The Korean Communications
in Statistics Vol. 10 No. 1, 2003
pp. 233-237

A Note on the Asymptotic Property of S^2 in Linear Regression Model with Correlated Errors[1)]

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Abstract

An asymptotic property of the ordinary least squares estimator of the disturbance variance is considered in the regression model with correlated errors. It is shown that the convergence in probability of S^2 is equivalent to the asymptotic unbiasedness. Beyond the assumption on the design matrix or the variance-covariance matrix of disturbances error, the result is quite general and simplify the earlier results.

keywords: convergence in probability, convergence in L_1 , uniform integrability, asymptotic unbiasedness, strictly stationary process

1. Introduction

Consider a familiar regression model,

$$y_t = \boldsymbol{x}_t' \boldsymbol{\beta} + \varepsilon_t, \qquad t = 1, 2, \ldots, n \tag{1.1}$$

where y_t is an observation, $\boldsymbol{x}_t$ a k -vector of nonstochastic regressors, $\boldsymbol{\beta}$ a k -vector of unknown parameters and ε_t a disturbance error. For the model, the method of least squares plays a central role in the inference of parameters. However, the correlated disturbance error reduce the efficiency of the Ordinary Least Squares(OLS) estimators. Thus many researchers have studied the efficiency of OLS estimator relative to Generalized Least Squares(GLS) estimator, or the limiting behavior of OLS estimator. Although the prime interest for this topic might be the OLS estimator of regression parameter itself, Kramer (1991), Kramer and Berghoff (1991), Baltagi and Kramer (1994), Song (1994) and Lee and Kim (1996) have studied the asymptotic properties of OLS estimator of disturbance errors under various assumptions on the variance-covariance matrix of disturbance errors or the nonstochastic regressor matrix. In all those articles, the asymptotic unbiasedness or the convergence in probability of S^2 were

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discussed separately. Thus once they proved the convergence in probability of S^2 , the asymptotic unbiasedness was proved by different frame work.

It is well known that the L_1 -convergence is more stringent convergence concept compared with the convergence in probability. Thus in general, it required more sever assumptions to establish the L_1 -convergence. Also it is not easy to show directly that a certain estimator converges in L_1 to the variance of the disturbance errors. This make it hard to establish the necessary conditions for L_1 -convergence of S^2 . On the other hand the necessary conditions for the convergence in probability can be somewhat generalized.

In this note we will show the equivalence of L_1 -convergence and convergence in probability by the uniform integrability of the sequence of S^2 . Thus once we obtain the convergence in probability result, the asymptotic unbiasedness of S^2 can be achieved.

2. The Result

Let $E(\underline{\varepsilon_n}\underline{\varepsilon_n}') = \sigma_\varepsilon^2 V_n$ where $\underline{\varepsilon_n} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)'$. Neudecker(1977) provided the useful inequality for proving the asymptotic unbiasedness of S^2 ,

$$0 \le \frac{1}{n-k} \sum_{i=1}^{n-k} \lambda_i \le E\left(\frac{S^2}{\sigma_\varepsilon^2}\right) \le \frac{1}{n-k} \sum_{i=k+1}^{n} \lambda_i \le \frac{n}{n-k}. \tag{2.1}$$

where λ_i 's are the increasing sequence in magnitude of eigenvalues of V_n . Since the upper bound of $E(S^2/\sigma_\varepsilon^2)$ tends to 1 as $n \to \infty$, we can show the asymptotic unbiasedness, if the mean of $n-k$ smallest eigenvalues of V_n tends to 1 as $n \to \infty$. Hence the proof of asymptotic unbiasedness rely on the eigenvalues of V_n . However, it is not easy to calculate the eigenvalue of V_n in general. Once we have the result of the convergence in probability, this situation can be overcome by utilizing simple probabilistic results.

Let $\{X_n\}_{n=1}^{\infty}$ and X be random variables defined on a probability space (Ω, F, P) . It is well known that $X_n \to X$ in L_p if and only if $X_n \to X$ in probability and $E|X_n|^p \to E|X|^p$. Also suppose that X_n is uniformly integrable. If $X_n \to X$ in probability, then $E|X| < \infty$ and $X_n \to X$ in L_1 . Now S_n^2 , as a function of n , is a sequence of random variables defined on some probability space. If we can show that the sequence S_n^2 is uniformly integrable, the convergence in probability and L_1 convergence can be used interchangeably.

However, in spite of simple definition of uniform integrability, it is not easy to check whether a sequence is uniformly integrable or not. Thus we refer some propositions which can be find in the contexts of most probability theory such as Chung (1974) and are useful in what follows.

Proposition 1. Let $\{Y_n\}$ be a sequence of identically distributed random variables with $E(|Y_1|) < \infty$. Then $\{Y_n\}$ is uniformly integrable.

Proposition 2. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables on a probability space (Ω, F, P) . Suppose that $\{Y_n\}$ is uniformly integrable and $|X_n| \le |Y_n|$ for all n . Then $\{X_n\}$ is also uniformly integrable.

Proposition 3. Let $\{X_n\}$ be uniformly integrable. Then $\{\frac{1}{n}\sum_{t=1}^{n} X_n\}$ is uniformly integrable as well.

Now we are ready to prove our assertion.

Theorem 2.1. Consider model (1.1). If ε_t , $t = 1, 2, \ldots$ are identically distributed random variables with finite second moment, then the sequence of OLS estimator $\{S_n^2\}$ is uniformly integrable.

Proof: Since $y_t - E(y_t)$'s are identically distributed random variables, so are $(y_t - E(y_t))^2$ and $\{(y_t - E(y_t))^2\}_{t=1}^{\infty}$ is an uniformly integrable sequence. Hence by Proposition 3, $\{\frac{1}{n}\sum_{t=1}^{n}(y_t - E(y_t))^2\}$ is uniformly integrable.

Let $\boldsymbol{y}_n$ be an $n \times 1$ vector of observations and $\boldsymbol{X}_n$ be corresponding design matrix. We have seen that $\frac{1}{n}(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta})'(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta})$ is an uniformly integrable sequence. Define $\boldsymbol{P}_{X_n} = \boldsymbol{X}_n(\boldsymbol{X}_n'\boldsymbol{X}_n)^{-}\boldsymbol{X}_n$ and $\boldsymbol{M}_{X_n} = \boldsymbol{I} - \boldsymbol{P}_{X_n}$. Then the OLS estimator of disturbance variance σ_ε^2 can be written as $S_n^2 = \boldsymbol{y}'_n\boldsymbol{M}_{X_n}\boldsymbol{y}_n/(n-k)$. Since $\boldsymbol{P}_{X_n}$ is a nonnegative definite matrix and $\boldsymbol{X}_n'\boldsymbol{M}_{X_n} = \boldsymbol{0}$, we have

$$
\begin{aligned}
0 \le \frac{1}{n}\boldsymbol{y}'_n\boldsymbol{M}_X\boldsymbol{y}_n &= \frac{1}{n}(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta})'\boldsymbol{M}_{X_n}(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta}) \\
&\le \frac{1}{n}(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta})'(\boldsymbol{y}_n - \boldsymbol{X}_n\boldsymbol{\beta}).
\end{aligned}
$$

This shows that $\frac{1}{n}\boldsymbol{y}'_n\boldsymbol{M}_X\boldsymbol{y}_n$ is uniformly integrable. But $\frac{1}{n}\boldsymbol{y}'_n\boldsymbol{M}_X\boldsymbol{y}_n - S_n^2 \to 0$ almost surely, we can conclude that S_n^2 is uniformly integrable as well.

Example 2.1 Suppose that $\frac{1}{n}\underline{\varepsilon_n}'\underline{\varepsilon} \xrightarrow{p} \sigma_\varepsilon^2$ and $\lambda_n^{\max}=o(n)$ where $\lambda_n^{\max}$ is the largest eigenvalue of V_n . Then $(n-k)S_n^2=\underline{\varepsilon_n}'\underline{\varepsilon_n}-\underline{\varepsilon_n}'P_{X_n}\underline{\varepsilon_n}$ and

$$
\begin{aligned}
\mathrm{E}(\underline{\varepsilon_n}'\mathbf{P}_{\mathbf{X}_n}\underline{\varepsilon_n}) &= \sigma_\varepsilon^2 \mathrm{tr}(\mathbf{P}_{\mathbf{X}_n}\mathbf{V}_n) \\
&\le \sigma_\varepsilon^2 k\lambda_n^{\max},
\end{aligned}
$$

we can conclude $\frac{1}{n}\underline{\varepsilon_n}'P_{X_n}\underline{\varepsilon_n}\to 0$ in probability and hence $S_n^2\to\sigma_\varepsilon^2$ in probability. This result is due to Kramer and Berghoff (1991). Because of the uniform integrability of S_n^2 , The convergence in probability also implies that S_n^2 is an asymptotic unbiased estimator of σ_ε^2 .

Example 2.2 Suppose the diturbance error in model (1.1) is generated by

$$\varepsilon_t=\theta\varepsilon_{t-1}+\delta_t,\quad t=2,3,\ldots,n$$

where δ_t 's are independently and identically distributed random variables with mean 0, and variance σ_δ^2 , $\mathrm{Cov}(\varepsilon_s,\delta_t)=0$ for all $t>s$, and $|\theta|<1$. The variance-covariance matrix of $\underline{\varepsilon_n}$ is given by

$$
\mathrm{Var}(\underline{\varepsilon_n})=\sigma_\varepsilon^2\mathbf{V}_n=\sigma_\varepsilon^2\begin{pmatrix}
1 & \theta & \theta^2 & \cdots & \theta^{n-1} \\
\theta & 1 & \theta & \cdots & \theta^{n-2} \\
\theta^2 & \theta & 1 & \cdots & \theta^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta^{n-1} & \theta^{n-2} & \theta^{n-3} & \cdots & 1
\end{pmatrix}.
$$

It is well known that the eigenvalue of V_n is bounded above. Also it can be shown that $\mathrm{Var}(\underline{\varepsilon_n}'\underline{\varepsilon_n}/n)=o(n)$ if $\mathrm{E}(\delta_t^4)<\infty$. Thus S^2 converges to σ_ε^2 in probability. Hence by the Theorem 2.1, $S_n^2\to\sigma_\varepsilon^2$ in L_1 which in turn implies that S_n^2 is asymptotically unbiased. This might simplify the earlier result of Song(1994).

3. Strictly Stationary Error Process

Consider model (1.1). Suppose that the sequence of the disturbance error is strictly stationary with $\varepsilon_t=\sum_{i=0}^{\infty}a_i\delta_{t-i}$, where δ_t is a sequence of *i.i.d.* random variables with mean zero and finite variance, and a_i is an absolutely summable sequence with $\sum_{i=0}^{\infty}a_i z^i\neq 0$ for $|z|\le 1$ in the complex plane. It is well known that the spectral density of ε_t is bounded from below and bounded from above. One such process is the well-known ARMA process.

For the model, Lee and Kim (1996) showed the asymptotic unbiasedness of S^2 under fairly mild assumptions on regressor matrix. Note that, in view of Theorem 4.2.1. and Theorem 6.2.1. of Fuller (1976), the conditions in Example 2.1 are satisfied if we assume the normality of δ_t . Thus a more stringent convergence result can be obtained without the assumption on regressor matrix. That is, S_n^2 converges in L_1 , and hence it is an asymptotic unbiased estimator. An example of this kind is shown in Example 2.2. when the error process are produced by $\varepsilon_t = \theta\varepsilon_{t-i} + \delta_t$. See also Song(1994).

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[Received March 2003, Accepted April 2003]