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CURVATURE BOUNDS OF EUCLIDEAN CONES OF SPHERES

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ABSTRACT. In this paper, we obtain the optimal condition of the curvature bounds guaranteeing that Euclidean cones over Aleksandrov spaces of curvature bounded above preserve the curvature bounds, by considering the Euclidean cone CS_r^n over n-dimensional sphere S_r^n of radius r. More precisely, we show that for $r < 1$, the Euclidean cone CS_r^n of S_r^n is a CBB(0) space, but not a CBA(κ)-space for any real $\kappa \in R$.

1. Introduction

In the middle of 20th century, based on ideas developed in the study of the intrinsic geometry of convex surfaces, the theory of spaces of curvature $\leq \kappa$ and $\geq \kappa$ was introduced by A. D. Aleksandrov. Even though the notion of upper and lower curvature bounds on metric spaces is defined without assumption of differentiability, it is natural to construct useful tools, which are familiar in Riemannian geometry, defined by means of intrinsic metric of the space. For instance, the concept of tangent cone at a point was defined as the Euclidean cone over the direction space at a point in a metric space of curvature bounded above in the sense of A. D. Aleksandrov. It corresponds to the tangent space of a point p in a Riemannian manifold, being itself an R_0-domain if p is a manifold point ([1]). Later Nikolaev showed in [5] that the completed tangent cone of a space X of curvature bounded above by κ, is an R_0-domain without assumption that X is locally compact or complete. From now on, a CBA(κ) space means an Aleksandrov space of

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curvature bounded above by κ. Similarly, a CBB(κ) space means an Aleksandrov space of curvature bounded below by κ.

The motivation of our study lies in the question whether the Euclidean cones over Aleksandrov spaces of curvature bounded above or below preserve the curvature bounds. Concerning this question, some facts are already known, and one of them is the following:

PROPOSITION. *X is a $CBA(1)$ space if and only if the Euclidean cone CX over X is $CBA(0)$. In general, X is a $CBA(1)$ space if and only if the κ-cone $C_\kappa X$ is a $CBA(\kappa)$ space.*

For details, we refer to [3]. The definition of κ-cone is given in Section 2 for convenience of reader. The concept of κ-cone $C_\kappa X$ over the direction space at a point of X plays an important role in the proof of Jung's theorem for CBA(κ)-spaces ([4]). On the other hand, for an Aleksandrov space X of curvature bounded below, we have a similar fact that X is a CBB(1) space if and only if CX is a CBB(0) space ([2]).

From the proposition above, the Euclidean cone CS^n_r over n-dimensional sphere S^n_r of radius r is $CBA(0)$ for $r \geq 1$. Hence it is natural to ask whether there is a number $\kappa \in R$ such that the cone CS^n_r is $CBA(\kappa)$ for $r < 1$. This paper gives a negative answer, i.e. for $r < 1$, the Euclidean cone CS^n_r is not of curvature bounded above. And we conclude that the curvature bounds conditions in the proposition above is optimal by considering the Euclidean cone over S^n_r. The main theorem of this paper is the following.

THEOREM. *For n-dimensional sphere S^n_r with radius r endowed with the length metric,*

$$\begin{cases} CS^n_r \text{ is } CBA(0) & \text{if} \quad r \geq 1\,, \\ CS^n_r \text{ is } CBB(0) & \text{if} \quad r < 1\,. \end{cases}$$

Furthermore, there is no real κ such that CS^n_r is $CBA(\kappa)$ for $r < 1$.

In order to show this theorem, we collect the preliminaries about Aleksandrov spaces in Section 2.

2. Spaces of curvature bounded above

We start off with some preliminaries about CBA(κ) spaces. Let X be a metric space with a metric d_X. A geodesic in X is a continuous map $\sigma : I \to X$ such that for some real number $\xi \geq 0$, every t in the interval $I \subset R$ has a neighborhood $B(t) \subset I$ such that $d_X(\sigma(t_1), \sigma(t_2)) = \xi e(t_1, t_2)$ for all $t_1, t_2 \in B(t)$, where e denotes the standard Euclidean distance. If one can take $B(t) = I$ for some $t \in I$, then σ is called a minimizer. The geodesic from x to y will be denoted by $\sigma(x,y)$. For a real number κ, we denote by M_κ the simply connected surface of constant curvature κ.

A triangle in X consists of three geodesic segments $\sigma_1, \sigma_2, \sigma_3$ in X whose endpoints match. A triangle with edges σ_1, σ_2, σ_3 will be denoted by $\triangle = \triangle(\sigma_1, \sigma_2, \sigma_3)$ and a triangle with vertices v_1, v_2, v_3 will be denoted by $\triangle = \triangle(v_1, v_2, v_3)$. For $\triangle$ in X, a triangle $\overline{\triangle} = \triangle(\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_3})$ in M_κ is called a comparison triangle for $\triangle$ if $L(\overline{\sigma_i}) = L(\sigma_i), 1 \leq i \leq 3$, where L denotes the arclength. A comparison triangle exists and is unique up to congruence if $L(\sigma_i) + L(\sigma_j) \geq L(\sigma_k)$ and the perimeter $P(\triangle)$ of $\triangle$ satisfies that $P(\triangle) = \sum_i L(\sigma_i) < \frac{2\pi}{\sqrt{\kappa}}$. From now on we use the convention $\frac{2\pi}{\sqrt{\kappa}} = \infty$ for $\kappa \leq 0$. One says that $\triangle$ is κ-thin, respectively κ-fat, if $d_X(x,y) \leq d_{M_\kappa}(\overline{x}, \overline{y})$, respectively $d_X(x,y) \geq d_{M_\kappa}(\overline{x}, \overline{y})$ for all points x, y on $\triangle$ and the corresponding points $\overline{x}, \overline{y}$ on $\overline{\triangle}$ in M_κ with the standard length metric d_{M_κ}. Here the corresponding point $\overline{x}$ of x means that if x is on $\sigma(v_1, v_2)$, then $L(\sigma(v_1, x)) = L(\sigma(\overline{v_1}, \overline{x}))$. For real κ an open subset U of X is called a R_κ-domain if for all $x, y \in U$ there is a geodesic $\sigma(x,y) : [0,1] \to U$ of length $d_X(x,y)$ and all triangles in U are κ-thin. We say that X has curvature at most κ, denoted by $CBA(\kappa)$, if every point $x \in X$ is contained in a R_κ-domain. On the other hand, a metric space X is said to have curvature bounded below by κ, denoted by $CBB(\kappa)$, if every point $x \in X$ has an open neighborhood $B(x)$ such that the comparison triangle $\overline{\triangle}$ in M_κ for every triangle $\triangle(p,q,r)$ in $B(x)$ has the following property: For every point z on the minimizer $\sigma(q,r)$ and for corresponding point $\overline{z} \in \sigma(\overline{q}, \overline{r})$, $d_X(p,z) \geq d_{M_\kappa}(\overline{p}, \overline{z})$, which is called the Aleksandrov convexity property.

We define the function $sin_\kappa : R \to R$ by

$$sin_\kappa(x) = \begin{cases} \frac{\sin(\sqrt{\kappa} x)}{\sqrt{\kappa}} & \text{if} \quad \kappa > 0\,, \\ x & \text{if} \quad \kappa = 0\,, \\ \frac{\sinh(\sqrt{-\kappa} x)}{\sqrt{-\kappa}} & \text{if} \quad \kappa < 0\,. \end{cases}$$

The Euclidean cone CX over a metric space (X, d_X) is homeomorphic to the quotient space of $X \times [0, \infty)$ obtained by contracting the set $X \times 0$ to a point o, called the vertex of CX. From now on, we denote by o the vertex of CX. We define the distance function d_{CX} on CX by

$$d_{CX}^2(\overline{x}, \overline{y}) = s^2 + t^2 - 2st \cos(\min\{d_X(x,y), \pi\}),$$

where $\overline{x} = (x, s)$, $\overline{y} = (y, t) \in CX$. We denote $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$, and ∞ if $\kappa \leq 0$. The κ-cone $C_\kappa X$ over X is the metric space $C_\kappa X = \{\overline{x} = (x, s) \in CX : s \leq D_\kappa/2\}$ endowed with the metric d_κ defined by

$$sin_\kappa^2 \frac{d_\kappa(\overline{x}, \overline{y})}{2} = sin_\kappa^2 \frac{s-t}{2} + sin_\kappa s \ sin_\kappa t \ \sin^2 \frac{\min\{d_X(x, y), \pi\}}{2},$$

where $\overline{x} = (x, s)$ and $\overline{y} = (y, t)$. Then $d_\kappa(\overline{x}, \overline{y}) \in [0, D_\kappa]$ for all $s, t \in [0, D_\kappa/2]$ and $x, y \in X$.

In proving the main theorem, we need the following fact ([3]).

PROPOSITION 2.1. *If σ is a minimizer in X of the length $L(\sigma) < \pi$, then $p^{-1}(\sigma)$ is a convex subset in CX isomorphic to a Euclidean sector (without vertex) with the angle $L(\sigma)$.*

We say that a metric d_X on X is interior if for every $x, y \in X$ and for each $\varepsilon > 0$ there exists an ε-midpoint $z \in X$, that is, $d_X(x, z), d_X(z, y) \leq \frac{1}{2} d_X(x, y) + \varepsilon$. It is easy to check that d_X is interior at the distances $< \pi$ if and only if d_{CX} is interior. But we remark that although d_X is interior, the restriction $d_{CX}|_{X\times t}$ of d_{CX} to the subspace $X \times t$ of CX may not be interior. For example, let $x, y \in R^{n+1}$ be points such that $e(x, y) \geq 2\pi$. Let $z \in R^{n+1}$. Then without loss of generality we can assume that $e(x, z) \geq \pi$. So

$$\begin{aligned} d_{CX}|_{X\times t}((x,t),(z,t)) &= 2t \sin\left(\frac{\min\{e(x,z),\pi\}}{2}\right) \\ &= 2t > t + \varepsilon \\ &= \frac{1}{2} d_{CX}|_{X\times t}((x,t),(y,t)) + \varepsilon \end{aligned}$$

for $\varepsilon < t$. In this case, if ε is sufficiently small then the ε-midpoint of (x, t) and (y, t) is sufficiently close to the vertex of CX.

EXAMPLE 2.1. Let RP^1 be a space obtained by identifying x and its antipodal point $-x$ of x in a circle S_1^1. Then RP^1 is $CBA(1)$. However some couple of points in RP^1 can get two midpoints in RP^1, so that some ball of radius $\frac{\pi}{2}$ may not be a 1-domain. And the cone CRP^1 has bigons in any small ball. Thus CRP^1 is not $CBA(\kappa)$.

3. Curvature bounds of Euclidean cones CS_r^n

For a real number $r > 0$ and a point $x = (\overline{x}, \alpha) \in CS_r^n - \{o\}$, we define a function r_x from the set $\mathcal{X} = CS_r^n - \{o, x\}$ to itself as follows : For $y = (\overline{y}, \beta) \in \mathcal{X}$, let $\Pi(x, o, y)$ be a flat $\{(\overline{z}, \gamma)\}$, where $\overline{z}$ is on a minimizer

$\sigma(\overline{x},\overline{y})$ and $\gamma \geq 0$, spanned by $\overrightarrow{ox} = \{(\overline{x},t) | t \geq 0\}$ and $\overrightarrow{oy}$. Then the point $r_x(y)$ is in the flat $\Pi(x,o,y)$ and satisfies that $\angle(x,o,r_x(y)) = \min\{r\angle(x,o,y),\pi\}$ and $d_{CS_r^n}(o,r_x(y)) = d_{CS_r^n}(o,y)$. Here we call $\overrightarrow{ox}$ the ray emanating from o and passing through x. Then we can easily check that $d_{CS_r^n}(x,y) = e(x,r_x(y)) = e(r_y(x),y)$, and then obtain the following main result.

THEOREM 3.1. *For n-dimensional sphere S_r^n with radius r endowed with the length metric,*

$$\begin{cases} CS_r^n \text{ is } CBA(0) & \text{if} \quad r \geq 1 \\ CS_r^n \text{ is } CBB(0) & \text{if} \quad r < 1. \end{cases}$$

Furthermore, there is no real κ such that CS_r^n is $CBA(\kappa)$ for $r < 1$.

In the case that $r \geq 1$, CS_r^n is $CBA(0)$ from the proposition in Section 1, since S_r^n is a $CBA(1)$ space. We start off the proof of Theorem 3.1 with the followings. For three points x,y,z in X we say that z is between x and y if $d_X(x,z) + d_X(z,y) = d_X(x,y)$. And z is called a midpoint of x and y if $d_X(x,z) = d_X(z,y) = \frac{1}{2}d_X(x,y)$.

LEMMA 3.1. *If z is between $x,y \in CS_r^n$, then z is in the interior of the angle $\angle(x,o,y)$.*

Proof. Let $r > 1$. Put $z = \overline{xr_x(y)} \cap \overline{o\left(\frac{r^2}{r+1}\right)_x(y)}$. Then

$$\begin{aligned} (1) \quad d_{CS_r^n}(x, {r_x}^{-1}(z)) + d_{CS_r^n}({r_x}^{-1}(z), y) &= e(x,z) + e(z, r_x(y)) \\ &= e(x, r_x(y)) \\ &= d_{CS_r^n}(x,y). \end{aligned}$$

Here (1) comes from the fact that $d_{CS_r^n}({r_x}^{-1}(z),y) = e({r_x}^{-1}(z), r_{{r_x}^{-1}(z)}$ $(y))$ and

$$\begin{aligned} \angle(r_{{r_x}^{-1}(z)}(y), o, {r_x}^{-1}(z)) &= r\left(1 - \frac{r}{r+1}\right)\angle(x,o,y) \\ &= \frac{r}{r+1}\angle(x,o,y) \\ &= \angle(z,o,r_x(y)). \end{aligned}$$

Now let $r < 1$. Let z be a point on the intersection of the line segment $\overline{xr_x(y)}$ and $\overline{or_x(r_x(y))}$. Then

$$\begin{aligned} d_{CS_r^n}(x,y) &= e(x, r_x(y)) \\ &= e(x,z) + e(z, r_x(y)) \\ &= d_{CS_r^n}(x, {r_x}^{-1}(z)) + e(r_y({r_x}^{-1}(z)), y) \\ &= d_{CS_r^n}(x, {r_x}^{-1}(z)) + d_{CS_r^n}({r_x}^{-1}(z), y). \end{aligned}$$

The third equality comes from the fact that

$$\begin{aligned} \angle(z, o, r_x(y)) &= (r - r^2)\angle(x, o, y) \\ &= r(1-r)\angle(x,o,y) \\ &= r\angle(y, o, r_y({r_x}^{-1}(z))). \end{aligned}$$

This completes the proof. □

From the routine computations:

$$\begin{cases} d_{CS_r^n}(x, {r_x}^{-1}(w)) = e(x,w) = \frac{1}{2}e(x, r_x(y)) = \frac{1}{2}d_{CS_r^n}(x,y) \\ d_{CS_r^n}({r_x}^{-1}(w), y) = e(w, r_x(y)) = \frac{1}{2}e(x, r_x(y)) = \frac{1}{2}d_{CS_r^n}(x,y), \end{cases}$$

we obtain the following result.

COROLLARY 3.1. *Let w be a point on $\overline{xr_x(y)}$ that bisects the Euclidean length of $\overline{xr_x(y)}$. Then ${r_x}^{-1}(w)$ is a midpoint of x and y.*

From Corollary 3.1 we obtain a minimizer on CS_r^n by connecting the midpoints as follows. Let $x_0, x_1 \in CS_r^n$. Then we can choose the midpoint $x_{\frac{1}{2}}$ of x_0 and x_1. Then by the similar procedure, we have the midpoints $x_{\frac{1}{4}}$ of x_0 and $x_{\frac{1}{2}}$, $x_{\frac{3}{4}}$ of $x_{\frac{1}{2}}$ and x_1, and so on. Now it is easy to find a minimizer obtained by connecting those points $x_0, \cdots, x_{\frac{1}{4}}, \cdots, x_{\frac{1}{2}}, \cdots, x_{\frac{3}{4}}, \cdots, x_1$, since S_r^n is complete.

Now we first show that if $r < 1$, then CS_r^n is not $CBA(\kappa)$ for any real κ. Let $\overline{\triangle}$ be a triangle such that the perimeter $P(\triangle) < 2\pi r$, where

$$\overline{x} = (x,s), \overline{y} = (y,t), \overline{z} = (z,u) \in CS_r^n - \{o\}.$$

Let $\overline{\triangle}$ be a comparison triangle for $\triangle$ on S_r^n. Since $r < 1$, we can choose a comparison triangle $\tilde{\triangle}$ for $\triangle$ on S_1^2. Let $\overline{\overline{x}}$ be a point on the ray $\overrightarrow{o\tilde{x}}$ such that $e(o, \overline{\overline{x}}) = s$ and let $\overline{\overline{y}}$ be a point on the ray $\overrightarrow{o\tilde{y}}$ such that $e(o, \overline{\overline{y}}) = t$ and let $\overline{\overline{z}}$ be a point on the ray $\overrightarrow{o\tilde{z}}$ such that $e(o, \overline{\overline{z}}) = u$. Then $\overline{\overline{\triangle}}$ is a Euclidean comparison triangle of $\overline{\triangle}$. Let $\overline{w}$ be a point on $\overline{\overline{y}r_{\overline{y}}(\overline{z})}$ that bisects the length of $\overline{\overline{y}r_{\overline{y}}(\overline{z})}$ and let w be a corresponding point of $\overline{w}$ on $\triangle$. Then for a corresponding point $\tilde{w}$ of w on $\tilde{\triangle}$ we get

$d_{S^n_r}(x,w) > d_{S^2_1}(\tilde{x},\tilde{w})$. Choose a point $\tilde{v}$ on a minimizer $\sigma(\tilde{x},\tilde{w})$ such that $d_{S^2_1}(\tilde{x},\tilde{v}) = d_{S^n_r}(x,w)$. Then trivially the ray $\overrightarrow{o\tilde{v}}$ passes through the interior of the triangle $\overline{\overline{\triangle}}$. So for a corresponding point $\overline{\overline{w}}$ of w on $\overline{\overline{\triangle}}$ $d_{CS^n_r}(\overline{x},\overline{w}) > e(\overline{\overline{x}},\overline{\overline{w}})$. So we have the result.

Finally we complete the proof of the theorem by showing that CS^n_r is $CBB(0)$ for $r < 1$. For each $x \in CS^n_r - \{o\}$, there exists an open ball $B(x,\varepsilon)$ about x and of radius ε such that for any triangle in $B(x,\varepsilon)$ is 0-fat by Proposition 2.1. Now we assume that $x = o$. Choose a triangle $\triangle(x,y,z)$ in $B(o,\frac{1}{2})$. First assume that $\triangle(x,y,z)$ does not contain the origin o. Then we have the result from the fact that $\triangle(x,y,z)$ has a triangulation consisting of each triangle such that the perimeter $< 2\pi r$. Secondly, if $\triangle(x,y,z)$ contains the origin o, say $\triangle(x,y,z) = \triangle(o,x,y)$. From the same reason above we assume that $\angle(x,o,y) < \pi$. Without loss of generality we assume that $d_{CS^n_r}(o,x) < d_{CS^n_r}(o,y)$. Choose a point $z \in \overline{oy}$ such that $d_{CS^n_r}(o,z) = d_{CS^n_r}(o,x)$. Then for a triangle $\triangle(x,y,z)$ we get the result. Now it remains to prove the result for a triangle $\triangle(o,x,z)$. We know that $\triangle(o,x,r_x(z))$ is a Euclidean comparison triangle for $\triangle(o,x,z)$. Let $w \in \overline{xr_x(z)}$. Then ${r_x}^{-1}(w)$ is a corresponding point of w on $\sigma(x,z)$. Then trivially

$$d_{CS^n_r}(o,{r_x}^{-1}(w)) > e(o,\overline{{r_x}^{-1}(w)}),$$

for a corresponding point $\overline{{r_x}^{-1}(w)}$ of ${r_x}^{-1}(w)$. So the proof is complete. □

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