Bull. Korean Math. Soc. **40** (2003), No. 2, pp. 289–300

A SIMPLE NASH-MOSER IMPLICIT FUNCTION THEOREM IN WEIGHTED BANACH SPACES

Sanghyun Cho[†] and Jaeseo Choi[‡]

Abstract. We prove a simplified version of the Nash-Moser implicit function theorem in weighted Banach spaces. We relax the conditions so that the linearized equation has an approximate inverse in different weighted Banach spaces in each recurrence step.

During the last several decades, "Nash-Moser implicit function theorem" helped to resolve several difficult problems of solvability for nonlinear problems, especially, nonlinear partial differential equations [6, 11].

Usually nonlinear partial differential equations (or nonlinear problems in general) can be transformed into solving the problem :

$$\phi(u) = 0,$$

where ϕ involves the variables x, the unknown function $u(x)$ and its derivatives up to the order m.

To prove implicit function theorem in infinite dimensional spaces (as spaces of functions usually are), we first linearize the equation, and then solve the linear equation so that we get the recursive solutions with appropriate recurrence estimates. The simplest one is known as Picard's iterative scheme. However when ϕ involves the derivatives of u up to order m, the Picard's scheme can not be convergent unless the linearized equation gets m derivatives (as does elliptic equations). To overcome this difficulty, Nash [11] and Moser [10] proposed another scheme involving smoothing operators so that the solution of the linearized equation

Received July 3, 2002.
2000 Mathematics Subject Classification: 47J07.
Key words and phrases: Nash-Moser, Banach spaces, Sobolev spaces.
(†)Partially supported by R01-1999-00001 from KOSEF and by R14-2002-044-01000-0 by KOSEF.

could be estimated inductively in each Sobolev space of order s. Later Hörmander proposed improved schemes [6, 7, 8] to get optimal results with respect to the regularity of the solution. However, these schemes are too complicated and rather frightening for the uninitiated reader.

In [12], Saint Raymond established a simplified version of C^∞ existence theorem so that the number of derivatives that are used (provided that it is finite) does not matter, and this method is useful when we are working on C^∞ category [1, 2, 3]. Raymond used the scheme proposed by Moser [10] which consists alternately in using Newton's scheme and Nash's smoothing operators closer and closer to the identity :

$$v_k = -\psi(u_k)\phi(u_k),\ u_{k+1} = u_k + S_k v_k,$$

and proved the convergence in C^∞ category. Here S_k is a smoothing operator and $\psi(u)$ is the right inverse of $(\partial\phi/\partial u)(u)$, that is,

$$\phi'(u)\psi(u) = I, \tag{1}$$

as an operator (I=identity). Also we need an estimate for $v = -\psi(u)\phi(u)$, so called "tame estimates":

$$|v|_s \leq C_s(\|\phi(u)\|_{s+d} + |u|_{s+d}\|\phi(u)\|_d), \tag{2}$$

in Hamilton [4].

In some cases, however, we have to deal with the case that a linearized equation has right inverse with error terms of second order, c.f.,[1, 2, 3], or, in each recurrence step, we have to solve a linearized equation with different weights in each weighted Sobolev space. For example, when we try to prove an embedding problem of a Cauchy-Riemann structure, the (approximate) linearized equation becomes an inhomogeneous Cauchy-Riemann equation on compact pseudoconvex almost complex manifolds (close to being integrable). In this case, we can not get an elliptic regularity for the solution up to the boundary. Therefore, we have to use weighted estimates for $\overline{\partial}$ [5, 9] with different weights in each Sobolev space. We then use these weighted estimates for $\overline{\partial}$ (as in (5) below) in each recurrence step in the process of Nash-Moser iteration.

In this paper we prove the Nash-Moser implicit function theorem in weighted Banach spaces. We relax the conditions in (1) and (2) as mentioned above. That is, in each k-th recurrence step, $\phi(u)+\phi'(u)(v) = 0$ has a solution with an error that depends on $(\|\phi(u)\|_{3d}^{(k)})^{1+\varepsilon}$, for $\varepsilon > 0$, where d is a positive integer, and can be estimated as

$$|v|_s^{(k)} \leq C_s t_k^{E(s)}(\|\phi(u)\|_{s+d}^{(k)} + |u|_{s+d}\|\phi(u)\|_d^{(k)}), \tag{3}$$

where $\{t_k\}$ is an increasing sequence of positive integers and the norms $|\cdot|_s^{(k)}$ (or $\|\cdot\|_s^{(k)}\|$), defined in each recurrence spaces, are Sobolev norms of order s with weight $e^{-t_k\lambda}$, where λ is a C^∞ function, and $E(s)$ is an integer depending on s. We state the main theorem as follows.

THEOREM 1. *Let $\{t_k\}_{k\geq 0}$ be a strictly increasing sequence of positive integers and suppose that $\mathcal{B}_s^k$ and B_s^k, $s,k\geq 0$, are families of Banach spaces with the following properties.*

(i) *For each fixed k, $\mathcal{B}_s^k\subset\mathcal{B}_t^k$ and $B_s^k\subset B_t^k$ if $s>t$.*

(ii) *If $|\cdot|_s^{(k)}$ and $\|\cdot\|_s^{(k)}$ denote the norms on $\mathcal{B}_s^k$ and B_s^k, respectively, then*

$$|\cdot|_s^{(k)}\geq|\cdot|_t^{(k)} \text{ and } \|\cdot\|_s^{(k)}\geq\|\cdot\|_t^{(k)} \text{ if } s>t.$$

(iii) *For each fixed k, if $\mathcal{B}_\infty^k=\bigcap_{s>0}\mathcal{B}_s^k$ and $B_\infty^k=\bigcap_{s>0}B_s^k$, then there exists an open set $U_k\subset\mathcal{B}_\infty^k$ with*

$$U_k=\{u\in\mathcal{B}_\infty^k;|u-u_0|_{3d}^{(k)}<\delta\},$$

where d is a positive integer, $u_0\in\mathcal{B}_\infty^0$, and δ is a given positive number. Furthermore, there is a C^2-map $\phi:U_k\longrightarrow B_\infty^k$ such that if $u\in U_k$ and $v,w\in\mathcal{B}_\infty^k$, then

$$\begin{aligned}\|\phi(u)\|_s^{(k)}&\leq C_s(1+|u|_{s+d}^{(k)}),\quad s\geq d\\ \|\phi'(u)(v)\|_{2d}^{(k)}&\leq C_1|v|_{3d}^{(k)},\\ \|\phi''(u)(v,w)\|_{2d}^{(k)}&\leq C_2|v|_{3d}^{(k)}|w|_{3d}^{(k)},\end{aligned}$$

(when one deals with (nonlinear) partial differential equations of order m, these estimates classically hold for $d>m+n/2$).

(iv) *There is a positive number $\varepsilon>0$ with the following properties : for each k, and for all $u\in U_k$, there exists a linear operator $\psi_k(u):B_\infty^k\longrightarrow\mathcal{B}_{k+T(\epsilon,d)+2d}^k$, such that*

$$\|\phi(u)-\phi'(u)\psi_k(u)\phi(u)\|_{2d}^{(k)}\leq C_1(\|\phi(u)\|_{3d}^{(k)})^{1+\varepsilon}, \tag{4}$$

and $v_k:=-\psi_k(u)\phi(u)$ satisfies, for each s, $d\leq s\leq k+T(\epsilon,d)+2d$, the following estimates (so called "tame estimate") :

$$|v_k|_s^{(k)}\leq C_s t_k^{E(s)}(\|\phi(u)\|_{s+d}^{(k)}+|u|_{s+d}^{(k)}\|\phi(u)\|_d^{(k)}), \tag{5}$$

where $E(s)$ is a polynomial in s, C_s is a constant, and $T(\epsilon, d)$ is an integer, for example, the smallest integer bigger than or equal to $3d+3+\frac{120(d+1)}{\epsilon}(2d+3)$.

(v) *There is $\theta_0 > 1$ such that*

$$\theta_0^{(1+\frac{\epsilon}{4})(t_k-t_{k+1})}|\cdot|_s^{(k)} \le |\cdot|_s^{(k+1)} \le \theta_0^{t_k-t_{k+1}}|\cdot|_s^{(k)},$$
$$\theta_0^{(1+\frac{\epsilon}{4})(t_k-t_{k+1})}\|\cdot\|_s^{(k)} \le \|\cdot\|_s^{(k+1)} \le \theta_0^{t_k-t_{k+1}}\|\cdot\|_s^{(k)}, \tag{6}$$

where $\epsilon > 0$ is the number in (iv) *satisfying* (4), *and hence $\mathcal{B}_s^k \subset \mathcal{B}_s^{k+1}$, $B_s^k \subset B_s^{k+1}$, for each $s \ge 0$.*

(vi) *For each k, there are smoothing operators $S_\theta : \bigcup_{s=0}^\infty \mathcal{B}_s^k \to \mathcal{B}_\infty^k$ and $\tilde{S}_\theta : \bigcup_{s=0}^\infty B_s^k \to B_\infty^k$ for all $\theta > 1$ such that for each real number s, t, there are a constant $C_{s,t}$ and an integer $E(s,t)$ such that*

$$|S_\theta v|_s^{(k)} \le C_{s,t} t_k^{E(s,t)} \theta^{s-t} |v|_t^{(k)}, \; s \ge t,$$
$$|v - S_\theta v|_s^{(k)} \le C_{s,t} t_k^{E(s,t)} \theta^{s-t} |v|_t^{(k)}, \; s \le t,$$

and the similar estimates hold for $\tilde{S}_\theta$.

Then there exist an integer B and a small number $b > 0$ such that if $\|\phi(u_0)\|_B^{(0)} < b$, for some $u_0 \in U_0$, then there exists an element $u \in U_0$ such that $\phi(u) = 0$.

REMARK 2. (a) Since $\|\phi(u)\|_{3d}^{(k)} \le 1$, we may assume that $0 < \epsilon \le 1$.
(b) In many cases, we approximate the non-linear problem up to second order error terms, and hence $\epsilon = 1$ in these cases.
(c) If λ is a smooth bounded function, we can modify λ so that $1 \le \lambda \le 1 + \frac{\epsilon}{4}$. Let $\mathcal{B}_s^k$ be the weighted Sobolev space of order s on a bounded domain $\Omega \subset \mathbb{C}^n$ with weighted norm :

$$(\|f\|_s^{(k)})^2 = \sum_{|\alpha| \le s} \int_\Omega |D^\alpha f|^2 e^{-t_k \lambda} dV, \; f \in \mathcal{B}_s^k.$$

Then (6) holds with $\theta_0 = e$.

By choosing a subsequence if necessary, we may assume that the sequence $\{t_k\}_{k \ge 0}$ satisfies

$$t_{k+1} \ge \frac{3}{2} t_k, \quad k \ge 0, \tag{7}$$

and t_0 is sufficiently large. Also, we will use a sequence of real numbers $\{\theta_k\}$ defined inductively as follows :

$$\theta_k = \theta_0^{t_k}, \quad k \geq 1, \tag{8}$$

and will use the corresponding smoothing operators S_{θ_k}. In the sequel, we set

$$\tau_0 = (1 + 2\epsilon/3) > 1, \quad \text{and} \quad \tau_k = 1 + \frac{(10-k)}{15}\epsilon, \; k = 1, 2, \ldots, 5,$$

and hence $1 < \tau_5 < \tau_4 < \tau_3 < \tau_2 < \tau_1 < \tau_0$. For a convenience, we set

$$T := T(\epsilon, d) := [\![3d + 3 + \frac{120(d+1)}{\epsilon}(2d+3)]\!], \tag{9}$$

where $[\![\Gamma]\!]$ denotes the smallest integer bigger than or equal to Γ. We first prove the following Lemma which is a crucial step in the proof of Theorem 1.

LEMMA 3. *With the same assumption as in the theorem and with the smoothing operators S_{θ_k} of the remark, the sequences*

$$v_k = -\psi_k(u_k)\phi(u_k), \quad u_{k+1} = u_k + S_{\theta_{k+1}} v_k,$$

are well defined if $\|\phi(u_0)\|_{2d}^{(0)} \leq \theta_0^{-2t_0}$ for sufficiently large t_0 ; more precisely, there exist constants $(U_t)_{t \geq d}$, and V (independent of k) such that for $k \geq 0$,

$$\begin{aligned}
&(i)_k \quad |u_k - u_0|_{3d}^{(k)} < \delta \text{ and } \|\phi(u_k)\|_{2d}^{(k)} \leq \theta_k^{-\tau_0},\\
&(ii)_k \quad |v_k|_{3d+3}^{(k)} \leq V\theta_k^{-\tau_3},\\
&(iii)_k \quad (1 + |u_{k+1}|_{s+2d}^{(k+1)}) \leq U_s \theta_{k+1}^{2d+1/2}(1 + |u_k|_{s+2d}^{(k)}),\\
&\qquad\quad d \leq s \leq k + T + 2d.
\end{aligned}$$

Proof. Since the property $(i)_k$ implies that the sequence u_k and v_k are well defined, it is sufficient to prove $(i)_k$, $(ii)_k$ and $(iii)_k$ inductively. The property $(i)_0$ is true by assumption.

Proof of $(ii)_{k+1}$. The tame estimate (5) gives, for every s, $d \leq s \leq k + T + 2d$, that

$$|v_k|_s^{(k)} \leq C_s t_k^{E(s)} \left(\|\phi(u_k)\|_{s+d}^{(k)} + |u_k|_{s+d}^{(k)} \|\phi(u_k)\|_{2d}^{(k)} \right). \tag{10}$$

For $s = d$, and using $(i)_k$ and (10), we have

(11)
$$|v_k|_d^{(k)} \le C_d t_k^{E(d)} \left(1 + |u_k - u_0|_{2d}^{(k)} + |u_0|_{2d}^{(k)}\right) \|\phi(u_k)\|_{2d}^{(k)} \le V_0 \theta_k^{-\tau_1},$$

because $t_k^{E(d)} \theta_k^{\tau_1 - \tau_0}$ is bounded.

Let T be the number defined in (9) and set $N = 4(2d+1)$. From the tame estimate (5) and the properties of $\|\phi(u_k)\|_s^{(k)}$ stated in (iii) of Theorem 1, it follows, for $d \le s \le k + T + 2d$, that

$$\begin{aligned} |v_k|_s^{(k)} &\le C_s t_k^{E(s)} \left(\|\phi(u_k)\|_{s+d}^{(k)} + |u_k|_{s+d}^{(k)} \|\phi(u_k)\|_d^{(k)}\right) \\ &\le C_s t_k^{E(s)} \left(C_{s+d}(1 + |u_k|_{s+2d}^{(k)}) + C_d(1 + |u_k|_{2d}^{(k)})|u_k|_{s+d}^{(k)}\right) \\ &\le C_s t_k^{E(s)} \left(C_{s+d} + C_{2d}(1 + \delta + |u_0|_{3d}^{(0)})\right) \cdot (1 + |u_k|_{s+2d}^{(k)}). \end{aligned} \tag{12}$$

The estimate

$$(1 + |u_j|_{T+2d}^{(j)}) \le (1 + |u_0|_{T+2d}^{(0)}) \theta_j^{N - \frac{1}{4}} \tag{13}$$

holds obviously for $j = 0$. Moreover, if it holds for some $j < k$, we obtain from (7), (13) and $(iii)_j$ that

$$\begin{aligned} (1 + |u_{j+1}|_{T+2d}^{(j+1)}) &\le U_T\, \theta_{j+1}^{2d+1/2} (1 + |u_j|_{T+2d}^{(j)}) \\ &\le U_T \theta_{j+1}^{-1/4} (1 + |u_0|_{T+2d}^{(0)}) \theta_j^N \theta_{j+1}^{2d+1-\frac{1}{4}} \\ &\le U_T \theta_{j+1}^{-1/4} (1 + |u_0|_{T+2d}^{(0)}) \theta_{j+1}^{N-\frac{1}{4}}, \end{aligned}$$

because $\theta_j^N \le \theta_{j+1}^{2N/3}$ and $(2d+1) + 2N/3 < N$. Therefore, by induction for $j \le k$ that, (13) holds provided t_0 (and hence t_1) is sufficiently large so that $\theta_0^{-t_1/4} U_T \le 1$.

Thanks to (13), we may write (12) as :

$$\begin{aligned} |v_k|_T^{(k)} &\le C_T t_k^{E(s)} \left(C_{T+d} + C_{2d}(1 + \delta + |u_0|_{3d}^{(0)})\right) \\ &\quad \cdot (1 + |u_0|_{T+2d}^{(0)}) \theta_k^{N - \frac{1}{4}} \le V_1 \theta_k^N, \end{aligned} \tag{14}$$

because $t_k^{E(s)} \theta_k^{-1/4}$ is bounded, for $d \le s \le k + T$.

Combining (9), (11), (14) and the properties (vi) in Theorem 1 of the smoothing operators, the interpolation formula, with $\bar{\theta}_k = \theta_k^{\frac{\tau_1-\tau_2}{(2d+3)}}$, can be written as

(15)

$$\begin{aligned}
&|v_k|_{3d+3}^{(k)} \\
&\leq |S_{\bar{\theta}_k} v_k|_{3d+3}^{(k)} + |v_k - S_{\bar{\theta}_k} v_k|_{3d+3}^{(k)} \\
&\leq C_{3d+3,d} t_k^{E(3d+3,d)} \bar{\theta}_k^{2d+3} |v_k|_d^{(k)} + C_{3d+3,T} t_k^{E(3d+3,T)} \bar{\theta}_k^{3d+3-T} |v_k|_T^{(k)} \\
&\leq C_{3d+3,d} V_0 t_k^{E(3d+3,d)+E(d)} \theta_k^{-\tau_2} + C_{3d+3,T} V_1 t_k^{E(3d+3,T)} \theta_k^{-4(2d+2)+N} \\
&\leq V \theta_k^{-\tau_3},
\end{aligned}$$

because $t_k^{E(3d+3,T)} \theta_k^{-4(2d+2)+N+\tau_3}$ and $t_k^{E(3d+3,d)+E(d)} \theta_k^{-\tau_2+\tau_3}$ are bounded. This proves $(ii)_{k+1}$.

Proof of $(iii)_{k+1}$. Now we want to estimate $|u_{k+1}|_{s+2d}^{(k+1)}$ in terms of $|v_k|_s^{(k)}$. Since $u_{k+1} = u_k + S_{\theta_{k+1}} v_k$, it follows, for $d \leq s \leq k+T$, that

$$\begin{aligned}
|u_{k+1}|_{s+2d}^{(k+1)} &\leq |u_k|_{s+2d}^{(k+1)} + |S_{\theta_{k+1}} v_k|_{s+2d}^{(k+1)} \\
&\leq |u_k|_{s+2d}^{(k)} + C_{s+2d,s} t_k^{E(s+2d,s)} \theta_{k+1}^{2d} |v_k|_s^{(k)}.
\end{aligned}$$

Thus one obtains from (13) that

$$\begin{aligned}
1 + |u_{k+1}|_{s+2d}^{(k+1)} &\leq W_s t_k^{E(s+2d,s)+E(s)} \theta_{k+1}^{2d} (1 + |u_k|_{s+2d}^{(k)}) \\
&\leq U_s \theta_{k+1}^{2d+1/2} (1 + |u_k|_{s+2d}^{(k)}),
\end{aligned}$$

because $t_k^{E(s+2d,s)+E(s)} \theta_{k+1}^{-\frac{1}{2}}$, $d \leq s \leq k+T+2d$, is bounded. This proves $(iii)_{k+1}$ with constants U_s does not depend on k.

Proof of $(i)_{k+1}$. Since $u_k - u_0 = \sum_{j<k} S_{\theta_{j+1}} v_j$, $\forall t \in [0,1]$, one can write, from (6) and $(ii)_k$, that

$$\begin{aligned}
|u_k + tS_{\theta_{k+1}} v_k - u_0|_{3d}^{(k+1)} &\leq \sum_{j \leq k} |S_{\theta_{j+1}} v_j|_{3d}^{(k+1)} \\
&\leq C_{3d,3d} \theta_0^{t_k - t_{k+1}} t_k^{E(3d,3d)} \sum_{j \leq k} |v_j|_{3d}^{(j)} \\
&\leq C_{3d,3d} V \theta_0^{-t_{k+1}/3} t_k^{E(3d,3d)} \sum_{j \leq k} \theta_j^{-\tau_3} \\
&\leq C_{3d,3d} V S \theta_0^{-t_{k+1}/4},
\end{aligned}$$

where $S = \sum_{j\leq\infty} \theta_j^{-\tau_3} < \infty$ is a constant. By choosing t_0 (and hence t_1) sufficiently large so that $C_{3d,3d}VS\theta_0^{-t_1/4} < \delta$, we have

$$|u_{k+1} - u_0|_{3d}^{(k+1)} = |\sum_{j<k+1} S_{\theta_{j+1}} v_j|_{3d}^{(k+1)} < \delta,$$

and this is the first part of $(i)_{k+1}$.

By virtue of Taylor's formula, we can write ;

$$\begin{aligned}\phi(u_{k+1}) &= \phi(u_k) + \phi'(u_k)S_{\theta_{k+1}}v_k \\ &\quad + \int_0^1 (1-t)\phi''(u_k + tS_{\theta_{k+1}}v_k)(S_{\theta_{k+1}}v_k, S_{\theta_{k+1}}v_k)dt \\ &= \phi_1 + \phi_2 + \phi_3,\end{aligned}$$

where

$$\begin{aligned}\phi_1 &= \phi(u_k) + \phi'(u_k)v_k \\ \phi_2 &= \phi'(u_k)(S_{\theta_{k+1}}v_k - v_k) \\ \phi_3 &= \int_0^1 (1-t)\phi''(u_k + tS_{\theta_{k+1}}v_k)(S_{\theta_{k+1}}v_k, S_{\theta_{k+1}}v_k)dt.\end{aligned}$$

First, we estimate ϕ_1. For this, we use the following two estimates ;

$$\|\phi(u_k)\|_{2d}^{(k)} \leq \theta_k^{-(1+2\varepsilon/3)} = \theta_k^{-\tau_0} \quad \text{and} \quad \|\phi(u_k)\|_{T+d}^{(k)} \leq A\theta_k^N. \tag{16}$$

Note that the second inequality comes from the properties (iii) in Theorem 1 and (14) with $s = T + d$. From (4), we have $\|\phi_1\|_{2d}^{(k+1)} = \|\phi(u_k) + \phi'(u_k)(v_k)\|_{2d}^{(k+1)} \leq C_1(\|\phi(u_k)\|_{3d}^{(k+1)})^{1+\varepsilon}$. Setting $\bar{\theta}_k = \theta_k^{\frac{2\varepsilon}{3d}}$, it follows from (16) that

$$\begin{aligned}&\|\phi(u_k)\|_{3d}^{(k)} \\ &= \|\tilde{S}_{\bar{\theta}_k}\phi(u_k)\|_{3d}^{(k)} + \|\phi(u_k) - \tilde{S}_{\bar{\theta}_k}\phi(u_k)\|_{3d}^{(k)} \\ &\leq C_{3d,2d}t_k^{E(3d,2d)}\bar{\theta}_k^d\|\phi(u_k)\|_{2d}^{(k)} + C_{3d,T+d}t_k^{E(3d,T+d)}\bar{\theta}_k^{3d-T-d}\|\phi(u_k)\|_{T+d}^{(k)} \\ &= C_{3d,2d}t_k^{E(3d,2d)}\theta_k^{-1} + C_{3d,T+d}t_k^{E(3d,T+d)}A\theta_k^{-1} \\ &= (C_{3d,2d}t_k^{E(3d,2d)} + C_{3d,T+d}t_k^{E(3d,T+d)}A)\theta_k^{-1}.\end{aligned}$$

Hence it follows from (6) that

$$\begin{aligned}
&\|\phi_1\|_{2d}^{(k+1)}\\
&\le C_1(\|\phi(u_k)\|_{3d}^{(k+1)})^{1+\varepsilon}\\
&= C_1\theta_0^{(1+\varepsilon)(t_k-t_{k+1})}(\|\phi(u_k)\|_{3d}^{(k)})^{1+\varepsilon}\\
&\le C_1\left(C_{3d,2d}t_k^{E(3d,2d)}+C_{3d,T+d}t_k^{E(3d,T+d)}A\right)^{1+\varepsilon}\theta_0^{(1+\varepsilon)(t_k-t_{k+1})}\theta_k^{-(1+\varepsilon)}\\
&\le C_1\left(C_{3d,2d}t_k^{E(3d,2d)}+C_{3d,T+d}t_k^{E(3d,T+d)}A\right)^{1+\varepsilon}\theta_0^{-(t_{k+1})(1+\varepsilon)}\\
&\le \frac{1}{3}\theta_{k+1}^{-(1+2\varepsilon/3)},
\end{aligned}$$

by choosing t_0 sufficiently large.

Next, let us estimate ϕ_2. From (6), (15) and the property (iii) of Theorem 1, we obtain that

$$\begin{aligned}
\|\phi_2\|_{2d}^{(k+1)} &\le \theta_0^{t_k-t_{k+1}}C_1|S_{\theta_{k+1}}-v_k|_{3d}^{(k)}\\
&\le C_1C_{3d,3d+3}t_k^{E(3d,3d+3)}\theta_{k+1}^{-3}|v_k|_{3d+3}^{(k)}\theta_0^{t_k-t_{k+1}}\\
&\le C_1C_{3d,3d+3}Vt_k^{E(3d,3d+3)}\theta_{k+1}^{-3}\theta_k^{-\tau_3}\theta_0^{t_k-t_{k+1}}\\
&\le \frac{1}{3}\theta_{k+1}^{-(1+2\varepsilon/3)},
\end{aligned}$$

provided t_0 is sufficiently large.

Finally, we estimate ϕ_3. By choosing t_0 sufficiently large, we have, from (15), that

$$\begin{aligned}
\|\phi_3\|_{2d}^{(k+1)} &\le (|S_{\theta_{k+1}}v_k|_{3d}^{(k+1)})^2 \le \theta_0^{2(t_k-t_{k+1})}(|S_{\theta_{k+1}}v_k|_{3d}^{(k)})^2\\
&\le \left(C_{3d,3d}t_k^{E(3d,3d)}|v_k|_{3d}^{(k)}\right)^2\theta_0^{2(t_k-t_{k+1})}\\
&\le \left(C_{3d,3d}Vt_k^{E(3d,3d)}\right)^2\theta_0^{2(t_k-t_{k+1})}\theta_k^{-2\tau_3}\\
&\le \frac{1}{3}\theta_{k+1}^{-(1+2\varepsilon/3)}.
\end{aligned}$$

If we combine the estimates of ϕ_1, ϕ_2 and ϕ_3, the second part of $(i)_{k+1}$ follows. □

We define a constant $M=\frac{\tau_4+N+1}{\tau_3-\tau_4}=\frac{15}{\epsilon}(\frac{6}{15}\epsilon+4(2d+1)+1)$, and for each integer $k\ge 0$, set

$$\Lambda(k)=\frac{k+T+(2+M)d-1}{M+1}. \tag{17}$$

The proof of the property $(ii)_k$ of Lemma 3 can be modified to prove an estimate for $|v_k|_s^{(k)}$ for every $s \geq d$.

LEMMA 4. *There exist constants $(V_s)_{s\geq d}$ such that the sequence $\{v_k\}$ of Lemma 3 satisfies, for every $k \geq 0$ and $d \leq s \leq \Lambda(k)$, that*

$$|v_k|_s^{(k)} \leq V_s t_k^{\max(E(s,d),E(s,D))} \theta_k^{-\tau_4},$$

where $D = [\![\frac{(s-d)}{(\tau_3-\tau_4)}(\tau_4 + N + 1) + s]\!]$.

Proof. Keeping the value $N = 4(2d+1)$, we obtain from (7) and $(iii)_k$ of Lemma 3 that

$$\begin{aligned}(1 + |u_{k+1}|_{s+2d}^{(k+1)})\theta_{k+1}^{-N} &\leq U_s \theta_{k+1}^{2d+1/2}(1 + |u_k|_{s+2d}^{(k)})\theta_{k+1}^{-N} \\ &\leq U_s \theta_{k+1}^{-1/2}(1 + |u_k|_{s+2d}^{(k)})\theta_{k+1}^{-N+2d+1} \\ &\leq U_s \theta_{k+1}^{-1/2}(1 + |u_k|_{s+2d}^{(k)})\theta_k^{-N},\end{aligned}$$

for $d \leq s \leq k + T + 2d$. Therefore for each fixed s, the sequence $(1 + |u_k|_{s+2d}^{(k)})\theta_k^{-N}$ is bounded, and hence there exists a constant K_s such that

$$(1 + |u_k|_{s+2d}^{(k)})\theta_k^{-N} \leq K_s.$$

Substituting this into (13), we obtain, for $d \leq s \leq k + T + 2d$, that

$$\begin{aligned}|v_k|_s^{(k)} &\leq C_s(C_{s+d} + C_{2d}(1 + \delta + |u_0|_{3d}^{(0)}))K_s t_k^{E(s)}\theta_k^N \\ &= W_s' t_k^{E(s)}\theta_k^N \leq W_s \theta_k^{N+1}, \qquad (18)\end{aligned}$$

where $N = 4(2d+1)$ does not depend on s. Now, if $d \leq s \leq \Lambda(k)$, the definition of $\Lambda(k)$ in (17) shows that $D \leq k + T + 2d$. Therefore for $d \leq s \leq \Lambda(k)$, we rewrite our interpolation formula with $\bar{\theta}_k = \theta_k^{\frac{(\tau_3-\tau_4)}{(s-d)}}$;

$$\begin{aligned}|v_k|_s^{(k)} &\leq |S_{\bar{\theta}_k} v_k|_s^{(k)} + |v_k - S_{\bar{\theta}_k} v_k|_s^{(k)} \\ &\leq C_{s,d} t_k^{E(s,d)} \bar{\theta}_k^{s-d} |v_k|_d^{(k)} + C_{s,D} t_k^{E(s,D)} \bar{\theta}_k^{s-D} |v_k|_D^{(k)} \\ &\leq C_{s,d} t_k^{E(s,d)} V \theta_k^{-\tau_4} + C_{s,D} t_k^{E(s,D)} W_D \theta_k^{-\tau_4} \\ &\leq V_s t_k^{\max(E(s,d),E(s,D))} \theta_k^{-\tau_4},\end{aligned}$$

where we have used the estimate (18). □

Proof of the Theorem 1. Let u_k and v_k be as in Lemma 3. From lemma 4 we have, for any $j \geq 0$ and $d \leq s \leq \Lambda(j)$, that

$$\begin{aligned}|S_{\theta_{j+1}} v_j|_s^{(j)} &\leq C_{s,s} t_j^{E(s,s)} |v_j|_s^{(j)} \\ &\leq C_{s,s} V_s t_j^{E(s,s)+\max(E(s,d),E(s,D))} \theta_j^{-\tau_4} \leq A_s \theta_j^{-\tau_5}.\end{aligned}$$

By (6) one thus obtains, for $d \leq s \leq \Lambda(j)$, that

$$\begin{aligned}|S_{\theta_{j+1}} v_j|_s^{(0)} &\leq \theta_0^{(1+\epsilon/4)(t_j-t_0)} |S_{\theta_{j+1}} v_j|_s^{(j)} \\ &\leq C_{s,s} A_s \theta_0^{(1+\epsilon/4)t_j} \theta_j^{-\tau_5} \\ &= C_{s,s} A_s \theta_j^{-\epsilon/12}\end{aligned} \tag{19}$$

because $(1+\epsilon/4) - \tau_5 = -\epsilon/12$. By virtue of (17), we also have

$$\Lambda(j) < s \text{ if and only if } j < \Lambda^{-1}(s) = s(M+1) - (2+M)d - T + 1. \tag{20}$$

Now for each fixed $s \geq 0$, the sequence $u_k = u_0 + \sum_{j<k} S_{\theta_{j+1}} v_j$ is convergent in $\mathcal{B}_s^0$ because

$$|\sum S_{\theta_{j+1}} v_j|_s^{(0)} \leq |\sum_{j<\Lambda^{-1}(s)} S_{\theta_{j+1}} v_j|_s^{(0)} + |\sum_{j \geq \Lambda^{-1}(s)} S_{\theta_{j+1}} v_j|_s^{(0)} < \infty,$$

where the first sum in the right is finite sum by (20), and the second sum in the right is finite by (19). Moreover, the limit $u \in \mathcal{B}_\infty^0$ of the sequence u_k satisfies

$$\begin{aligned}\|\phi(u)\|_{2d}^{(0)} &\leq \|\phi(u_k)\|_{2d}^{(0)} + \|\int_0^1 \phi'(u_k + t(u-u_k))(u-u_k)dt\|_{2d}^{(0)} \\ &\leq \|\phi(u_k)\|_{2d}^{(0)} + C_1 |u-u_k|_{3d}^{(0)},\end{aligned}$$

and by (6), one obtains that

$$\|\phi(u_k)\|_{2d}^{(0)} \leq \theta_0^{(1+\epsilon/4)(t_k-t_0)} \|\phi(u_k)\|_{2d}^{(k)} \leq \theta_k^{-5\epsilon/12},$$

for all k. Therefore, by taking limit for $k = \infty$, it follows that $\phi(u) = 0$ and this proves Theorem 1 with $B = 2d$ and $b = \theta_0^{-2t_0}$. □

References

[1] D. Catlin, *Sufficient conditions for the extension of CR structures*, J. Geom. Anal. **4** (1994), 467–538.

[2] D. Catlin and S. Cho, *Extension of CR structures on three dimensional compact pseudoconvex CR manifolds*, preprint.

[3] S. Cho, *Embedding of pseudoconvex CR manifolds with Levi-forms one degenerate eigenvalue*, Pacific J. of Math. **207** (2002), 311–344.

[4] R. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[5] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$–operator*, Acta Math. **113** (1965), 89–152.

[6] ———, *The boundary problems of physical geodesy*, Arch. Rat. Mech. Anal. **62** (1976), 1–52.

[7] ———, *Implicit function theorems*, Lectures at Standford University, Summer Quarter (1977).

[8] ———, *On the Nash-Moser implicit function theorem*, Annales Acad. Sci. Fenniae, Series A.I. Math. **10** (1985), 255–259.

[9] J. J. Kohn, *Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds*, Trans. Amer. Math. Soc. **181** (1973), 273–292.

[10] J. Moser, *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci. **47** (1961), 1824–1831.

[11] J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.

[12] S. X. Raymond, *A simple Nash-Moser implicit function theorem*, Enseign. Math. (2) **35** (1989), 217–226.

Department of Mathematics, Sogang University, Seoul 121-742, Korea
E-mail: shcho@sogang.ac.kr
jschoi@sogang.ac.kr