International Journal of Reliability and Applications
Vol. 4, No. 1, pp. 41-50, 2003

Comparison of Parameter Estimation for Weibull Distribution

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Abstract. This paper represents the first comprehensive comparison of the Newton-Raphson's method and Simple Iterative Procedure (SIP) in the maximum likelihood estimation of the two-parameter Weibull distribution. Computer simulation is employed to compare these two methods for multiply censored, singly censored data (Type I or Type II censoring) and complete data. Results indicate the Newton-Raphson's with the Menon's estimated value, as an initial point remains the effective iterative procedure for estimating the parameters.

Key Words : *maximum likelihood estimation (MLE), Menon's method, Newton-Raphons' method, simple iterative procedure (SIP), two-parameter Weibull distribution*

1. INTRODUCTION

The two-parameter Weibull distribution is often used in life testing and reliability theory, because it models either increasing or decreasing failure rate in a simple manner. Many authors have proposed various methods in order to obtain the estimates of these two parameters--the shape parameter β and the scale parameter θ (see Cohen (1965), Thoman et al. (1969), Lowless (1982), Nelson (1982), Bain and Engelhard (1991), Keats

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et al. (1997)). The method of maximum likelihood is the most general method of estimating distribution parameters. In order to find the maximum likelihood estimation of the two parameters, the nonlinear equations must be solved. The Newton-Raphson method is used in many studies. Qiao and Tsokos (1994) introduced a new method "Simple Iterative Procedure (SIP)" to solve the two-parameter Weibull distribution. The simulation is used to compare the effects of the Newton-Raphson method with the Simple Iterative Procedure for complete data and censored data.

Most numerical iterative procedure is unstable and may be diverge without an appropriate initial point. Menon (1963) proposed a straightforward method to estimate the shape parameter β of the two-parameter Weibull distribution. This estimated value would be used as a starting point in these two procedures. It takes less computation time than the arbitrary starting point in both methods. The simulation results show that the Newton-Raphson method with the estimated initial point is superior to the Simple Iterative Procedure for nearly all cases with complete data. The Newton-Raphson method also holds promise for use with censored data.

2. ESTIMATION OF THE TWO-PARAMETER WEIBULL DISTRIBUTION

The two-parameter Weibull probability density function is

$$f(t;\theta,\beta) = \frac{\beta t^{\beta-1}}{\theta^{\beta}} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right] \quad (1)$$

where $t \geq 0, \theta > 0, \beta > 0$. The scale parameter θ is also called the "characteristic life". Then the Weibull cumulative distribution function is

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \quad t > 0 \quad (2)$$

2.1 Multiply Censored Data

The likelihood function for the multiply censored data is given by

$$L = f(t_{1,f},\ldots,t_{r,f},t_{1,s},\ldots,t_{m,s}) = C\prod_{i=1}^{r} f(t_{i,f})\prod_{j=1}^{m}\left[1 - F(t_{j,s})\right] \quad (3)$$

where C is a constant, $f(\cdot)$ is the density function and $F(\cdot)$ is the distribution function. There are r failures at times $t_{1,f}, t_{2,f}, \cdots, t_{r,f}$ and m units with censoring times $t_{1,s}, t_{2,s}, \cdots, t_{m,s}$.

With f(t) and F(t) given by (1), by (2), respectively, the logarithm of the likelihood function becomes

$$\ln L = \ln C + r\ln\beta - r\beta\ln\theta + (\beta-1)\sum_{i=1}^{r}\ln t_{i,f} - \frac{\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}}{\theta^{\beta}} \tag{4}$$

Upon differentiating (4) with respect to β and θ equating each result to zero, two equations must be simultaneous satisfied to obtain the estimates of β and θ. These equations are given by

$$\frac{\partial \ln L}{\partial\beta} = \frac{r}{\beta} - r\ln\theta + \sum_{i=1}^{r}\ln t_{i,f} + \frac{\ln\theta}{\theta^{\beta}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}\right) - \frac{1}{\theta^{\beta}}\left(\sum_{i=1}^{r} t_{i,f}^{\beta}\ln t_{i,f} + \sum_{j=1}^{m} t_{j,s}^{\beta}\ln t_{j,s}\right) = 0 \tag{5}$$

$$\frac{\partial \ln L}{\partial\theta} = -\frac{r\beta}{\theta} + \beta\theta^{-\beta-1}\left(\sum_{i=1}^{r} t_{i,f}^{\beta} + \sum_{j=1}^{m} t_{j,s}^{\beta}\right) = 0$$

In the above form, we have

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{r}\ln t_{i,f}^{\hat{\beta}} + \sum_{j=1}^{m}\ln t_{j,s}^{\hat{\beta}}}{r}\right)^{1/\hat{\beta}} \tag{6}$$

and $\hat{\beta}$ is the solution of

$$\frac{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}}\ln t_{i,f} + \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}}\ln t_{j,s}\right)}{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}}\right)} - \frac{1}{\hat{\beta}} = \frac{1}{r}\sum_{i=1}^{r}\ln t_{i,f} \tag{7}$$

If we define

$$S_{1,f} = \sum_{i=1}^{r}\ln t_{i,n}, S_{2,f}^{(l)} = \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}}, \qquad S_{2,s}^{(l)} = \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}} \tag{8}$$

$$S_{3,f}^{(l)} = \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}}\ln t_{i,f}, S_{3,s}^{(l)} = \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}}\ln t_{j,s}$$

$$S_{4,f}^{(l)} = \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln^2 t_{i,f}, S_{4,s}^{(l)} = \sum_{j=1}^{m} t_{j,s}^{\hat{\beta}} \ln^2 t_{j,s}$$

The Newton-Raphson iterative procedure is employed. Equation (7) becomes

$$\hat{\beta}_{l+1} = \hat{\beta}_l - \frac{\left\{ \frac{\hat{\beta}_l}{r} S_{1,f} (S_{2,f}^{(l)} + S_{2,s}^{(l)}) - \hat{\beta}_l (S_{3,f}^{(l)} + S_{3,s}^{(l)}) + (S_{2,f}^{(l)} + S_{2,s}^{(l)}) \right\}}{\left\{ \frac{1}{r} S_{1,f} \left((S_{2,f}^{(l)} + S_{2,s}^{(l)}) + \hat{\beta}(S_{3,f}^{(l)} + S_{3,s}^{(l)}) \right) - \hat{\beta}_l (S_{4,f}^{(l)} + S_{4,s}^{(l)}) \right\}} \tag{9}$$

The Simple Iterative Procedure (Qiao and Tsokos, 1994) is employed. Equation (7) becomes

$$\hat{\beta}_{l+1} = \frac{\hat{\beta}_l + \dfrac{r(S_{2,f}^{(l)} + S_{2,s}^{(l)})}{r(S_{3,f}^{(l)} + S_{3,s}^{(l)}) - S_{1,f}(S_{2,f}^{(l)} + S_{2,s}^{(l)})}}{2}. \tag{10}$$

The Menon's estimate of β (Menon, 1963) is a straightforward method; therefore the estimated starting point is given by

$$\hat{\beta}_0 = \left(\frac{\dfrac{6}{\pi^2} \left(\sum_{i=1}^{r} \ln^2 t_{i,f} - \dfrac{\left(\sum_{i=1}^{r} \ln t_{i,f} \right)^2}{r} \right)}{r-1} \right)^{-1/2} \tag{11}$$

2.2 Type I or Type II Censoring Data

The likelihood function for the first r observations from a sample size n drawn from the two-parameter Weibull distribution in both Type I and Type II censoring is given by

$$L = f(t_{1,n},\dots,t_{r,n}) = C\prod_{i=1}^{r} f(t_{i,n})[1-F(t_*)]^{n-r} \tag{12}$$

where $t_* = t_0$, the time of cessation of the test for Type I censoring, and $t_* = t_r$, the time of the rth failure for Type II censoring. The maximum likelihood equations for the parameters θ and β are given by

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + (n-r)t_*^{\hat{\beta}}}{r} \right)^{1/\hat{\beta}} \tag{13}$$

and $\hat{\beta}$ is the solution of

$$\frac{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln t_{i,f} + (n-r)t_*^{\hat{\beta}} \ln t_* \right)}{\left(\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + (n-r)t_*^{\hat{\beta}} \right)} - \frac{1}{\hat{\beta}} = \frac{1}{r}\sum_{i=1}^{r} \ln t_{i,f} \tag{14}$$

Therefore, equations (8) become

$$\begin{aligned} S_{1,f} &= \sum_{i=1}^{r} \ln t_{i,n}, S_{2,f}^{(l)} = \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}}, \qquad S_{2,s}^{(l)} = (n-r)t_*^{\hat{\beta}} \\ S_{3,f}^{(l)} &= \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln t_{i,f}, S_{3,s}^{(l)} = (n-r)t_*^{\hat{\beta}} \ln t_* \\ S_{4,f}^{(l)} &= \sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln^2 t_{i,f}, S_{4,s}^{(l)} = (n-r)t_*^{\hat{\beta}} \ln^2 t_* \end{aligned} \tag{15}$$

then equation (9) becomes

$$\hat{\beta}_{l+1} = \hat{\beta}_l - \frac{\left\{ \frac{\hat{\beta}_l}{r} S_{1,f}(S_{2,f}^{(l)} + S_{2,s}^{(l)}) - \hat{\beta}_l(S_{3,f}^{(l)} + S_{3,s}^{(l)}) + (S_{2,f}^{(l)} + S_{2,s}^{(l)}) \right\}}{\left\{ \frac{1}{r} S_{1,f}\left((S_{2,f}^{(l)} + S_{2,s}^{(l)}) + \hat{\beta}(S_{3,f}^{(l)} + S_{3,s}^{(l)}) \right) - \hat{\beta}_l(S_{4,f}^{(l)} + S_{4,s}^{(l)}) \right\}} \tag{16}$$

and equation (10) becomes

$$\hat{\beta}_{l+1} = \frac{\hat{\beta}_l + \dfrac{r(S_{2,f}^{(l)} + S_{2,s}^{(l)})}{r(S_{3,f}^{(l)} + S_{3,s}^{(l)}) - S_{1,f}(S_{2,f}^{(l)} + S_{2,s}^{(l)})}}{2} \tag{17}$$

2.3 Completed Censored Data

Simply replace r with n in the $S_{i,f}^{l}$ equations (8), i=1,2,3,4 and ignore (treat as zero) the $S_{i,s}^{l}$ portions of equations (8). The maximum likelihood equations for the parameters θ and β are given by

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{n} \ln t_{i,f}^{\hat{\beta}}}{n} \right)^{1/\hat{\beta}} \tag{18}$$

and $\hat{\beta}$ is the solution of

$$\frac{\sum_{i=1}^{n} t_{i,f}^{\hat{\beta}} \ln t_{i,f}}{\sum_{i=1}^{n} t_{i,f}^{\hat{\beta}}} - \frac{1}{\hat{\beta}} = \frac{1}{n} \sum_{i=1}^{n} \ln t_{i,f} \tag{19}$$

then equation (9) becomes

$$\hat{\beta}_{l+1} = \hat{\beta}_l - \frac{\left\{ \dfrac{\hat{\beta}_l}{n} S_{1,f} S_{2,f}^{(l)} - \hat{\beta}_l S_{3,f}^{(l)} + S_{2,f}^{(l)} \right\}}{\left\{ \dfrac{1}{n} S_{1,f} \left(S_{2,f}^{(l)} + \hat{\beta}_l S_{3,f}^{(l)} \right) - \hat{\beta}_l S_{4,f}^{(l)} \right\}} \tag{20}$$

and equation (10) becomes

$$\hat{\beta}_{l+1} = \frac{\hat{\beta}_l + \dfrac{n S_{2,f}^{(l)}}{n S_{3,f}^{(l)} - S_{1,f} S_{2,f}^{(l)}}}{2} \tag{21}$$

3. SIMULATION RESULTS

In this section, complete and censored data were used to compare the effects of the Newton-Raphson procedure and the Simple Iterative Procedure described in section 2.

The tolerance=0.001 is given by the stopping criterion for both methods. For complete data, the Weibull with different parameters θ, β and sample size was used to generate 10,000 random samples from IMSL subroutine DRNWIB (IMSL, 1993), a FORTRAN-based procedure for generating Weibull variates. 10,000 random samples were also generated, using DRNWIB for each of three censored cases of sample size 20 with 12 failures and 8-censored units. The three censored data patterns are (1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0), (0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1) and (1 0 1 0 1 0 1 0 0 0 0 0 0 1 0 1 0 1 0 1) where 0 is a failure unit and 1 is nonfailure unit. These patterns represent just three of 20! / 12! patterns possible with 12 failures among 20 units and hence are not intended to represent this case. They do represent left, right and multiply censored patterns, respectively. The computer simulation was implemented in a FORTRAN program.

The results of the simulation are shown in Tables 1 - 2. Table 1 reports simulation results for the average convergence steps including the average estimated parameter θ and β with complete data. Table 2 reports simulation results for the average convergence steps including the average estimated parameter θ and β with censored data.

Table 1, which reports the average convergence steps with complete data, suggests that the Newton-Raphson with the Menon's estimate value is best. With respect to no specific starting point, the Simple Iterative Procedure converged in all cases except two cases and the Newton-Raphson procedure failed to converge in some cases. However, both methods with the Menon's estimate value can converge faster in all cases. The computation of the Menon's estimate is a straightforward. Therefore, the convergence of the Newton-Raphson procedure is not a problem in this case.

Table 2, which reports the average convergence steps with censored data, suggests that the Newton-Raphson with the Menon's estimate is best. With respect to no specific starting point, both methods failed to converge in all three cases. However, our enthusiasm must be tempered by the fact that we examined only three censored failure patterns with one degree of censoring (12 of 20 failures in each case). There is a need to study how the method performs as censoring increases. Also these two iterative procedures are based on the maximum likelihood estimation method. The simulation results of the average estimated values of the parameters by these two procedures with complete or censored data are shown very close.

4. CONCLUSION

In this paper, an initial starting point of the iteration can be derived from the Menon's estimate on equation (11) for the parameter estimation of the two-parameter Weibull distribution. Based on simulation studies, it is concluded that the Newton-Raphson procedure with the estimated initial starting point is still the best method with the complete or censored data. The comparison study is useful and has many good properties for estimation of Weibull parameters under complete and censored data conditions. Further research with a more efficient method to estimate the two-parameter Weibull distribution is indicated.

Table 1. The average convergence steps with complete data in 10,000 simulation runs.

sample size	θ	β	$\hat{\beta}_0$	Newton-Raphson	SIP
25	3.0	2.0	5.0	fail	6.61
			3.0	4.40	5.37
			0.1	8.19	9.38
			Menon's estimate	2.96	3.90
50	3.0	2.0	5.0	fail	6.57
			3.0	4.40	5.41
			0.1	8.07	9.19
			Menon's estimate	2.84	3.69
25	5.0	0.5	10.0	fail	fail
			5.0	fail	8.07
			0.1	5.97	6.28
			Menon's estimate	2.70	3.05
50	5.0	0.5	10.0	fail	fail
			5.0	fail	7.99
			0.1	5.98	6.14
			Menon's estimate	2.66	2.88
75	100.0	8.0	10.0	3.86	5.55
			5.0	4.44	4.84
			Menon's estimate	2.99	4.39
100	7.0	12.0	10.0	3.80	5.05
			5.0	5.08	6.47
			Menon's estimate	3.02	4.58

Table 2. The average convergence steps with censored data in 10,000 simulation runs.

sample size=20	θ	β	$\hat{\beta}_0$	Newton-Raphson	SIP
Case 1	100	0.5	3.0	fail	Fail
			Menon's estimate	5	5
	100	1.0	3.0	fail	fail
			Menon's estimate	5	6
	100	2.0	3.0	fail	fail
			Menon's estimate	6	7
Case 2	100	0.5	3.0	fail	fail
			Menon's estimate	3	3
	100	1.0	3.0	fail	fail
			Menon's estimate	4	4
	100	2.0	3.0	fail	fail
			Menon's estimate	4	4
Case 3	100	0.5	3.0	fail	fail
			Menon's estimate	3	3
	100	1.0	3.0	fail	fail
			Menon's estimate	3	3
	100	2.0	3.0	fail	fail
			Menon's estimate	3	4

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