Journal of the Society of Korea Industrial and Systems Engineering
Vol, 26, No. 1, pp.79－84, March 2003.

수리를 최소로 하는 최적교체모델

박태근* · 최동순**

*한국철도연구원 · **전주공업대학 시스템정보경영과

Optimal Replacement Model for Minimal Repairs

Tae-Geun Park* · Dong-Soon Choi**

*Korea Railroad Research Institute;
**Dept. of Industrial and Systems Engineering, Jeonju Technical College

종래의 연구들은 주로 시간이 경과함에 따라 수리비용과 고장시간 간격이 고정된 상태에서 최적교환시각(T)을 구하는 조건을 발견하는데 중점을 두었으나, 대부분의 시스템은 시간이 경과할수록 고장시간간격이 좁아지고 수리비용은 증가하는 것이 일반적이다.

본 논문에서는 위의 두 조건을 만족하는 보다 현실적인 모델을 구축하였으며, 또 일정시간 내에 2개의 시스템이 존재할 때 어느 조건 하에서 시스템이 확률적으로 우월한가를 분석하는 연구를 수행하였다. 즉, 시스템은 시간이 경과함에 따라 확률 P[N=k]로서 완전수리를, 1-P[N=k]로서 소수리를 행하는 모델을 고려하였다. 여기서 N은 연속된 완전수리 사이의 소수리의 수를 나타낸다. 또한 초기고장에 있어서 수리에 의해 새로운 시스템이 되는 확률이 높고, 고장횟수가 증가함에 따라 완전수리가 행해지는 확률이 낮아지는, 보다 현실에 가까운 모델을 구축하였다. 모델을 일반화하기 위해 수리비용은 확률변수로 가정하였다.

Keywords : Minimal Repair, Hazard Rate, Basic composition Formular, Non-Homogeneous Poisson Process, Discrete Probability Function, Replacement, IFR, Discrete DMRL

1. Introduction

Minimal and perfect repair are useful assumptions for mathematical models to represent practical maintenance activities. The minimal repair means that the failed system recovers its function properly but carries its age, while the perfect repair restores entire system into the new condition so that it behaves as a new system. Barlow and Hunter [1] proposed the maintenance problem with minimal repairs between planned replacements, and derived planned replacement period T which minimizes the total long-run expected cost per system time. Uematsu, Ohi, Kowada, and Nishida [7] introduced the random variable N which treats the number of failure between consecutive perfect repairs, i.e., the system is perfectly repaired with probability $\Pr[N=k]$ or is minimally done with probability $1-\Pr[N=k]$. When a system fails, it is perfectly or minimally repaired with probability $\Pr[N=k]$ or $1-\Pr[N=k]$, respectively.

In this paper, we consider an imperfect repair [5] where, at the k-th failure, the system is perfectly repaired with probability $\Pr[N=k]$ or is minimally done with $1-\Pr[N=k]$. The random variable N represent the number of failures between consecutive perfect repairs. We investigate the properties of the distribution of time between perfect repairs and its hazard rate function. In two replacement models, it is introduced that the cost of imperfect repair is random variable and depends on age of the system. And the time required for performing maintenance activities are negligible. We discuss the planned replacement period which minimizes the total expected cost per unit time in the steady state.

2. Formulation of Models

Let X_k be the interval time between $(k-1)$-st minimal repair and k-th one and $f(x_1, \cdots, x_n)$ be the common probability density function of the random vector $(X_1, \cdots, X_n)$. We define the partial sums such that

$$T_N = \sum_{i=1}^{N} X_i \quad \cdots (1)$$

where $T_0=0$. T_N may be called the real age of the system at the N-th failure as it is the elapsed time since the system was put in operation. It is easy to see that

$$\Pr[N(t) \geq k] = \Pr[T_k \leq t] \quad \cdots (2)$$

By conditioning on N,

$$\begin{aligned} \overline{F}_N(t) &= \Pr[T_N > t] \\ &= \sum_{l=0}^{\infty} \Pr[T_N > t \mid N=l] \Pr[N=l] \\ &= \sum_{l=0}^{\infty} \Pr[T_l > t] \Pr[N=l] \end{aligned}$$

From (2)

$$\begin{aligned} \overline{F}_N(t) &= \sum_{l=0}^{\infty} \Pr[N(t) < l] \Pr[N=l] \\ &= \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N>k] \end{aligned}$$

But

$$\Pr[N(t)=k] = \frac{1}{k!}\left[\int_0^t \lambda(x)dx\right]^k \exp\left[-\int_0^t \lambda(x), dx\right]$$

Hence, we obtain

$$\overline{F}_N(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\int_0^t \lambda(x)dx\right]^k \exp\left[-\int_0^t \lambda(x)dx\right] \Pr[N>k] \quad \cdots (3)$$

3. Stochastic Comparisons between T_{N_1} and T_{N_2}

In this section we consider stochastic comparisons between T_{N_1} and T_{N_2}. N and N' are positive integer valued random variables independent of $\{N(t), t \geq 0\}$.

Lemma 3.1 If $\lambda(t)$ is increasing in t, $\Pr[N=k]$ is discrete IFR and $\Pr[N>k]$ is discrete DMRL, the $\lambda(t) - \lambda_N(t)$ is increasing in t.

$$\overline{F}_N(t) \geq \overline{F}(t) \quad \cdots (4)$$

for all $t > 0$, then

$$\lambda_N(t) \leq \lambda(t) \quad \cdots (5)$$

Proof : Differentiating $\lambda(t) - \lambda_N(t)$ with respect to t, we have

$$\frac{d}{dt}(\lambda(t) - \lambda_N(t)) = \lambda'(t) \cdot \frac{\lambda(t) \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N=k+1]}{\left\{\sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N>k]\right\}^2} \quad \cdots (6)$$

$$- [\lambda(t)]^2 \cdot \frac{\begin{pmatrix} \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N=k+1], \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N>k] \\ - \sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N>k+2], \sum_{k=0}^{\infty} \Pr[N(t)=k] \backslash Pr[N>k+1] \end{pmatrix}}{\left\{\sum_{k=0}^{\infty} \Pr[N(t)=k] \Pr[N>k]\right\}^2} \quad \cdots (7)$$

$$\geq 0$$

since $\lambda(t)$ is increasing in t, then (6) is positive. The numerator of (7) is

$$D=\begin{vmatrix} \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N=k+1] & \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N>k] \\ \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N=k+2] & \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N>k+1]\end{vmatrix}$$

$$\geq\begin{vmatrix} \sum_{k=0}^{\infty}\Pr[N(t)=k+1]\Pr[N=k+2] & \sum_{k=0}^{\infty}\Pr[N(t)=k+1]\Pr[N>k+1] \\ \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N=k+2] & \sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N>k+1]\end{vmatrix}$$

$$=\sum\sum_{0\leq k_1<k_2\leq\infty}\begin{vmatrix}\Pr[N(t)=k_1+1] & \Pr[N(t)=k_2+1] \\ \Pr[N(t)=k_1] & \Pr[N(t)=k_2]\end{vmatrix}\times\begin{vmatrix}\Pr[N=k_1+2] & \Pr[N=k_1+1] \\ \Pr[N=k_2+2] & \Pr[N>k_2+1]\end{vmatrix}$$

by the basic composition Fomular [2]. The first determinant is

$$\begin{vmatrix}\Pr[N(t)=k_1+1] & \Pr[N(t)=k_2+1] \\ \Pr[N(t)=k_1] & \Pr[N(t)=k_2]\end{vmatrix}\geq 0 \quad\cdots\cdots\cdots (8)$$

The second determent is ≤ 0 since $\Pr[N=k]$ is discrete IFR and $\Pr[N>k]$ is discrete DMRL. Thus $D\leq 0$. It follows that $\lambda(t)-\lambda_N(t)$ is increasing in t.

Remark 3.1 If $\Pr[N>k]=0$ for all $k>0$, then $\lambda_N(t)=\lambda(t)$.

Remark 3.2 If $\Pr[N>0]=1$, then $\lambda_N(t)/\lambda(t)=\Pr[N=1]$.

Theorem 3.1 Consider two probability $\Pr[N_1=k]$ and $\Pr[N_2=k]$ such that $\Pr[N_1=0]=\Pr[N_2=k]=0$. If for all $t\geq 0$ and $x\geq 0$, for $k_1\leq k_2$.

$$\frac{\Pr[N_1=k_2]}{\Pr[N_1=k_1]}\geq\frac{\Pr[N_2=k_2]}{\Pr[N_2=k_1]} \quad\cdots\cdots\cdots (9)$$

then

$$\frac{f_{N_1}(t+x)}{f_{N_2}(t+x)}\leq\frac{f_{N_1}(t)}{f_{N_2}(t)} \quad\cdots\cdots\cdots (10)$$

Theorem 3.2 Consider two probability $\Pr[N_1>k]$ and $\Pr[N_2>k]$ such that $\Pr[N_1>0]=\Pr[N_2>0]=1$. If for all $t\geq 0$ and $x\geq 0$, for $k_1\leq k_2$.

$$\frac{\Pr[N_1>k_2]}{\Pr[N_1>k_1]}\geq\frac{\Pr[N_2>k_2]}{\Pr[N_2>k_1]} \quad\cdots\cdots\cdots (11)$$

then

$$\frac{\overline{F}_{N_1}(t+x)}{\overline{F}_{N_2}(t+x)}\leq\frac{\overline{F}_{N_1}(t)}{\overline{F}_{N_2}(t)} \quad\cdots\cdots\cdots (12)$$

Theorem 3.3 Consider two probability $\Pr[N_1>k]$ and $\Pr[N_2>k]$ such that $\Pr[N_1>0]=\Pr[N_2>0]=1$. If for all $t\geq 0$ and $x\geq 0$, for $k_1\leq k_2$.

$$\frac{\Pr[N_1>k_2]}{\Pr[N_1>k_1]}\geq\frac{\Pr[N_2>k_2]}{\Pr[N_2>k_1]} \quad\cdots\cdots\cdots (13)$$

then

$$\frac{\int_0^{\infty}\overline{F}_{N_1}(t+x)dx}{\overline{F}_{N_2}(t+x)}\leq\frac{\int_0^{\infty}\overline{F}_{N_1}(t)dx}{\overline{F}_{N_2}(t)} \quad\cdots\cdots\cdots (14)$$

4. The Model

The system is replaced whenever the number of imperfect repairs reaches n times in succession. We constitute the total time W and the total cost C until the unit completely replaced.

$$W = \begin{cases} \sum_{i=1}^{N} X_{i,} & \text{if } N < n \\ \sum_{i=1}^{n} X_{i,} & \text{if } N \geq n \end{cases} \quad \cdots\cdots (15)$$

and

$$C = \begin{cases} \sum_{i=1}^{N} C_{i,} & \text{if } N < n \\ \sum_{i=1}^{n-1} C_i + D, & \text{if } N \geq n \end{cases} \quad \cdots\cdots (16)$$

For convenience, we shall use the following notations.

$E[C_i] = \alpha_i$, $\Pr[N=k] = \beta_k$ and $\sum_{k=n}^{\infty} \beta_k = \gamma_n$

Theorem 4.1 The total long-run expected cost per system time can be obtained by using the theory of renewal process and is equal to,

$$K(n) = \frac{\sum_{k=1}^{n} \gamma_k \alpha_n + \gamma_n D}{\sum_{k=1}^{n} \gamma_k \beta_k} \quad \cdots\cdots (17)$$

Proof : From renewal theory, we have that

$$\lim_{t \to \infty} \frac{K(t)}{t} = \frac{E[C]}{E[W]}$$

where $E[C]$ and $E[W]$ are the expected cost per renewal cycle and the expected time of a renewal cycle, respectively. We compute $E[W]$ first.

$$E[W] = \sum_{k=1}^{n-1} E\left[\sum_{i=1}^{N} X_i \mid N=k\right] \Pr[N=k] + \sum_{k=n}^{\infty} E\left[\sum_{i=1}^{n} X_i \mid N=k\right] \Pr[N=k]$$

$$= \sum_{k=1}^{n} \gamma_k \beta_k$$

Next we compute $E[C]$.

$$E[C] = \sum_{k=1}^{n-1} E\left[\sum_{i=1}^{N} C_i \mid N=k\right] \Pr[N=k] + \sum_{k=n}^{\infty} E\left[\sum_{i=1}^{n} C_i + D \mid N=k\right] \Pr[N=k]$$

$$= \sum_{k=1}^{n} \gamma_k \alpha_k + \gamma_n D$$

Theorem 4.2 If $F(t)$ is DMRL and β_k is discrete DFR in k and α_k is nondecreasing in k, then $\hat{K}(n)$ is nonincreasing in n,

$$\hat{K}(n) = \frac{\frac{\gamma_n}{\gamma_{n+1}} \sum_{k=1}^{n+1} \gamma_k \beta_k - \sum_{k=1}^{n} \gamma_k \beta_k}{\alpha_{n+1} \sum_{k=1}^{n} \gamma_k \beta_k - \beta_{n+1} \sum_{k=1}^{n} \gamma_k \alpha_k} \quad \cdots\cdots (18)$$

Proof : For our model we have

$$K(n+1) - K(n) = \frac{\sum_{k=1}^{n+1} \gamma_k \alpha_k + \gamma_{n+1} D}{\sum_{k=1}^{n+1} \gamma_k \beta_k} - \frac{\sum_{k=1}^{n} \gamma_k \alpha_k + \gamma_n D}{\sum_{k=1}^{n} \gamma_k \beta_k}$$

{the numerator of $K(n+1) - K(n)$}

$$= \gamma_{n+1}\left[\alpha_{n+1} \sum_{k=1}^{n} \gamma_k \beta_k - \beta_{n+1} \sum_{k=1}^{n} \gamma_k \alpha_k - D\left(\frac{\gamma_n}{\gamma_{n+1}} \sum_{k=1}^{n+1} \gamma_k \beta_k - \sum_{k=1}^{n} \gamma_k \beta_k\right)\right]$$

where $a(n) = \alpha_{n+1} \sum_{k=1}^{n} \gamma_k \beta_k - \beta_{n+1} \sum_{k=1}^{n} \gamma_k \alpha_k$

and $b(n) = \frac{\gamma_n}{\gamma_{n+1}} \sum_{k=1}^{n+1} \gamma_k \beta_k - \sum_{k=1}^{n} \gamma_k \beta_k$

If $F(t)$ is DMRL, μ_n is nonincreasing in n,

$$a(n+1) - a(n) = \alpha_{n+2} \sum_{k=1}^{n+1} \alpha_k \beta_k - \alpha_{n+1} \sum_{k=1}^{n} \gamma_k \beta_k - \left(\beta_{n+2} \sum_{k=1}^{n+1} \gamma_k \beta_k - \beta_{n+1} \sum_{k=1}^{n} \gamma_k \beta_k\right) \geq 0$$

since α_n is nondecreasing and β_n is discrete DFR. Thus, $a(n)$ is nondecreasing in n.

Furthermore, if β_k is discrete DFR in k,

$$b(n+1)-b(n) = \left(\frac{\gamma_{n+1}}{\gamma_{n+2}} - \frac{\gamma_n}{\gamma_{n+1}}\right)\sum_{k=1}^{n+1}\gamma_k\beta_k + \gamma_{n+1}(\beta_{n+2}-\beta_{n+1}) \le 0,$$

thus $b(n)$ is nonincreasing in n. Hence $\hat{K}(n)=b(n)/a(n)$ is nonincreasing in n.

Theorem 4.3 Let $F(t)$ is DMRL and P_k is discrete DFR, and that

$$\hat{K}(1) > \frac{1}{\hat{D}} > \hat{K}(\infty), \quad \cdots\cdots (19)$$

there exists at least one finite positive period n^* which minimizes the total long-run expected cost per unit time $K(n)$.

Proof : For the infinity-horizon case we want to find a n that minimizes $K(n)$. From theorem 4.2, {the numerator of $K(n+1)-K(n)$}

$$= \gamma_{n+1}\left[\alpha_{n+1}\sum_{k=1}^{n}\gamma_k\beta_k - \beta_{n+1}\sum_{k=1}^{n}\gamma_k\alpha_k - D\left(\frac{\gamma_n}{\gamma_{n+1}}\sum_{k=1}^{n+1}\gamma_k\beta_k - \sum_{k=1}^{n}\gamma_k\beta_k\right)\right]$$

is nondecreasing n.

$$\hat{K}(1) = \frac{\left(\frac{1-\gamma_2}{\gamma_2}\right)\beta_1+\beta_2}{\alpha_2\beta_1-\beta_2\alpha_1}$$

and

$$\hat{K}(\infty) = \frac{q_0\overline{\gamma\beta}}{\alpha\overline{\beta\gamma}-\beta\overline{\gamma\alpha}}$$

where

$$q_0 = \lim_{n\to\infty}(\Pr[N\ge n]/\Pr[N\ge n+1]) \ge 1,$$

$$\lim_{n\to\infty}\beta_n=\beta,\quad \lim_{n\to\infty}\alpha_n=\alpha,\quad \sum_{k=1}^{\infty}\gamma_k\beta_k=\overline{\gamma\beta},$$

$$\sum_{k=1}^{\infty}\beta_k\gamma_k=\overline{\beta\gamma},\quad \sum_{k=1}^{\infty}\gamma_k\alpha_k=\overline{\gamma\alpha} \quad \text{and}$$

$$\alpha\overline{\beta\gamma} \ne \beta\overline{\gamma\alpha}.$$

Since $\hat{K}(1), > 1/\hat{D} > \hat{K}(\infty)$ we can verify

$$\hat{K}(2)-\hat{K}(1)<0$$

and

$$\lim_{n\to\infty}[\hat{K}(n+1)-\hat{K}(n)]>0$$

Hence theorem 4.3 is proved.

Example 4.1 When $\Pr[N=k]=p(1-p)^{k-1}$ (geometric distribution), $F(t)$ is DMRL, and $\hat{K}(1) > 1/\hat{D} > \hat{K}(\infty)$, then there exists uniquely a finite optimal n^* which minimizes $K(n)$, where

$$\hat{K}(1) = \frac{\frac{p}{1-p}(1+(1-p)^2)-p}{\alpha_2-\alpha_1(1-p)}$$

and

$$\hat{K}(\infty) = \frac{\frac{1}{2+p}(q_0-1)}{\frac{\alpha}{1+p}-\beta A}$$

$$A = \sum_{k=1}^{\infty}\alpha_k(1-p)^{k-1}$$

5. Conclusion

In this paper we developed the optimal replacement model depending on age. In developing the model, we supposed the random process that determines the time and cost as a discrete probability function instead of continuous. In our model the number of minimal repairs is proposed instead of age of the system, i.e., the system is replaced whenever the number of minimal repairs reaches n times in succession. In policy we discuss the property of $\Pr[N=k]$ under which an optimum replacement period exists for minimizing the total long-run average cost per unit time. The basic composition formular is used in deriving many of the results.

Bibliography

[1] Barlow, R. and Hunter, L., "Optimum Preventive Maintenance Policies," Operations Research, Vol. 8, pp. 90--100 (1960).

[2] Karlin, S., "Total Positivity I," Stanford University Press, (1975).

[3] Kijima, M. and Sumita, U., "A Useful Generalization Renewal Theory : Counting Processes Governed by Non-Negative Markovian Increments," Journal of Applied Probability, Vol. {23}, pp. 71--88 (1986).

[4] Morimura, H., "On Some Preventive Maintenance Policies for IFR," Journal of the Operations Research Society of Japan, Vol. 12, No. 3, pp. 94--124 (1970).

[5] Nakagawa, T. and Kowada, M., "Analysis of a System with Minimal Repair and its Application to Replacement Policy," European Journal of Operations Research, 12, pp176-182. (1983)

[6] Ross, S., "Applied Probability Models with Optimization Applications," Holden-Day, San Francisco, (1970).

[7] Uematsu, K., Ohi, F., Kowada, M. and Nishida, T., "Optimal Replacement Models with Imperfect Repair," Mathematica Japonica, Vol. {49}, No. 3, pp. 385--393 (1999).