Bull. Korean Math. Soc. **40** (2003), No. 2, pp. 183–193

HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC TYPE FUNCTIONAL EQUATION

Sang Han Lee and Kil-Woung Jun

Abstract. In this paper, we prove the stability of a quadratic type functional equation

$$a^2 f\left(\frac{x+y+z}{a}\right) + a^2 f\left(\frac{x-y+z}{a}\right) + a^2 f\left(\frac{x+y-z}{a}\right)$$
$$+ a^2 f\left(\frac{-x+y+z}{a}\right) = 4f(x) + 4f(y) + 4f(z).$$

1. Introduction

In 1940, S. M. Ulam ([9]) posed the following question on the stability of homomorphisms: Given a group G_1, a metric group G_2 with a metric d and a number $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d(f(x), h(x)) < \epsilon$$

for all $x \in G_1$? This question became a source of the stability theory in the Hyers-Ulam sense. The case of approximately additive mappings was solved by D. H. Hyers ([1]) under the assumption that G_1 and G_2 are Banach spaces.

In 1978, Th. M. Rassias ([7]) generalized the result of Hyers as follows: Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \le p < 1$ be fixed. If f satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \le \theta(||x||^p + ||y||^p)$$

Received August 2, 2002.
2000 Mathematics Subject Classification: Primary 39B72.
Key words and phrases: quadratic functional equation, stability.

for some $\theta \geq 0$ and all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$||A(x) - f(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

The quadratic function $f(x) = x^2$ is a solution of the following functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

and

$$f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) \\ = 4f(x) + 4f(y) + 4f(z).$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is said to be a quadratic function.

In this paper we deal with a quadratic type functional equation

$$a^2 f\left(\frac{x+y+z}{a}\right) + a^2 f\left(\frac{x-y+z}{a}\right) + a^2 f\left(\frac{x+y-z}{a}\right) \\ + a^2 f\left(\frac{-x+y+z}{a}\right) = 4f(x) + 4f(y) + 4f(z).$$

Throughout this paper a is a nonzero real constant.

2. Solutions of a quadratic type functional equation

Throughout this section X and Y will be real linear spaces. Given a function $f : X \to Y$, consider the following equation

(2.1)
$$a^2 f\left(\frac{x+y+z}{a}\right) + a^2 f\left(\frac{x-y+z}{a}\right) + a^2 f\left(\frac{x+y-z}{a}\right) \\ + a^2 f\left(\frac{-x+y+z}{a}\right) = 4f(x) + 4f(y) + 4f(z).$$

LEMMA 1. *If an even function $f : X \to Y$ satisfies* (2.1) *for all $x, y, z \in X$ and $f(0) = 0$, then f is quadratic.*

Proof. Note that $f(-x) = f(x)$ for all $x \in X$ since f is an even function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f\left(\frac{x}{a}\right) = f(x) \tag{2.2}$$

for all $x \in X$. Using (2.2) in (2.1) we have

$$\begin{aligned} &f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) \\ &= 4f(x) + 4f(y) + 4f(z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.3) we deduce $2f(x) + 2f(y) = f(x+y) + f(x-y)$ for all $x, y \in X$. This shows that f is quadratic. □

LEMMA 2. *If an odd function $f : X \to Y$ satisfies* (2.1) *for all $x, y, z \in X$, then f is additive.*

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd function. Putting $y = z = 0$ in (2.1) we have

$$a^2 f\left(\frac{x}{a}\right) = 2f(x) \tag{2.4}$$

for all $x \in X$. Using (2.4) in (2.1) we have

$$\begin{aligned} &f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) \\ &= 2f(x) + 2f(y) + 2f(z) \end{aligned} \tag{2.5}$$

for all $x, y, z \in X$. Putting $z = 0$ in (2.5) we deduce $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. This shows that f is additive. □

REMARK. In Lemma 2, an additive mapping f is nonzero in general. But if a is a rational number and $a \neq 2$ in (2.1), then $f \equiv 0$.

THEOREM 3. *If a function $f : X \to Y$ satisfies* (2.1) *for all $x, y, z \in X$ and $f(0) = 0$, then there exist an additive mapping $A : X \to Y$ and a quadratic function $Q : X \to Y$ such that*

$$f(x) = Q(x) + A(x) \tag{2.6}$$

for all $x \in X$.

Proof. Let $A(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $A(-x) = -A(x)$ and A satisfies (2.1) for all $x, y, z \in X$. By Lemma 2, A is additive.

Let $Q(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $Q(0) = 0$, $Q(-x) = Q(x)$ and Q satisfies (2.1) for all $x, y, z \in X$. By Lemma 1, Q is quadratic. Clearly, we have $f(x) = Q(x) + A(x)$ for all $x \in X$.□

3. Stability of a quadratic type functional equation

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Recall that a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv, tw) = t^p H(u, v, w)$ for all nonnegative real numbers t, u, v and w. Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. We may assume that H is homogeneous of degree p. Given a function $f : X \to Y$, we set

$$Df(x,y,z) := a^2 f\left(\frac{x+y+z}{a}\right) + a^2 f\left(\frac{x-y+z}{a}\right) + a^2 f\left(\frac{x+y-z}{a}\right) + a^2 f\left(\frac{-x+y+z}{a}\right) - 4f(x) - 4f(y) - 4f(z)$$

for all $x, y, z \in X$.

THEOREM 4. *Assume that $\delta \geq 0$, $p \in (0, \infty) \setminus \{1\}$ and $\delta = 0$ when $p > 1$. Let an odd function $f : X \to Y$ satisfy*

$$\|Df(x,y,z)\| \leq \delta + H(\|x\|, \|y\|, \|z\|) \tag{3.1}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2}\delta + \frac{1}{|2 - 2^p|} h(x) \tag{3.2}$$

for all $x \in X$, where $h(x) = \frac{1}{4}(H(\|x\|, \|x\|, 0) + H(\|2x\|, 0, 0))$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd function. Putting $y = z = 0$ in (3.1) and then replacing x by $2x$ we have

$$\left\|a^2 f\left(\frac{2x}{a}\right) - 2f(2x)\right\| \leq \frac{1}{2}(\delta + H(\|2x\|, 0, 0)) \tag{3.3}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (3.1) we have

$$\left\|a^2 f\left(\frac{2x}{a}\right) - 4f(x)\right\| \le \frac{1}{2}(\delta + H(||x||, ||x||, 0)) \tag{3.4}$$

for all $x \in X$. By (3.3) and (3.4), we have

$$||f(2x) - 2f(x)|| \le \frac{1}{2}\delta + h(x) \tag{3.5}$$

for all $x \in X$, where $h(x) = \frac{1}{4}(H(||x||, ||x||, 0) + H(||2x||, 0, 0))$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 1$. By (3.5), we have

$$\left\|f(x) - \frac{f(2x)}{2}\right\| \le \frac{1}{4}\delta + \frac{1}{2}h(x) \tag{3.6}$$

for all $x \in X$. Using (3.6) we have

$$\begin{aligned}\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right\| &= \frac{1}{2^n}\left\|f(2^n x) - \frac{f(2\cdot 2^n x)}{2}\right\| \\ &\le \frac{1}{2^{n+2}}\delta + \frac{1}{2}2^{(p-1)n}h(x)\end{aligned} \tag{3.7}$$

for all $x \in X$ and all positive integers n. By (3.7), we have

$$\left\|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\right\| \le \sum_{k=m}^{n-1}\frac{1}{2^{k+2}}\delta + \sum_{k=m}^{n-1}\frac{1}{2}2^{(p-1)k}h(x) \tag{3.8}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence for all $x \in X$ since the right side of (3.8) converges to zero when $m \to \infty$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

$$\begin{aligned}||DA(x,y,z)|| &= \lim_{n\to\infty} 2^{-n}||Df(2^n x, 2^n y, 2^n z)|| \\ &\le \lim_{n\to\infty} 2^{-n}\delta + 2^{(p-1)n}H(||x||, ||y||, ||z||) \\ &= 0\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2, it follows that A is additive. By (3.6) and (3.7), we have

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \le \sum_{k=0}^{n-1} \frac{1}{2^{k+2}}\delta + \sum_{k=0}^{n-1} \frac{1}{2} 2^{(p-1)k} h(x) \tag{3.9}$$

for all $x \in X$ and all positive integers n. Taking the limit in (3.9) as $n \to \infty$, we get (3.2).

Now, let $A' : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned} ||A(x) - A'(x)|| &= 2^{-n}||A(2^n x) - A'(2^n x)|| \\ &\le 2^{-n}(||A(2^n x) - f(2^n x)|| + ||A'(2^n x) - f(2^n x)||) \\ &\le 2^{-n}\delta + \frac{2}{|2-2^p|} 2^{(p-1)n} h(x) \end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n\to\infty} \left(2^{-n}\delta + \frac{2}{|2-2^p|} 2^{(p-1)n} h(x)\right) = 0,$$

we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

(II) The case $p > 1$. Replacing x by $\frac{x}{2}$ in (3.5) we have

$$||2f(2^{-1}x) - f(x)|| \le 2^{-p} h(x) \tag{3.10}$$

for all $x \in X$. Using (3.10) we have

$$||2^n f(2^{-n}x) - 2^{n+1} f(2^{-(n+1)}x)|| \le 2^{-p} 2^{(1-p)n} h(x) \tag{3.11}$$

for all $x \in X$ and all positive integers n. By (3.10) and (3.11), we have

$$||2^n f(2^{-n}x) - f(x)|| \le \sum_{k=0}^{n-1} 2^{(1-p)k} 2^{-p} h(x)$$

for all $x \in X$ and all positive integers n. The rest of the proof is similar to the corresponding part of the case $0 < p < 1$. □

THEOREM 5. *Assume that $\delta \geq 0$, $p \in (0,\infty) \setminus \{2\}$ and $\delta = 0$ when $p > 2$. Let an even function $f : X \to Y$ satisfy (3.1) for all $x, y, z \in X$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that*

$$||f(x) - Q(x)|| \leq \frac{1}{4}\delta + \frac{1}{|4 - 2^p|}h(x) \tag{3.12}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||x||, ||x||, 0) + \frac{1}{4}H(||2x||, 0, 0)$.

Proof. Putting $y = x$ and $z = 0$ in (3.1) we have

$$\left|\left|a^2 f\left(\frac{2x}{a}\right) - 4f(x)\right|\right| \leq \frac{1}{2}(\delta + H(||x||, ||x||, 0)) \tag{3.13}$$

for all $x \in X$. Putting $y = z = 0$ in (3.1) and then replacing x by $2x$ we have

$$\left|\left|a^2 f\left(\frac{2x}{a}\right) - f(2x)\right|\right| \leq \frac{1}{4}(\delta + H(||2x||, 0, 0)) \tag{3.14}$$

for all $x \in X$. By (3.13) and (3.14), we have

$$||f(2x) - 4f(x)|| \leq \frac{3}{4}\delta + h(x) \tag{3.15}$$

for all $x \in X$, where $h(x) = \frac{1}{2}H(||x||, ||x||, 0) + \frac{1}{4}H(||2x||, 0, 0)$.

We divide the remaining proof by two cases.

(I) The case $0 < p < 2$. By (3.15), we have

$$\left|\left|f(x) - \frac{f(2x)}{4}\right|\right| \leq \frac{3}{16}\delta + \frac{1}{4}h(x) \tag{3.16}$$

for all $x \in X$. Using (3.16) we have

$$\begin{aligned} \left|\left|\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}\right|\right| &= \frac{1}{4^n}\left|\left|f(2^n x) - \frac{f(2 \cdot 2^n x)}{4}\right|\right| \\ &\leq \frac{3}{16}4^{-n}\delta + \frac{1}{4}2^{(p-2)n}h(x) \end{aligned} \tag{3.17}$$

for all $x \in X$ and all positive integers n. By (3.16) and (3.17), we have

$$\left|\left|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\right|\right| \leq \sum_{k=m}^{n-1} \frac{3}{16}4^{-k}\delta + \sum_{k=m}^{n-1} \frac{1}{4}2^{(p-2)k}h(x) \tag{3.18}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. We have $Q(0) = 0$, $Q(-x) = Q(x)$ and

$$\begin{aligned} ||DQ(x,y,z)|| &= \lim_{n\to\infty} 4^{-n}||Df(2^n x, 2^n y, 2^n z)|| \\ &\le \lim_{n\to\infty} (4^{-n}\delta + 2^{(p-2)n} H(||x||, ||y||, ||z||)) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X$. By Lemma 1, it follows that Q is quadratic. Putting $m = 0$ in (3.18) and letting $n \to \infty$ we have (3.12). The proof of the uniqueness of Q is similar to the proof of Theorem 4.

(II) The case $p > 2$. Replacing x by $\frac{x}{2}$ in (3.15) we have

$$||4f(2^{-1}x) - f(x)|| \le 2^{-p} h(x) \tag{3.19}$$

for all $x \in X$. Using (3.19) we have

$$||4^n f(2^{-n}x) - 4^{n+1} f(2^{-(n+1)}x)|| \le 2^{-p} 2^{(2-p)n} h(x) \tag{3.20}$$

for all $x \in X$. By (3.19) and (3.20), we have

$$||4^n f(2^{-n}x) - f(x)|| \le \sum_{k=0}^{n-1} 2^{(2-p)k} 2^{-p} h(x)$$

for all $x \in X$ and all positive integers n. The rest of the proof is similar to the corresponding part of the case $p < 2$. □

THEOREM 6. *Let $\delta \ge 0$ and $p \in (0, \infty) \setminus \{1, 2\}$. Assume that $\delta = 0$ if $p > 1$ and $||(a^2 - 3)f(0)|| = 0$ if $p > 2$. If a function $f : X \to Y$ satisfy (3.1) for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} &||f(x) - f(0) - Q(x) - A(x)|| \\ &\le \frac{3}{4}\delta + ||(a^2 - 3)f(0)|| + \frac{1}{|4 - 2^p|} h_1(x) + \frac{1}{|2 - 2^p|} h_2(x), \end{aligned} \tag{3.21}$$

(3.22)
$$\left\|\frac{f(x)+f(-x)}{2}-f(0)-Q(x)\right\| \le \frac{1}{4}\delta + \|(a^2-3)f(0)\| + \frac{1}{|4-2^p|}h_1(x),$$

and

$$\left\|\frac{f(x)-f(-x)}{2}-A(x)\right\| \le \frac{1}{2}\delta + \frac{1}{|2-2^p|}h_2(x) \tag{3.23}$$

for all $x \in X$, *where* $h_1(x) = \frac{1}{2}H(\|x\|,\|x\|,0) + \frac{1}{4}H(\|2x\|,0,0)$ *and* $h_2(x) = \frac{1}{4}(H(\|x\|,\|x\|,0) + H(\|2x\|,0,0))$.

Proof. Let $q_1(x) := \frac{1}{2}(f(x)+f(-x))$ for all $x \in X$. Then $q_1(0) = f(0)$, $q_1(-x) = q_1(x)$ and

$$\|Dq_1(x,y,z)\| \le \delta + H(\|x\|,\|y\|,\|z\|)$$

for all $x, y, z \in X$.

Let $q(x) := q_1(x) - q_1(0)$ for all $x \in X$. Then $q(0) = 0$, $q(-x) = q(x)$ and

$$\begin{aligned}\|Dq(x,y,z)\| &= \|Dq_1(x,y,z) - (4a^2-12)q_1(0)\| \\ &\le \|Dq_1(x,y,z)\| + \|(4a^2-12)q_1(0)\| \\ &\le \delta + \|(4a^2-12)f(0)\| + H(\|x\|,\|y\|,\|z\|)\end{aligned}$$

for all $x, y, z \in X$. By Theorem 5, there exists a unique quadratic function $Q : X \to Y$ satisfying (3.22).

Let $g(x) := \frac{1}{2}(f(x)-f(-x))$ for all $x \in X$. Then $g(-x) = -g(x)$ and

$$\|Dg(x,y,z)\| \le \delta + H(\|x\|,\|y\|,\|z\|)$$

for all $x, y, z \in X$. By Theorem 4, there exists a unique additive mapping $A : X \to Y$ satisfying (3.23). Clearly, we have (3.21) for all $x \in X$. □

Define a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(a,b,c) = (a^p + b^p + c^p)\theta$ where $\theta \ge 0$ and $p \in (0,\infty)$. Then H is homogeneous of degree p. Thus we have the following corollaries.

COROLLARY 7. *Assume that* $\delta \ge 0$, $p \in (0,\infty)\setminus\{1\}$ *and* $\delta = 0$ *when* $p > 1$. *Let an odd function* $f : X \to Y$ *satisfy*

$$\|Df(x,y,z)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{1}{2}\delta + \frac{2+2^p}{4|2-2^p|}\theta\|x\|^p$$

for all $x \in X$.

COROLLARY 8. *Assume that* $\delta \geq 0$, $p \in (0,\infty)\setminus\{2\}$ *and* $\delta = 0$ *when* $p > 2$. *Let an even function* $f : X \to Y$ *satisfy*

$$||Df(x,y,z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$ *and* $f(0) = 0$. *Then there exists a unique quadratic function* $Q : X \to Y$ *such that*

$$||f(x) - Q(x)|| \leq \frac{1}{4}\delta + \frac{4+2^p}{4|4-2^p|}\theta||x||^p$$

for all $x \in X$.

COROLLARY 9. *Let* $\delta \geq 0$ and $p \in (0,\infty)\setminus\{1,2\}$. *Assume that* $\delta = 0$ if $p > 1$ and $||(a^2-3)f(0)|| = 0$ if $p > 2$. *If* a function $f : X \to Y$ satisfy

$$||Df(x,y,z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, *then there exist* a *unique quadratic function* $Q : X \to Y$ *and* a *unique additive mapping* $A : X \to Y$ *such that*

$$\begin{aligned} &||f(x) - f(0) - Q(x) - A(x)|| \\ &\leq \frac{3}{4}\delta + ||(a^2-3)f(0)|| + \left(\frac{4+2^p}{4|4-2^p|} + \frac{2+2^p}{4|2-2^p|}\right)\theta||x||^p, \end{aligned}$$

$$\left|\left|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\right|\right| \leq \frac{1}{4}\delta + ||(a^2-3)f(0)|| + \frac{4+2^p}{4|4-2^p|}\theta||x||^p,$$

and

$$\left|\left|\frac{f(x)-f(-x)}{2} - A(x)\right|\right| \leq \frac{1}{2}\delta + \frac{2+2^p}{4|2-2^p|}\theta||x||^p$$

for all $x \in X$.

References

[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[2] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aeq. Math. **44** (1992), 125–153.

[3] S. -M. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. **126** (1998), 3137–3143.

[4] ______, *Quadratic functional equations of Pexider type*, Int. J. Math. Math. Sci. **24** (2000), 351–359.
[5] Y. -H. Lee and K. -W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
[6] J. C. Parnami and H. L. Vasudeva, *On Jensen's functional equation*, Aeq. Math. **43** (1992), 211–218.
[7] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[8] T. Trif, *Hyers-Ulam-Rassias stability of a Jensen type functional equation*, J. Math. Anal. Appl. **250** (2000), 579–588.
[9] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science Editions, Wiley, New York, 1964.

Sang Han Lee, Department of Cultural Studies, Chungbuk Provincial University of Science & Technology, Okcheon 373-807, Korea
E-mail: shlee@ctech.ac.kr

Kil-Woung Jun, Department of Mathematics, Chungnam National University, Taejeon 305-764, Korea
E-mail: kwjun@math.chungnam.ac.kr