\beta\lambda\right), \qquad 1 \le \alpha \le 4, \end{cases} \tag{44}$$

where c_α and B_α^β are smooth functions on $\mathcal{U} \cap \overline{\mathcal{U}}$ and the 4×4 matrix $(B_\alpha^\beta(x))$ is an element of the semi-orthogonal group $O(q-1, 4-q+1)$ for each $x \in \mathcal{U} \cap \overline{\mathcal{U}}$. Then by using (44) and the first equation of the compound Frenet equation for both F and $\overline{F}$ we obtain

$$\begin{cases} \overline{h} = \dfrac{d^2 t}{d\bar{t}^2}\dfrac{d\bar{t}}{dt} + h\dfrac{dt}{d\bar{t}} + (c_2\tau_1 - c_1\kappa_1)\left(\dfrac{dt}{d\bar{t}}\right)^2, \\ (\overline{\kappa}_1, \overline{\tau}_1,\ 0,\ 0)\left(B_\alpha^\beta(x)\right) = (\kappa_1, \tau_1,\ 0,\ 0)\left(\dfrac{dt}{d\bar{t}}\right)^2. \end{cases} \tag{45}$$

The screen vector bundle transformation (44) of the form

$$(B_\beta^\alpha(x)) = \left(\begin{array}{cc|cc} B_1^1 & B_1^2 & 0 & 0 \\ B_2^1 & B_2^2 & 0 & 0 \\ \hline 0 & 0 & B_3^3 & B_3^4 \\ 0 & 0 & B_4^3 & B_4^4 \end{array}\right).$$

is called *canonical.* Using above relations and the method of Proposition 6.4, we obtain the following proposition.

Proposition 6.5. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$ and F, $\overline{F}$ be two Frenet frames with respect to $(t, S(TC^\perp), \mathcal{U})$ and $(\bar{t}, \overline{S}(TC^\perp), \overline{\mathcal{U}})$ and their respective curvature functions. If $\kappa_1 \neq 0$ for all t, then the type of Frenet equations is invariant of the canonical screen vector bundle transformations.*

The following properties of the compound Frenet equations hold:

(a) The vanishing of the first curvature κ_1 on a neighborhood is independent of both the parameter transformations on C and the screen vector bundle transformations.

(b) It is possible to find a parameter on C such that $h = 0$ in Frenet equations of all possible types, using the same screen bundle.

To prove (a) we let $\kappa_1 = 0$ (which implies that $\overline{\kappa}_1 = 0$) on $\mathcal{U} \cap \overline{\mathcal{U}}$. Then, there exists a point $x \in \mathcal{U} \cap \overline{\mathcal{U}}$ such that, for Type 1,

$$B_1^1(x) = \cdots = B_1^4(x) = 0$$

and, for Type 2 and Type 3,

$$B_1^1(x) - B_2^1(x) = \cdots = B_1^4(x) - B_2^4(x) = 0.$$

This implies that the first and the second rows of the matrix $(B_\alpha^\beta(x))$ are linearly dependent, which is not possible since this matrix belongs to $O(q-1, 4-q+1)$. Hence it follows from the relation of (45) that (a) holds.

To prove (b) we consider the following differential equation

$$\frac{d^2t}{dt^{*2}} - h^* \frac{dt}{dt^*} = 0$$

whose general solution comes from

$$t = a \int_{t_0^*}^{t*} \exp\left(\int_{s_0}^{s} h^*(t^*)dt^*\right) ds + b, \quad a, b \in R.$$

It follows from the relation (33) that any of these solutions, with $a \neq 0$, might be taken as special parameter on C, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where t is the general parameter as defined in above equation. We call p a *distinguished parameter* of C, in terms for which $h = 0$. It is important to note that when t is replaced by p in the compound Frenet equations (24), the first two equations become

$$\begin{cases} \nabla_{\frac{d}{dp}} \dfrac{d}{dp} &= \kappa_1 W_1 + \tau_1 W_2, \\ \nabla_{\frac{d}{dp}} N &= \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4, \end{cases} \tag{46}$$

and the other equations remain unchanged.

In case $\kappa_1 = 0$, then, since $\tau_1 = 0$ or $\tau_1 = -\kappa_1$, the first equation of (46) takes the following familiar form

$$\frac{d^2x^i}{dp^2} + \sum_{j,k=0}^{5} \Gamma_{jk}^i \frac{dx^j}{dp}\frac{dx^k}{dp} = 0, \quad i \in \{0, \ldots, 5\},$$

where Γ_{jk}^i are the Christoffel symbols of the second type induced by ∇. Hence C is a null geodesic of $\mathbf{M}$. The converse follows easily. Thus we have the following theorem.

Theorem 6.1. *Let C be a null curve of a semi-Riemannian manifold $\mathbf{M}_q$. Then C is a null geodesic of $\mathbf{M}$ if and only if the first curvature κ_1 vanishes identically on C.*

Suppose $F = \{\lambda, N, W_1, \ldots, W_4\}$ and $\overline{F} = \{\overline{\lambda}, \overline{N}, \overline{W}_1, \ldots, \overline{W}_4\}$ are two Frenet frames of C with their respective screen spaces. Then, we know from Propositions 6.4 and 6.5 that they both belong to one of Type 1, Type 2 and Type 3.

Lemma 4. *Let C be a null curve of $\mathbf{M}_q$, with $\kappa_1 \neq 0$, and two Frenet frames F and $\overline{F}$ of Type 1. Then, their curvature functions are related by*

$$(*)\quad \begin{cases} \overline{\kappa}_1 &= \kappa_1 B_1 \left(\dfrac{dt}{d\overline{t}}\right)^2, \\ \overline{\kappa}_2 &= B_1 \left\{ \kappa_2 + \overline{h} c_1 + \dfrac{d\kappa_1}{d\overline{t}} - \dfrac{1}{2}\kappa_1 \left(\dfrac{dt}{d\overline{t}}\right)^2 \displaystyle\sum_{\alpha=1}^{4} (c_\alpha)^2 - (c_2\kappa_4 + c_3\kappa_5)\dfrac{dt}{d\overline{t}} \right\}, \\ \overline{\kappa}_3 &= \kappa_3 B_2^2 + \overline{h} \displaystyle\sum_{\alpha=2}^{4} B_2^\alpha c_\alpha + \sum_{\alpha=2}^{4} B_2^\alpha \dfrac{dc_\alpha}{dt} + (c_1\kappa_4 - c_3\kappa_6 - c_4\kappa_7) B_2^2 \dfrac{dt}{d\overline{t}} \\ &\qquad + (c_1\kappa_5 + c_2\kappa_6 - c_4\kappa_8) B_2^3 \dfrac{dt}{d\overline{t}} + (c_2\kappa_7 + c_3\kappa_8) B_2^4 \dfrac{dt}{d\overline{t}}, \\ \overline{\kappa}_4 &= B_1 \left\{ B_2^2 \left(\kappa_4 + \kappa_1 \dfrac{dt}{d\overline{t}} c_2\right) + B_2^3 \left(\kappa_5 + \kappa_1 \dfrac{dt}{d\overline{t}} c_3\right) + B_2^4 \kappa_1 \dfrac{dt}{d\overline{t}} c_4 \right\} \dfrac{dt}{d\overline{t}}, \\ \overline{\kappa}_5 &= B_1 \left\{ B_3^2 \left(\kappa_4 + \kappa_1 \dfrac{dt}{d\overline{t}} c_2\right) + B_3^3 \left(\kappa_5 + \kappa_1 \dfrac{dt}{d\overline{t}} c_3\right) + B_3^4 \kappa_1 \dfrac{dt}{d\overline{t}} c_4 \right\} \dfrac{dt}{d\overline{t}}, \\ \overline{\kappa}_6 &= \begin{vmatrix} B_2^2 & B_2^3 & \kappa_8 \\ B_2^3 & B_3^3 & -\kappa_7 \\ B_2^4 & B_3^4 & \kappa_6 \end{vmatrix} \dfrac{dt}{d\overline{t}} + \displaystyle\sum_{\alpha=2}^{4} B_3^\alpha \dfrac{dB_2^\alpha}{d\overline{t}}, \\ \overline{\kappa}_7 &= \begin{vmatrix} B_2^2 & -\kappa_8 & B_4^2 \\ B_2^3 & \kappa_7 & B_4^3 \\ B_2^4 & -\kappa_6 & B_4^4 \end{vmatrix} \dfrac{dt}{d\overline{t}} + \displaystyle\sum_{\alpha=2}^{4} B_4^\alpha \dfrac{dB_2^\alpha}{d\overline{t}}, \\ \overline{\kappa}_8 &= \begin{vmatrix} \kappa_8 & B_3^2 & B_4^2 \\ -\kappa_7 & B_3^3 & B_4^3 \\ \kappa_6 & B_3^4 & B_4^4 \end{vmatrix} \dfrac{dt}{d\overline{t}} + \displaystyle\sum_{\alpha=2}^{4} B_4^\alpha \dfrac{dB_3^\alpha}{d\overline{t}}. \end{cases}$$

Proof. For the Type 1 it follows from (45) that

$$B_1^i = B_i^1 = 0 \ (i \neq 1), \quad B_1^1 = B_1 = \pm 1.$$

Therefore the general transformations relating the elements of F and $\overline{F}$ on $\mathcal{U} \cap \overline{\mathcal{U}}$ are given by

$$\begin{cases} \overline{\lambda} &= \dfrac{dt}{d\overline{t}}\lambda, \\ \overline{N} &= -\dfrac{1}{2}\dfrac{dt}{d\overline{t}}\displaystyle\sum_{i=1}^{4}\varepsilon_i(c_i)^2\lambda + \dfrac{d\overline{t}}{dt}N + \sum_{i=1}^{4}c_iW_i, \\ \overline{W}_1 &= B_1(W_1 + \dfrac{dt}{d\overline{t}}c_1\lambda), \\ \overline{W}_\alpha &= \displaystyle\sum_{\beta=2}^{4}B_\alpha^\beta(W_\beta - \dfrac{dt}{d\overline{t}}c_\beta\lambda), \quad \alpha \in \{2,3,4\}. \end{cases} \tag{47}$$

The relations (∗) follow by straightforward calculations from the Frenet equations of Type 1 and the use of (47). □

Theorem 6.2. *Let C be a null curve of $\mathbf{M}_q$, with a Frenet frame F of Type 1 and a screen vector bundle $S(TC^\perp)$ on $\mathcal{U} \subset C$ such that $\kappa_1 \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $\overline{S}(TC)$ which induces another Frenet frame $\overline{F}$ of Type 1 on $\mathcal{U}$ such that $\overline{\kappa}_4 = \overline{\kappa}_5 = 0$.*

Proof. Define the following vector fields in terms of the elements of F on $\mathcal{U}$:

$$\begin{cases} \overline{N} &= -\dfrac{1}{2}\left(\dfrac{\kappa_4^2+\kappa_5^2}{\kappa_1^2}\right)\lambda + N - \dfrac{\kappa_4}{\kappa_1}W_2 - \dfrac{\kappa_5}{\kappa_1}W_3, \\ \overline{W}_2 &= W_2 + \dfrac{\kappa_4}{\kappa_1}\lambda, \\ \overline{W}_3 &= W_3 + \dfrac{\kappa_5}{\kappa_1}\lambda, \\ \overline{W}_i &= W_i, \quad i \in \{1,4\}. \end{cases} \tag{48}$$

Let $\mathcal{U}^*$ be another coordinate neighborhood with parameter t^* on C such that $\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$. By Lemma 1 we have the following on $\mathcal{U} \cap \mathcal{U}^*$

$$\begin{cases} \kappa_1^* &= \kappa_1 A_1\left(\dfrac{dt}{dt^*}\right)^2, \\ \kappa_4^* &= \kappa_4 A_3\dfrac{dt}{dt^*}, \\ \kappa_5^* &= \kappa_5 A_4\dfrac{dt}{dt^*}, \\ W_i^* &= A_iW_i, \quad i \in \{1,2,3,4\}. \end{cases} \tag{49}$$

Define $\{\overline{N}^*, \overline{W}_1^*, \overline{W}_2^* \overline{W}_3^*, \overline{W}_4^*\}$ by (48) but on $\mathcal{U}^*$ with respect to F^*, induced by the same $S(TC^\perp)$ on $\mathcal{U}^*$. Then by using (30), (31), (48) and (49) we obtain

$$\overline{N}^* = \frac{dt^*}{dt}\overline{N},$$
$$\overline{W}_i^* = A_i \overline{W}_i, \quad i \in \{1, 2, 3, 4\}.$$

Hence there exists a vector bundle $\overline{S}(TC^\perp)$ spanned on $\mathcal{U}$ by $\{\overline{W}_1, \overline{W}_2^* \overline{W}_3^*, \overline{W}_4^*\}$ given by (48). Moreover, it is easy to check that this vector bundle is complementary to TC in $TC^\perp$. The null transversal vector (constructed in Theorem 2.1), with respect to $S(TC^\perp)$, is locally represented by $\overline{N}$ from (48). Finally taking into account that $t = \bar{t}$ and

$$c_2 = -\frac{\kappa_4}{\kappa_1}, \quad c_3 = -\frac{\kappa_5}{\kappa_1}$$

in the fourth and the fifth equations of Lemma 4, we obtain $\overline{\kappa}_4 = \overline{\kappa}_5 = 0$ which completes the proof. □

At this point we assume that the transformations (45) are *diagonal transformations*, that is, they satisfy $B_i^j = B_j^i = 0$ $(i \neq j)$. For this case, it follows from the sixth equation of Lemma 4 that $c_4 = 0$. Using this we obtain the following theorem.

Theorem 6.3. *Let C be a null curve of $\mathbf{M}_q$ with Frenet frame of Type 1 such that $\kappa_1 \neq 0$. Then, there exist a lightlike 2-surface which is invariant with respect to both the parameter transformations on C and the diagonal screen vector bundle transformations.*

Proof. Let C^* be an integral curve of the vector field W_4. Since, by Lemma 4, $c_4 = 0$ for a diagonal screen vector bundle transformation, the 2-surface $S = C \times C^*$ is always invariant with respect to this particular class of screen vector bundle transformations. S can neither be Lorentz nor definite because its two base vectors $\{\ell, W_4\}$ contain a single null vector ℓ. Therefore, S must be lightlike. This completes the proof. □

Remark 5. Theorem 6.3 can be used to define null sectional curvature of a null vector in $\mathbf{M}_q$ in a similar way as introduced by Beem-Ehrlich [1, p. 571] for a Lorentzian manifold. Also see O'Neill [10, pp. 152–153 & p. 163] on null geodesic in surfaces and lightlike particles and Harris [7] on triangle comparison theorem for Lorentz manifolds.

Now we consider the case when F and $\overline{F}$ are both of Type 2. Using the equations (10) and the method of the proposition 6.2, we have the following general result for $\mathbf{M}_q$.

$$\begin{cases} B_1^1 &= B_2^2, \\ B_2^1 &= B_1^2, \\ B_1^3 &= B_3^1 = B_1^4 = B_4^1 = B_2^3 = B_3^2 = B_2^4 = B_4^2 = 0, \\ \overline{L}_1 &= D\left\{L_1 - C_2\dfrac{dt}{d\overline{t}}\lambda\right\}, \\ \overline{L}_2 &= E\left\{L_2 - C_1\dfrac{dt}{d\overline{t}}\lambda\right\}, \end{cases} \tag{50}$$

where

$$D = B_1^1 - B_2^1,\ E = B_1^1 + B_2^1,\ C_1 = \frac{1}{\sqrt{2}}(c_2 - c_1) \text{ and } C_2 = \frac{1}{\sqrt{2}}(c_2 + c_1).$$

Lemma 5. *Let C be a null curve of $\mathbf{M}_q$ such that $\kappa_1 \neq 0$, and two Frenet frames F and $\overline{F}$ of Type 2. Then their curvature functions are related by*

$$\begin{cases} \overline{K}_1 &= K_1 E\Big(\dfrac{dt}{d\overline{t}}\Big)^2, \\ \overline{K}_2 &= E\Big\{K_2 + \overline{h}C_1 + \dfrac{dC_1}{d\overline{t}} - \dfrac{K_1}{2}\Big(\dfrac{dt}{d\overline{t}}\Big)^2 \sum \varepsilon_i (c_i)^2 - (C_1 k_4 + c_3 K_7 + c_4 K_8)\dfrac{dt}{d\overline{t}}\Big\}, \\ \overline{K}_3 &= D\Big\{K_3 + \overline{h}C_2 + \dfrac{dC_2}{d\overline{t}} + (C_2 k_4 - c_3 K_5 - c_4 K_6)\dfrac{dt}{d\overline{t}}\Big\}, \\ \overline{K}_4 &= \Big\{k_4 + K_1 C_2 \dfrac{dt}{d\overline{t}} - E\dfrac{dD}{dt}\Big\}\dfrac{dt}{d\overline{t}}, \\ \overline{K}_5 &= D\Big\{K_5 B_3^3 + K_6 B_3^4\Big\}\dfrac{dt}{d\overline{t}}, \\ \overline{K}_6 &= D\Big\{K_5 B_4^3 + K_6 B_4^4\Big\}\dfrac{dt}{d\overline{t}}, \\ \overline{K}_7 &= E\Big\{\Big(K_7 + K_1\dfrac{dt}{d\overline{t}}c_3\Big)B_3^3 + \Big(K_8 + K_1\dfrac{dt}{d\overline{t}}c_4\Big)B_3^4\Big\}\dfrac{dt}{d\overline{t}}, \\ \overline{K}_8 &= E\Big\{\Big(K_7 + K_1\dfrac{dt}{d\overline{t}}c_3\Big)B_4^3 + \Big(K_8 + K_1\dfrac{dt}{d\overline{t}}c_4\Big)B_4^4\Big\}\dfrac{dt}{d\overline{t}}. \end{cases} \tag{51}$$

Proof. The matrix $(B_j^i(x))$, in the relations (50), is made up of two 2×2 matrices (a Lorentz and an orthogonal). Therefore, using (50), the general transformations are given by

$$\begin{cases} \overline{\lambda} &= \dfrac{dt}{d\overline{t}}\,\lambda, \\ \overline{N} &= -\dfrac{1}{2}\dfrac{dt}{d\overline{t}}\displaystyle\sum_{i=1}^{4}\varepsilon_i(c_i)^2\lambda + \dfrac{d\overline{t}}{dt}N + C_1L_1 + C_2L_2 + c_3W_3 + c_4W_4, \\ \overline{L}_1 &= D\Big\{L_1 - \dfrac{dt}{d\overline{t}}\,C_2\overline{\lambda}\Big\}, \\ \overline{L}_2 &= E\Big\{L_2 - \dfrac{dt}{d\overline{t}}\,C_1\overline{\lambda}\Big\}, \\ \overline{W}_\alpha &= \displaystyle\sum_{\beta=3}^{4} B_\alpha^\beta\Big(W_\beta - \dfrac{dt}{d\overline{t}}\,c_\beta\lambda\Big). \end{cases} \tag{52}$$

Straightforward calculations from above relations and the use of (15) implies (51), which proves this lemma. □

By a procedure same as for the Theorem 6.2, one can prove the following:

Theorem 6.4. *Let C be a null curve of $\mathbf{M}_q$ with screen bundle space $S(TC^\perp)$ and a Frenet frame F of Type 2 such that $\kappa_1 \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $\overline{S}(TC^\perp)$ which induces another Frenet frame F on $\mathcal{U}$ such that $\overline{K}_7 = \overline{K}_8 = 0$ on $\mathcal{U}$.*

Next we consider the case when F and $\overline{F}$ are both of Type 3. Using the equations (24), we have the following general result for $\mathbf{M}_q$.

$$\begin{cases} \overline{\lambda} &= \dfrac{dt}{d\overline{t}}\,\lambda, \\ \overline{N} &= -\dfrac{1}{2}\dfrac{dt}{d\overline{t}}\displaystyle\sum_{i=1}^{4}\varepsilon_i(c_i)^2\lambda + \dfrac{d\overline{t}}{dt}N + \displaystyle\sum_{\alpha=1}^{4} C_\alpha L_\alpha, \\ \overline{L}_1 &= \Big\{L_1 - C_2\dfrac{dt}{d\overline{t}}\,\lambda\Big\}, \\ \overline{L}_2 &= E\Big\{L_2 - C_1\dfrac{dt}{d\overline{t}}\,\lambda\Big\}, \\ \overline{L}_3 &= G\Big\{L_3 - C_4\dfrac{dt}{d\overline{t}}\,\lambda\Big\}, \\ \overline{L}_4 &= H\Big\{L_4 - C_3\dfrac{dt}{d\overline{t}}\,\lambda\Big\}, \end{cases} \tag{53}$$

where

$$\begin{aligned} D &= B_1^1 - B_2^1, & E &= B_1^1 + B_2^1, & G &= B_3^3 - B_4^3, & H &= B_3^3 + B_4^3, \\ C_1 &= \tfrac{1}{\sqrt{2}}(c_2 - c_1), & C_2 &= \tfrac{1}{\sqrt{2}}(c_2 + c_1), & C_3 &= \tfrac{1}{\sqrt{2}}(c_4 - c_3), & C_4 &= \tfrac{1}{\sqrt{2}}(c_4 + c_3). \end{aligned}$$

By the procedure same as for the Lemma 5, one can prove the following:

Lemma 6. *Let C be a null curve of $\mathbf{M}_q$ such that $\kappa_1 \neq 0$, and two Frenet frames F and $\overline{F}$ of Type 3. Then the functions G and H are constant and their curvature functions are related by*

$$\left\{\begin{aligned}
\overline{K}_1 &= K_1 E\Big(\frac{dt}{d\bar{t}}\Big)^2,\\
\overline{K}_2 &= E\Big\{K_2+\bar{h}C_1+\frac{dC_1}{d\bar{t}}-\frac{K_1}{2}\Big(\frac{dt}{d\bar{t}}\Big)^2\sum\varepsilon_i(c_i)^2-(C_1K_4+C_3K_7+C_4K_6)\frac{dt}{d\bar{t}}\Big\},\\
\overline{K}_3 &= D\Big\{K_3+\bar{h}C_2+\frac{dC_2}{d\bar{t}}+(C_2K_4-C_3T_5)\frac{dt}{d\bar{t}}\Big\},\\
\overline{T}_3 &= H\Big\{T_3+\bar{h}C_3+\frac{dC_3}{d\bar{t}}+(C_1T_4+C_2T_6+C_3T_8)\frac{dt}{d\bar{t}}\Big\},\\
0 &= \bar{h}C_4+\frac{dC_4}{d\bar{t}}+(C_1T_5+C_2T_7-C_4T_8)\frac{dt}{d\bar{t}},\\
\overline{K}_4 &= \Big\{K_4-K_1C_2\frac{dt}{d\bar{t}}+E\frac{dD}{dt}\Big\}\frac{dt}{d\bar{t}},\\
\overline{T}_4 &= DHT_4\frac{dt}{d\bar{t}},\\
\overline{T}_5 &= DGT_5\frac{dt}{d\bar{t}},\\
\overline{T}_6 &= E\Big\{T_6+K_1C_3\frac{dt}{d\bar{t}}\Big\}H\frac{dt}{d\bar{t}},\\
\overline{T}_7 &= E\Big\{T_7-K_1C_4\frac{dt}{d\bar{t}}\Big\}G\frac{dt}{d\bar{t}},\\
\overline{T}_8 &= T_8\frac{dt}{d\bar{t}}.
\end{aligned}\right. \tag{54}$$

By the method of Theorem 6.2 and 6.4, one can prove the following:

Theorem 6.5. *Let C be a null curve of $\mathbf{M}_q$ with screen bundle space $S(TC^{\perp})$ and a Frenet frame F of Type 3 such that $\kappa_1 \neq 0$ on $\mathcal{U}$. Then there exists a screen vector bundle $\overline{S}(TC^{\perp})$ which induces another Frenet frame F on $\mathcal{U}$ such that $\overline{T}_6 = \overline{T}_7 = 0$ on $\mathcal{U}$.*

7. Concluding remark

In this paper we have shown that it is possible to find general compound Frenet equations (24), with a variety of Frenet frames of Types 1, 2 and 3, for a null curve in a 6-dimensional semi-Riemannian manifold of index q. This is only a step further of the earlier work of Duggal & Bejancu [4] on null curves of Lorentzian manifolds and of Duggal & Jin [5] on null curves of semi-Riemannian manifolds of index 2. However, the general case of null curves in semi-Riemannian manifolds of arbitrary dimension is still an open problem. We guess that this case is much more complicated and the null curve have $\min\{q, m+2-q\}$-type Frenet equations. We hope that the publication of this paper will help in solving the general case.

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