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ON THE STABILITY AND INSTABILITY OF A CLASS OF NONLINEAR NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This note presents sufficient conditions for Lyapunov's stability and instability of a class of nonlinear nonautonomous second-order ordinary differential equations. Such a class includes as particular cases a remarkably large number of differential equations with specific physical applications. Two successive nonlinear transformations are applied to the original differential equation in order to convert it into a more convenient form for stability analysis purposes. The obtained stability / instability conditions depend closely on the parameterization of the original differential equation.

1. Introduction

Consider the class of nonlinear non-autonomous ordinary differential equations (ODE), [1]:

$$y'' + \frac{a}{y}y'^2 + \frac{b}{x}y' + \frac{c}{x^2}y + dx^r y^s = 0 \tag{1}$$

where $(.)' \equiv d/dx$, and a, b, c, d, r and s are arbitrary constants ($r \neq -2$, $s \neq 1$). The importance of the class of models (1) arises since it includes as particular cases a number of equations useful in physical applications. For instance, it includes the Emden–Lane–Fowler equation ($a = c = 0$, $r = 0$, $d = 1$, $b = 2$), Bellman's equation ($a = b = c = 0$, $d > 0$), the Thomas–Fermi equation ($d = -1$, $r = -1/2$, $s = 3/2$), which is a particular case of Bellman's equation, the Langmuir–Blodgett equation, the Langmuir–Bogulavski equation ($a = c = 0$, $d = -1$, $b = -r = n$,

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$s = -1/2$), Ivey' s equation ($a = -1$, $b = s = 2$, $r = c = 0$, $d > 0$) and many others, [1]. The relevance of those equations is well–known. For instance, Bellman's equation includes that used in the analysis of the large deflection of an annular membrane, the Langmuir–Blodgett equation is related to the space–charge equation for cylinders and Ivey's equation which occurs in space– charge theory as well. The objective is to investigate Lyapunov's stability of (1), [2], when the argument x is interpreted as a time variable. Note that $(by'/x) \to 0$ as $x \to \infty$ and one gets after multiplying all the remaining terms by the independent variable $y, y' = y'' = 0 \Rightarrow (y^2/x^2 + dx^r y^{s+1}) \to 0$ as $x \to \infty$. Thus, $y = y' = 0$ is an equilibrium point of (1). In the following, sufficient stability and instability conditions of the ODE (1) are investigated, related to the equilibrium point, by first applying two nonlinear transformations.

2. Nonlinear transformations of the ODE

The subsequent two successive nonlinear transformations

$$y = zx^{\frac{2+r}{1-s}}; \qquad \omega(z) = xz^a \frac{dz}{dx} \tag{2}$$

and $\rho = \omega^{-1}$ convert (1), respectively, into the first-order classes of auxiliary ordinary differential equations (AODE) and inverse auxiliary ordinary differential equations (IAODE) below:

$$\frac{d\omega}{dz} = F(z) + \frac{G(z)}{\omega}; \qquad \frac{d\rho}{dz} = -\rho^2 F(z) - \rho^3 G(z), \tag{3}$$

where

$$F(z) = \left(1 - b - 2(a+1)\frac{2+r}{1-s}\right) z^a;$$

$$G(z) = \left((1-b)\frac{2+r}{1-s} - (a+1)\left(\frac{2+r}{1-s}\right)^2 - c\right) z^{2a+1} - dz^{2a+s}. \tag{4}$$

The above transformations make easy the stability and instability analysis through the use of Lyapunov's second methods. Note that the right–hand–side of the AODE has a singularity at $\omega = 0$. However, that of the IAODE is singularity-free and then it is more appropriate for stability analysis. After calculating dy/dx from (2), one gets directly

$$\frac{dy}{dx} = x^{\frac{1+r+s}{1-s}} \left(z^{-a}\omega(z) + \frac{2+r}{1-s} z\right) = x^{\frac{2+r}{1-s}} \frac{dz}{dx} + \frac{2+r}{1-s}\frac{y}{x}. \tag{5}$$

The investigation about the stability (or instability) of (1) is made through (3) since the IAODE being asymptotically stable (or unstable) implies and it is implied by the AODE being unstable (asymptotically stable). Then, the stability/ instability of the ODE will be discussed from the above intermediate stability/ instability results through the analysis of the relative growth rates of y and dy/dx, related to w, as x and z tend to infinity, via the auxiliary identity

$$\omega x^{\frac{2+r}{1-s}(a+1)} = y\left(xy^{a-1}\frac{dy}{dx} - \frac{2+r}{1-s}\right) \tag{6}$$

obtained directly from (5) and the first nonlinear transformation in (2).

3. Stability/ instability properties of the IAODE and AODE

Consider the Lyapunov's function candidate $V = 1/2\rho^2$ for the IAODE. Thus, along any trajectory of the IAODE, $V' \leq \overline{V'} = -\rho^4 G + |\rho^3 F|$. As a result, the following facts may be concluded:

Facts 1. If $F(z) \equiv 0$ and $G(z) < 0$ for all $z \neq 0$ then the IAODE (AODE) is unstable (globally asymptotically Lyapunov's stable–GALS) from Lyapunov's instability theorem [2] since $\omega = \rho^{-1}$. If $F(z) \equiv 0$ and $G(z) > 0$ for all $z \neq 0$ then the IAODE (AODE) is GALS (unstable) from Lyapunov's stability theorem, [2].

Facts 2. If $F(z) \neq 0$ for all $z \neq 0$ the sign of $\overline{V'}$ depends on its dominant right–hand–side term, which is always $-\rho^4 G(z)$ for sufficiently large as $|\rho|$ as $|z| \to \infty$. Thus, the IAODE (AODE) is still unstable (locally asymptotically Lyapunov's stable- LALS) if $G(z) < 0$ for $z \neq 0$ if $|\rho(0)|$ is sufficiently large (i.e., $|\omega(0)|$ is sufficiently small). If $G(z) > 0$ for $z \neq 0$ then the IAODE is globally Lyapunov's stable (GLS), but not necessarily asymptotically stable, with ultimate boundedness. In particular, if $|\rho\,(0)|$ is sufficiently small, so that $|\rho^3 F(z)| > |\rho^4 G(z)|$, then the zero equilibrium point of the IAODE may be locally unstable. However, the IAODE is GLS due to the ultimate boundedness property since $|\rho^3 F(z)| > |\rho^4 G(z)|$ for a sufficiently large $|\rho\,(0)|$ and the Lyapunov's function is continuously time-differentiable.

Fact 3. G(z) does not vanish as $|z| \to \infty$ if $a > 0$ from (4) since either $2a + 1 > 0$ or $2a + s > 0$ with $s \neq 1$. If $a < 0$ then G(z) does not vanish as $|z| \to \infty$ provided that $Max(1, s) > 2|a|$.

A convex and compact real interval may be defined along $\overline{V'} \equiv 0$ which contains inside the local instability region L_I for the IAODE so that ultimate boundedness takes place in its complementary set with the IAODE being GLS as established in Fact 3. The associated calculation is trivial but avoided in order to not overlength the manuscript. Fact 3 is relevant to ensure the Lyapunov's function candidate time-derivative to be either positive definite or negative definite in order to apply either the instability or the stability Lyapunov's theorems for the IAODE. From Facts 1–3, the following preliminary result is immediate and later used to obtain stability and instability properties of the ODE.

LEMMA 1. *The following items hold:*

(i) *The IAODE is unstable (so that $|\rho(z)| \to \infty$ as $|z| \to \infty$) and then the AODE is GALS (so that $\omega(z) \to 0$ as $|z| \to \infty$) if the following set of conditions holds:*

i1. $r \neq -2$; $s = 2n$ *(any $n \in Z_0^+ = Z^+ \cup \{0\}$; i.e., the set of nonnegative integers), $d > 0$.*

i2.

$$b = 1 - \frac{2(a+1)(2+r)}{1-s} \ ;$$

$$c = (1-b)\left(\frac{2+r}{1-s}\right) - (a+1)\left(\frac{2+r}{1-s}\right)^2 = (a+1)\left(\frac{2+r}{1-s}\right)^2 .$$

i3. *If $a < 0$ then $Max(1,s) > 2|a|$.*

(ii) *The IAODE is unstable for sufficiently large $|\rho(0)|$ (and, thus, $|\rho(z)| \to \infty$ as $|z| \to \infty$) if the following set of conditions holds:*

ii.1. $r \neq -2$; $s = 2(n-a) \neq 1$; $a = \dfrac{2n'-1}{2}$; $d \geq 0$ *for any* $n, n' \in Z_0^+$.

ii.2.

$$b \neq 1 - \frac{2(a+1)(2+r)}{1-s} \ ; \ c \geq (1-b)\left(\frac{2+r}{1-s}\right) - (a+1)\left(\frac{2+r}{1-s}\right)^2 .$$

ii.3. *If $a < 0$ then $Max(1,s) > 2|a|$ and simultaneous equalities for d and c do not hold.*

(iii) *The IAODE is GLS with ultimate boundedness if* (ii.1)–(ii.3) *hold except that the "$\geq$" inequalities for d and c are replaced with "$\leq$" inequalities and two equalities for d and c do not hold simultaneously.*

Proof. First note that the instability (asymptotic stability) of the IAODE implies and it is implied by the asymptotic stability (instability) of the AODE at the zero equilibrium.

(i) The instability of the IAODE follows from Lyapunov's instability theorem with $F(z) \equiv 0$ and $V(z) > 0$, $V'(z) > 0$ for all $z \neq 0$(Facts 1) and $G(z)$ does not vanish as $|z| \to \infty$ either for $a > 0$ or for $a < 0$ from (i.3) (Fact 3). The GALS property of the IAODE for $d < 0$ follows from Lyapunov's stability theorem with $G(z) = dz^{2a+s} > 0$ $(z \neq 0)$, with $(2a + s)$ being an even integer, leading to $V'(z) < 0$ for $z \neq 0$. If $d = 0$ then $F(z) = G(z) = V'(z) = 0$ and then both the IAODE and the AODE are GLS with bounded constant solutions $\rho(z) = \omega^{-1}(z) = \rho(0) = \omega^{-1}(0)$ for all nonzero $\rho(0)$.
(ii) The instability of the IAODE follows from Facts 2–3 from Lyapunov's instability theorem for large initial conditions in the same way as in (i) but with $F(z) \neq 0$ from (ii.2). Therefore, the AODE is LALS for small initial conditions $|\omega(0)|$. For small initial conditions, the dominance of $|G(z)|$ on $|F(z)|$ cannot be established so that the instability remains unproved. The IAODE is GLS with ultimate boundedness from Lyapunov's stability theorem if the constraints on d and c are changed from "$\geq$" to "$\leq$" with two simultaneous equalities being excluded in order to avoid $G(z)$ to vanish.

REMARK 1. Note that if the IAODE is (at least) GLS with ultimate boundedness, it is not ensured that the equilibrium of the AODE is GLS. In fact , if the equilibrium of the IAODE is LALS (with the IAODE being GLS), then the AODE is unstable for large initial conditions. However, if the equilibrium of the IAODE is locally unstable (with the IAODE being GLS) then the AODE is GLS.

4. The main result

The results of Lemma 1 are combined with extra results about relative growth of functions with the argument x in (6). In that way, sufficient conditions for stability and instability of the ODE are established in the following result:

THEOREM 1. *The following items hold:*
(i) *The ODE is GALS if the following conditions are fulfilled:*
(C1) *Conditions* (i.1)–(i.2) *of Lemma 1*(i) *hold:*
(C2)

$$\frac{2+r}{1-s} < -1 \ ; \ a = \frac{1}{n'} \left(n' \in Z_0^+\right) \ \textit{if} \ a \neq 0 \quad ; \quad \textit{or, alternatively,}$$

$$a \leq -1 \ ; \ \frac{2+r}{1-s} > 0 \ ; \ s > Max\,(2|a|, 1).$$

The ODE is also GALS under (C2) *if* (C1) *includes the case* $d = 0$ *and when* (C2) *includes* $a = 0$, *with* $\frac{2+r}{1-s} < -1$, *and then* $d > 0$ *in* (C1).

(ii) *The ODE is LALS if the following conditions are fulfilled:*
(C3) *Conditions* (ii.1)–(ii.2) *of Lemma 1.* (ii) *hold*
(C4) $\frac{2+r}{1-s} < -1$ *and* $n' = 1$; *i.e.,* $(a = 1/2)$ *if* $a > 0$ *or, alternatively,* $a \leq -1$; $\frac{2+r}{1-s} > 0$ *and* $s > Max\,(2|a|, 1)$ *if* $d \neq 0$ *unless* $c = 0$; *and* $d = 0$ and $c = \left(1 - b - (a+1)\left(\frac{2+r}{1-s}\right)\right)\left(\frac{2+r}{1-s}\right)$ *do not hold simultaneously.*

(iii) *The ODE is unstable if conditions* (i1)–(i3) *of Lemma 1.* (i) *hold except that* $d < 0$.

Proof. First, note that $s = 2n$ or $s > Max\,(2|a|, 1)$ ensure that (0, 0) is an equilibrium point of the ODE for $d = 0$. Also, note from (6) that if $\omega \to 0$ as $x \to \infty$ or if $|\omega|$ is bounded and the AODE is GLS and $\left(\frac{2+r}{1-s}\right)(a+1) < 0$ then $\omega x^{\frac{2+r}{1-s}(a+1)} \to 0$ so that $y\left(xy^{a-1}\frac{dy}{dx} - \frac{2+r}{1-s}\right) \to 0$ as $x, |z| \to \infty$.

(i) To prove this item two cases are analyzed separately, namely:

Case a. Assume $a > 0$ and that $y \to 0$ as $x \to \infty$. Thus, $z^a x \to \infty$ and $dz/dx = (1/z^a x)\omega \to 0$ as x, $|z| \to \infty$ from (2) so that $dy/dx \to 0$ as $x \to \infty$ from (5) if $(2+r)/(1-s) < 0$ since $x^{\frac{2+r}{1-s}}\frac{dz}{dx} \to 0$ and $\frac{2+r}{1-s}\frac{y}{x} \to \infty$ as $x \to \infty$ since $y \to 0$. The final conclusion is that if $y \to 0$ as $x \to \infty$ then $dy/dx \to 0$ as $x \to \infty$ and (0,0) is a stable equilibrium point of the ODE. Now, assume that y does not vanish asymptotically but $\omega x^{\frac{2+r}{1-s}(a+1)} \to 0$ as $x \to \infty$. Thus, $\frac{dy^a}{dx}$ cannot diverge as $x \to 0$ if $a \neq 0$. Direct calculations with (2) yield:

$$\frac{dy^a}{dx} = ay^{a-1}\frac{dy}{dx} = ay^{a-1}x^{\frac{2+r}{1-s}}\frac{dz}{dx} + a\frac{2+r}{1-s}\frac{y^a}{x}.$$

If y diverges then $\frac{dy^a}{y^a} \to \frac{2+r}{1-s}\frac{a}{x}dx$ as $x \to \infty$.

Thus, for sufficiently large finite $x_0 > 0$ such that $y^a(x_0) \neq 0$ (if $y^a(x) = 0$, for all $x \in [x_0, \infty)$ then the boundedness of $y^a(x)$ has been proved for $a > 0$), there exists a bounded vanishing function $\varepsilon : [x_0, \infty) \to R$ as $x \to \infty$ such that $\dfrac{dy^a}{y^a} = \dfrac{2+r}{1-s}\dfrac{a}{x}dx + \varepsilon(x)dx$ for $x \geq x_0$. After using direct integration with respect to x, one gets:

$$\left|\left|\frac{2+r}{1-s}\,a\,\frac{x}{x_0}\right| - \bar{\varepsilon}(x-x_0)\right| \leq \left|\frac{y^a(x)}{y^a(x_0)}\right| \leq \left|a\frac{2+r}{1-s}\frac{x}{x_0}\right| + \bar{\varepsilon}(x-x_0), \quad \text{all} \quad x \geq x_0.$$

As a result, $|y^a(x)|$ varies at most linearly with x and if it diverges then it cannot diverge faster than x. Thus, $|y(x)|$ varies at most with the same rate as $x^{1/a}$. Then, dy^a/dx cannot vanish asymptotically as $x \to \infty$ if $a = \dfrac{1}{n'} > 0 (n' \in Z^+)$ and Lemma 1 (i) holds with the AODE being GALS what leads to a contradiction if $|y(x)|$ diverges. As a conclusion, $|y(x)|$ remains bounded, even if it does not vanish asymptotically, and $\left|\dfrac{dy^a}{dx}\right| \to 0$ as $x \to \infty$. Also, $\left|\dfrac{dy}{dx}\right| \to 0$ as $x \to \infty$ since $dz/dx \to 0$ as $x \to \infty$ from (5). Now, $|y| = \left|zx^{\frac{2+r}{1-s}}\right| \leq Kx^{\frac{2+r}{1-s}+1}$ from (2) again since $dz/dx \to 0$ as $x \to \infty$. Using a similar reasoning for $|y^a|$, one concludes that $|z|$ diverges slower than linearly with x so that $y \to 0$ as $x \to \infty$ if $(2+r)/(1-s) + 1 < 0$ and the assumption that $y(x)$ does not vanish asymptotically is false. It has been proved that $y \to 0$, $dy/dx \to 0$ as $x \to \infty$ (and then the ODE is GALS or LALS) if $a = 1/n > 0\,(n \in Z^+)$ provided that $\omega \to 0$ as x, $|z| \to \infty$; i.e., the IAODE is globally unstable and the AODE is GALS from Lemma 1(i) or if the AODE is LALS from Lemma 1 (ii) (with $a = 1/2$). If ω is uniformly bounded then the ODE is also GALS from the last part of Lemma 1(i) with $d = 0$ and the IAODE and AODE being GLS. Thus, (i) has been proved for the first part of condition C2 ($a > 0$). In order to prove (i) with the second part of Conditions C2 , consider Case b below:

Case b. Assume $a \leq -1$. Thus, it cannot be directly ensured that $dz/dx \to 0$ as $x \to \infty$. Therefore, the right–hand–side of (6) vanishing asymptotically is considered without its factorization so that $\dfrac{x}{a+1}\dfrac{dy^{a+1}}{dx} - \dfrac{2+r}{1-s}y \to 0$ as $x \to \infty$. Thus, $\ln\dfrac{y^{a+1}(x)}{y^{a+1}(x_0)} < \infty$ as

$x \to \infty$ for some sufficiently large finite $x_0 > 0$ with $y^{a+1}(x_0) \neq 0$ after direct integration with respect to x. As from (7), one concludes that $\left|y^{a+1}(x)\right|$ varies no faster than linearly with $x^{1/(a+1)}$. Thus, one gets immediately that

$$\frac{dy^{a+1}}{dx} \to (a+1)\frac{2+r}{1-s}\frac{y}{x} \quad ; \quad \frac{dy}{dx} \to \frac{2+r}{1-s}\frac{y^{1-a}}{x}.$$

Since $a \leq -1, 1/(1+a) \to 0$ so that $y \to 0$ and $dy/dx \to \frac{2+r}{1-s}$ $\frac{y^{1+|a|}}{x} \to 0$ as $x \to \infty$ and the ODE is GALS for $a \leq -1$. The case $d = 0$ may be included since $\omega = \omega(0) < \infty$ from Lemma 1 (i) and then the left–hand–side of (6) still tends asymptotically to zero. It remains to be proved that the ODE is GALS with $a = 0$ (but $d > 0$) and $\frac{a+r}{1-s} < -1$. Under Lemma 1 (i) $|G(z)| \to \infty$ and as and the left–hand–side of (6) tends asymptotically to zero. Thus, its right–hand–side $x\frac{dy}{dx} - \frac{2+r}{1-s}y \to 0$ as $x \to \infty$ so that one concludes that y varies linearly with x. Thus, $\left|\frac{dy}{dx}\right| \leq K_y < \infty$. From (2) and (5) $dz/dx \to 0$ as $x \to \infty$ since $\frac{2+r}{1-s} \leq -1$ and $x^{-1}z$ is bounded as $x \to \infty$. Thus, $y \to \frac{1-s}{2+r}x\frac{dy}{dx} \to x^{\frac{2+r}{1-s}}x^{-1}z \to 0$ as $x \to \infty$ since $|x^{-1}z| < \infty$ and $\frac{2+r}{1-s} < -1$. Thus, the ODE is GALS. Item (i) has been fully proved.

(ii) follows in the same way for small initial conditions $|\omega(0)|$ if the AODE is LALS with $\omega \to 0$ as $x \to \infty$ (the IAODE being unstable for large $|\rho(0)|$) from the conditions of Lemma 1 (ii) and the above proof. For the case $a = \frac{2n'-1}{2} = \frac{1}{m} > 0$ $(m \in Z^+)$, it turns out that $m = n' = 2$ from Condition (ii.1) so that $a = 1/2$.

(iii) First, note that the AODE (IAODE) is GALS with exponential stability for $d < 0$ from Lemma 1 (i). The exponential instability follows from the fact that the Lyapunov function for the IAODE is quadratic and its time–derivative converges to zero exponentially from the form of its available overbounding function $\overline{V}'$. Therefore, $\omega \to \infty$ exponentially as $x \to \infty$ and then $\omega x^{\frac{2+r}{1-s}(a+1)}$ diverges for any finite values of r, s and a (even if $\frac{2+r}{1-s}(a+1) < 0$). Thus, either $|y| \to \infty$ as $x \to \infty$ from (6) and the instability of the AODE is already proved or $x\left|\frac{dy^a}{dx}\right| \to \infty$ as $x \to \infty$. Now, one gets from (5) that

$$\left|\frac{dy}{dx}\right| = \left|x^{\frac{1+r+s}{1-s}} z^{-a}\omega(z) + \frac{2+r}{1-s}y\right| \geq \left|x^{\frac{1+r+s}{1-s}} z^{-a}\omega(z)\right| - \left|\frac{2+r}{1-s}y\right|$$

provided that y is bounded since $\left|x^{\frac{1+r+s}{1-s}} z^{-a}\right|$ cannot diverge faster than exponentially while ω diverges exponentially as $x \to \infty$ and the AODE is still unstable. The proof of (iii) has been completed.

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