Bull. Korean Math. Soc. **40** (2003), No. 2, pp. 301–307

ON A QUASI FIXED-POINT THEOREM

Won Kyu Kim

Abstract. The purpose of this paper is to give a generalization of the quasi fixed-point theorem due to Lefebvre, and prove a new existence theorem of equilibrium in a generalized quasi-game.

1. Introduction

The fixed point theory of multimaps is an intensively developing research area in recent decades, and there have been numerous applications in nonlinear analysis, convex analysis, game theory and other diverse branches of modern mathematics. Also, maximal element existence results are useful tools for proving equilibrium existence theorems for generalized games, e.g., see [2-4, 7]. It is easy to see that fixed point theorems and maximal element existence theorems are equivalent. Among a number of fixed point theorems for upper semicontinuous multimaps, the Fan-Glicksberg fixed point theorem is well-known and very basic in many applications, e.g., see [1, 5].

Recently, Lefebvre [6] presents a version in the case of locally convex Hausdorff topological vector spaces of results on the existence of a fixed point for upper semicontinuous in some variables and also a maximal element for lower semicontinuous in the others. However, the basic setting in [6] is too strong to apply in general applications. In fact, since we will encounter many kinds of constraints and preferences in various economic situations, so we shall consider several types of constraints and preferences defined on general settings, and suitable existence results

Received June 1, 2002.
2000 Mathematics Subject Classification: Primary 47H10, 91A14.
Key words and phrases: quasi fixed-point theorem, generalized quasi-game, equilibrium, continuous selection.
This work was supported by a grant No. R05-2002-000-00046-0 from the Basic Research Program of KOSEF.

for such correspondences should be needed to analyze such economic models, e.g., see [1].

The main purpose of this paper is to give a generalization of the quasi fixed-point theorem due to Lefebvre, and as an application, we shall prove a new existence theorem of equilibrium in a generalized quasi-game with infinite number of agents.

2. Preliminaries

We first recall the following notations and definitions. Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $cl\, A$ the closure of A in X. If A is a subset of a vector space, we shall denote by $co\, A$ the convex hull of A. If A is a non-empty subset of a topological vector space X and $S, T : A \to 2^X$ are correspondences (or multimaps), then $co\, T$, $cl\, T$, $T \cap S : A \to 2^X$ are correspondences defined by $(co\, T)(x) = co\, T(x)$, $(cl\, T)(x) = cl\, T(x)$, and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a correspondence. A correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a correspondence $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$; and T is said to be *continuous* if T is both upper semicontinuous and lower semicontinuous.

Recall that a normal topological space in which each open set is an F_σ is called *perfectly normal*, and an F_σ set in a paracompact set is also paracompact.

We now introduce the following new definition of equilibrium in the game theory. Let I be any set of agents. For each $i \in I$, let X_i be a non-empty set of actions. A *generalized quasi-game* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples (X_i, A_i, P_i) where X_i is a non-empty topological vector space, $X := \Pi_{i \in I} X_i$ a choice set, $A_i : X \times X \to 2^{X_i}$ is a constraint correspondence and $P_i : X \times X \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $(\hat{x}, \hat{y}) \in X \times X$ such that for each $i \in I$, $\hat{x}_i \in cl\, A_i(\hat{x}, \hat{y})$ and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}, \hat{y}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call Γ an *n-person quasi-game.*

The following continuous selection theorem is essential in proving our main result:

LEMMA 1 [7]. *Let X be a non-empty paracompact Hausdorff topological space and Y be a Hausdorff topological vector space. Let $T : X \to 2^Y$ be a correspondence such that each $T(x)$ is non-empty convex and for each $y \in Y$, $T^{-1}(y)$ is open in X. Then T has a continuous selection, i.e., there exists a continuous map $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.*

3. A quasi fixed-point theorem

We begin with the following generalization of the quasi fixed-point theorem due to Lefebvre [6] :

THEOREM 1. *Let I and J be any (possibly uncountable) index sets. For each $i \in I$ and $j \in J$, let X_i and Y_j be non-empty compact convex subsets of locally convex Hausdorff topological vector spaces E_i and F_j, respectively. Let $X := \Pi_{i\in I} X_i$, $Y := \Pi_{j\in J} Y_j$, and $Z := X \times Y$; and denote $(x,y) \in Z$, where $x = (x_i) \in X$ and $y = (y_j) \in Y$.*

Let $\phi_i : Z \to 2^{X_i}$ be a multimap such that

(1) $\phi_i(x,y)$ is convex for each $(x,y) \in Z$;

(2) $\phi_i^{-1}(x_i)$ is (possibly empty) open in Z for each $x_i \in X_i$;

(3) the set $W_i := \{(x,y) \in Z \mid \phi_i(x,y) \neq \emptyset\}$ is perfectly normal;

and let $\psi_j : Z \to 2^{Y_j}$ be an upper semicontinuous multimap such that

(4) $\psi_j(x,y)$ is non-empty closed convex for each $(x,y) \in Z$.

Then there exists a point $(\bar{x},\bar{y}) \in Z$ such that for each $i \in I$, either $\phi_i(\bar{x},\bar{y}) = \emptyset$ or $\bar{x}_i \in \phi_i(\bar{x},\bar{y})$, and for each $j \in J$, $\bar{y}_j \in \psi_j(\bar{x},\bar{y})$.

Proof. We first endow $\Pi_{i\in I} E_i$ and $\Pi_{j\in J} F_j$ with the product topologies; and then $\Pi_{i\in I} E_i \times \Pi_{j\in J} F_j$ is also a locally convex Hausdorff topological vector space. For each $i \in I$, by the assumption (2), we have that $W_i = \cup_{x_i \in X_i} \phi_i^{-1}(x_i)$ is open. Since W_i is perfectly normal, W_i is an F_σ. Since W_i is an F_σ in a compact set Z, W_i is paracompact. By applying Lemma 1 to the restriction ϕ_i on W_i, we can obtain that $\phi_i|_{W_i} : W_i \to 2^{X_i}$ has a continuous selection $f_i : W_i \to X_i$, i.e., $f_i(x,y) \in \phi_i(x,y)$ for each $(x,y) \in W_i$.

For each $i \in I$, we define a multimap $\phi_i' : Z \to 2^{X_i}$ by

$$\phi_i'(x,y) := \begin{cases} f_i(x,y), & \text{if } \ x \in W_i, \\ X_i, & \text{if } \ x \notin W_i. \end{cases}$$

Then for each $(x,y) \in Z$, $\phi_i'(x,y)$ is a non-empty closed convex subset of X_i. Also, ϕ_i' is an upper semicontinuous multimap on Z. In fact, for each non-empty proper open subset V of X_i, we have

$$\begin{aligned} U &:= \{(x,y) \in Z \mid \phi_i'(x,y) \subset V\} \\ &= \{(x,y) \in W_i \mid \phi_i'(x,y) \subset V\} \cup \{(x,y) \in X \setminus W_i \mid \phi_i'(x,y) \subset V\} \\ &= \{(x,y) \in W_i \mid f_i(x,y) \in V\} \cup \{(x,y) \in X \setminus W_i \mid X_i \subset V\} \\ &= \{(x,y) \in W_i \mid f_i(x) \in V\} = f_i^{-1}(V) \cap W_i. \end{aligned}$$

Since W_i is open and f_i is a continuous map on W_i, U is open, and hence ϕ_i' is upper semicontinuous on Z.

Finally, we define a multimap $\Phi : Z \to 2^Z$ by

$$\Phi(x,y) := \Pi_{i\in I}\ \phi_i'(x,y) \times \Pi_{j\in J}\ \psi_j(x,y) \quad \text{for each } (x,y) \in Z.$$

Then, by Lemma 3 in [5], Φ is an upper semicontinuous multimap such that each $\Phi(x,y)$ is non-empty closed convex. Therefore, by the Fan-Glicksberg fixed point theorem, there exists a fixed point $(\bar{x},\bar{y}) \in Z$ such that $(\bar{x},\bar{y}) \in \Phi(\bar{x},\bar{y})$, i.e., for each $i \in I$, $\bar{x}_i \in \phi_i'(\bar{x},\bar{y})$, and for each $j \in J$, $\bar{y}_j \in \psi_j(\bar{x},\bar{y})$. If $(\bar{x},\bar{y}) \in W_i$ for some $i \in I$, then $\bar{x}_i = f_i(\bar{x},\bar{y}) \in \phi_i(\bar{x},\bar{y})$; and if $(\bar{x},\bar{y}) \notin W_i$ for some $i \in I$, then $\phi_i(\bar{x},\bar{y}) = \emptyset$. Therefore, we have that for each $i \in I$, either $\phi_i(\bar{x},\bar{y}) = \emptyset$ or $\bar{x}_i \in \phi_i(\bar{x},\bar{y})$. Also, for each $j \in J$, we already have $\bar{y}_j \in \psi_j(\bar{x},\bar{y})$. This completes the proof. □

REMARKS. (i) Theorem 1 generalizes a recent quasi fixed-point due to Lefebvre [6] in the following aspects :

(a) the index sets I and J need not be finite;

(b) X_i and Y_j need not be metrizable subsets of locally convex Hausdorff topological vector spaces E_i and F_j, respectively; however the set W_i must be perfectly normal.

(ii) Theorem 1 is slightly different from previous existence theorems on fixed points and maximal elements in [2-5, 7]. In fact, if $\phi_i(x,y) := X_i$ for each $i \in I$, then Theorem 1 reduces to the fixed point result for $\psi_j(x,\cdot)$; and if $\psi_j(x,y) := Y_j$ for each $j \in J$ and $x_i \notin \phi_i(x,y)$ for each $i \in I$, then Theorem 1 reduces to the maximal element existence result for $\phi_i(\cdot,y)$.

4. An equilibrium for a generalized quasi-game

As an application of Theorem 1, we shall prove the following new equilibrium existence theorem :

THEOREM 2. *Let* $\Gamma=(X_i, A_i, P_i)_{i\in I}$ *be a generalized quasi-game where* I *be a (possibly uncountable) set of agents such that for each* $i \in I$,

(1) X_i *is a non-empty compact convex subset of a locally convex Hausdorff topological vector space, and denote* $X := \Pi_{i\in I} X_i$ *and* $Z := X \times X$;

(2) *the multimap* $A_i : Z \to 2^{X_i}$ *is upper semicontinuous such that* $A_i(x,y)$ *is a non-empty convex subset of* X_i *for each* $(x,y) \in Z$;

(3) $A_i^{-1}(x_i)$ *is (possibly empty) open for each* $x_i \in X_i$;

(4) *the multimap* $P_i : Z \to 2^{X_i}$ *is such that* $(A_i \cap P_i)^{-1}(x_i)$ *is (possibly empty) open for each* $x_i \in X_i$;

(5) *the set* $W_i := \{(x,y) \in Z \mid (A_i \cap P_i)(x,y) \neq \emptyset\}$ *is perfectly normal;*

(6) *for each* $(x,y) \in W_i$, $x_i \notin co\, P_i(x,y)$.

Then there exists an equilibrium point $(\bar{x},\bar{y}) \in Z$ *for* Γ, *i.e., for each* $i \in I$,

$$\bar{x}_i \in cl\, A_i(\bar{x},\bar{y}) \quad \text{and} \quad A_i(\bar{x},\bar{y}) \cap P_i(\bar{x},\bar{y}) = \emptyset.$$

Proof. For each $i \in I$, we first define a multimap $\phi_i : Z \to 2^{X_i}$ by

$$\phi_i(x,y) := \begin{cases} co\,(A_i \cap P_i)(x), & \text{if } (x,y) \in W_i, \\ \emptyset, & \text{if } (x,y) \notin W_i; \end{cases}$$

and for each $j \in I$, define a multimap $\psi_j : Z \to 2^{X_j}$ by

$$\psi_j(x,y) := cl\, A_j(x,y), \qquad \text{for each } (x,y) \in Z.$$

Then ψ_j is upper semicontinuous such that $\psi_j(x,y)$ is a non-empty closed convex subset of X_j for each $(x,y) \in Z$; and for each $(x,y) \in Z$, $\phi_i(x,y)$ is a convex subset of X_i. Since $W_i = \cup_{x_i \in X_i} (A_i \cap P_i)^{-1}(x_i)$, by the assumptions (4) and (5), W_i is open and hence W_i is paracompact. By the assumption (4) and Lemma 5.1 in [7], we have

$$\begin{aligned} \phi_i^{-1}(x_i) &= \{(x,y) \in Z \mid x_i \in \phi_i(x,y)\} \\ &= \{(x,y) \in W_i \mid x_i \in \phi_i(x,y)\} \\ &= \left[co\,(A_i \cap P_i)\right]^{-1}(x_i) \cap W_i \end{aligned}$$

is open in Z for each $x_i \in X_i$. Therefore, the whole assumptions of Theorem 1 are satisfied so that there exists a point $(\bar{x},\bar{y}) \in Z$ such that for each $i \in I$, either $\phi_i(\bar{x},\bar{y}) = \emptyset$ or $\bar{x}_i \in \phi_i(\bar{x},\bar{y})$, and $\bar{y}_i \in \psi_i(\bar{x},\bar{y})$.

If $\bar{x}_i \in \phi_i(\bar{x},\bar{y})$ for some $i \in I$, then $\bar{x}_i \in \phi_i(\bar{x},\bar{y}) = co\,(A_i \cap P_i)(\bar{x},\bar{y}) \subset co\, P_i(\bar{x},\bar{y})$, which contradicts the assumption (6). Therefore, $\phi_i(\bar{x},\bar{y}) = \emptyset$ for each $i \in I$, and hence $(\bar{x},\bar{y}) \notin W_i$ so that $(A_i \cap P_i)(\bar{x},\bar{y}) = \emptyset$. On the other hand, for each $i \in I$, we already have $\bar{y}_i \in \psi_i(\bar{x},\bar{y}) = cl\, A_i(\bar{x},\bar{y})$. This completes the proof. □

REMARK. Theorem 2 is different from the previous equilibrium existence theorems as we mentioned before. In fact, in Theorem 2, the multimaps A_i and P_i should be defined on the product space $X \times X$ as a choice set, and so this new type equilibrium existence result can have another reasonable interpretation in the game theory.

Finally, we give a simple example of a generalized quasi-game with finite number of agents where the previous equilibrium existence theorems in [2-4, 7] can not be applicable but Theorem 2 is suitable:

EXAMPLE. Let $\Gamma = (X_k, A_k, P_k)_{k\in I}$ be a generalized quasi-game where for each $k \in I = \{1, 2, \cdots, n\}$, let $X_k = [0,1]$ be a compact convex choice set, $X := \Pi_{k\in I} X_k$, $Z := X \times X$, and the multimaps $A_k, P_k : Z \to 2^{X_k}$ be defined as follows : for each $k \in I$ and $(x,y) \in Z$ where $x = (x_1, x_2, \cdots, x_n) \in X$, $y = (y_1, y_2, \cdots, y_n) \in X$,

$$A_k(x,y) := \begin{cases} (1 - x_k y_k, 1], & \text{for each } \ x_k y_k \neq 1, \\ \{1\}, & \text{for each } \ x_k y_k = 1, \end{cases}$$

$$P_k(x,y) := [0, x_k y_k), \qquad \text{for each } \ (x,y) \in Z.$$

Then each $A_k(x,y)$ is a non-empty convex subset of X_k. Note that $P_k(x,O) = P_k(O,y) = \emptyset$ for each $k \in I$. And, we also note that each $A_k^{-1}(t)$ is open for each $t \in X_k$. In fact, if $t = 1$, then $A_k^{-1}(1) = Z$ is open. If $0 \leq t < 1$, then we have

$$\begin{aligned} \{(x,y) \in Z \mid (x,y) \in A_k^{-1}(t)\} &= \{(x,y) \in Z \mid t \in A_k(x,y)\} \\ &= \{(x,y) \in Z \mid 1 - x_k y_k < t \leq 1\} \\ &= \{(x,y) \in Z \mid 1 - t < x_k y_k \leq 1\} \end{aligned}$$

is open in Z. Similarly, we can show that $(A_k \cap P_k)^{-1}(t)$ is open for each $t \in X_k$. Since each X_k is metrizable, the set W_i is perfectly normal. And it is clear that $x_k \notin co\ P_k(x,y) = [0, x_k y_k) \subset [0, x_k)$ for each $k \in I$. Therefore, all hypotheses of Theorem 2 are satisfied so that there exists an equilibrium $(\bar{x},\bar{y}) \in Z$ for a generalized quasi-game Γ, where $\bar{x} = (1, 1, \cdots, 1) \in X$ and $\bar{y} = (\frac{1}{3}, \frac{1}{3}, \cdots, \frac{1}{3}) \in X$, such that $\bar{x}_k \in A_k(\bar{x}, \bar{y})$ and $A_k(\bar{x}, \bar{y}) \cap P_k(\bar{x}, \bar{y}) = \emptyset$ for each $k \in I$.

Finally, it should be noted that this generalized quasi-game Γ is different from the previous generalized games in [2, 4, 7] so that Theorem 4 in Ding-Tan [4], Corollary 3 in Borglin-Keiding [2], Theorem 6.1 in Yannelis-Prabhakar [7] can not be applicable in this generalized quasi-game.

References

[1] J. P. Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam, 1979.

[2] A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorem*, J. Math. Econom. **3** (1976), 313–316.

[3] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. **38** (1952), 886–893.

[4] X. P. Ding and K. -K. Tan, *On equilibria of non-compact generalized games*, J. Math. Anal. Appl. **177** (1993), 226–238.

[5] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.

[6] I. Lefebvre, *A quasi fixed-point theorem for a product of u.s.c. or o.l.s. correspondences with an economic applications*, Set-Valued Anal. **9** (2001), 273–288.

[7] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), 233–245.

Department of Mathematics Education and Institute of Science Education, Chungbuk National University, Cheongju 361-763, Korea
E-mail: wkkim@chungbuk.ac.kr