

Some Results on Availability of Repairable Component and Repairable Coherent System

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Abstract

Availability is an important measure of performance of a repairable component. In this paper, the explicit expression for the availability of a repairable component, which is subject to the policy II(Age Replacement Policy) of Barlow and Hunter (1960), is obtained and the existence of the steady state availability is shown. The steady state availabilities of the model are also obtained for the cases when the mean of the minimal repair time is increasing at a geometric rate or linearly increasing. In order to show the importance and the utility of the obtained result, we also consider an illustrative example of the repairable coherent system whose components are repairable, and the obtained results are applied to derive the steady state availability of the whole system. In this situation, we can see that the condition of the existence of the steady state availability for each component is essential. Some remarks on the optimal replacement policy that maximizes the steady state availability are also given.

Keywords : Availability, Limiting efficiency, Repairable component, Minimal repair, Replacement

1. Introduction

Consider a component which can be in one of two states, namely 'up(on)' and 'down(off)'. By 'up(on)' we mean the component is still functioning and by 'down(off)' we mean the component is not functioning; in the latter case the component is being repaired or replaced, depending on whether the component is repairable or not. Let $X(t)=1$ if the component is up at time t , and $X(t)=0$ if it is not. An important characteristic of a repairable component is *availability*. The availability at time t is defined by

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$$A(t) = P(X(t) = 1) = E[X(t)],$$

which is the probability that the component is operational at time t . It is very difficult to obtain an explicit expression for $A(t)$ except for a few simple cases. However, in practice when $t \rightarrow \infty$, usually the convergence of $A(t)$ is rapid, thus the limit $A = \lim_{t \rightarrow \infty} A(t)$ is good enough for measuring the performance of the component *provided this limit exists*. This limit is called the *steady state availability* of the component. Some other kinds of availability which are useful in practical applications can be found in Birolini (1985, 1994) and Hoyland and Rausand (1994). Another measure of performance associated with a repairable component is *efficiency*, which is defined by

$$Eff_t = \frac{E(U(t))}{t},$$

where $U(t)$ is the total amount of functioning time during $(0, t]$, that is, $U(t) = \int_0^t X(u) du$. This measure can be interpreted as the expected proportion of time that the component is in the operational state during $(0, t]$. The *limiting efficiency* is defined by

$$Eff_\infty = \lim_{t \rightarrow \infty} Eff_t = \lim_{t \rightarrow \infty} \frac{E(U(t))}{t},$$

which was first considered by Barlow and Hunter (1960), and then this measure means the long-run expected proportion of time that the component is in the up(operational) state.

Note that if $A = \lim_{t \rightarrow \infty} A(t)$ exists, then Eff_∞ also exists and the relationship $\lim_{t \rightarrow \infty} A(t) = Eff_\infty$ holds (Barlow and Proschan, 1975, ch. 7). However, the $A = \lim_{t \rightarrow \infty} A(t)$ does not always exist even if Eff_∞ exists, which can be easily seen by considering the following simple example.

Example 1.

Consider a component that is activated and functioning at time $t=0$. Whenever the component fails, it is repaired completely. Let X_1, X_2, \dots denote the successive lifetimes of the component. Likewise we assume the corresponding repair times Y_1, Y_2, \dots . Furthermore, we assume that $X_i = \mu$ a.s., $Y_i = \nu$, a.s. for all $i=1, 2, \dots$, where μ and ν are fixed constants; that is, the lifetimes and repair times are all fixed constants and they are mutually independent. In this case, it obviously holds that

$$Eff_\infty = \lim_{t \rightarrow \infty} \frac{E(U(t))}{t} = \frac{\mu}{\mu + \nu}.$$

However, the availability $A(t)$ has its value 0 and 1 periodically as $t \rightarrow \infty$, thus its limit, the

steady state availability, does not exist.

Numerous authors have studied the following replacement policy: The component is renewed(or replaced) every time its age reaches at T . For each intervening failure only minimal repair is done(Age Replacement Policy). Barlow and Hunter (1960) first proposed this model and showed that there exists optimal policy maximizing limiting efficiency when the failure rate of the component is increasing failure rate(IFR). However, the existence of the steady state availability of the model has not been reported.

In this paper, the existence of the steady state availability is shown and some remarks on the importance of the property are given. Furthermore, the steady state availabilities of the model are also obtained for the cases when the mean of the minimal repair time is increasing. Also we consider the coherent repairable system whose components are repairable, and it is shown that the obtained results can be applied to derive the steady state availability of the whole system.

2. Model Description

We consider a component with failure rate $\lambda(t)$. Let this component be renewed every time its age reaches at T . For each intervening failure only minimal repair is done. Assume that during the repair of the system it does not age, and we also assume that after each renewal the component state is as good as new state. Now we introduce the notations and random variables that are needed.

T_i : the time of the completion of the i^{th} renewal with $T_0 \equiv 0$, $i = 1, 2, \dots$.

N_i' : the total number of failures in the i^{th} renewal period, $(T_{i-1}, T_i]$, $i = 1, 2, \dots$.

N_i : $N_i \equiv N_i' + 1$, $i = 1, 2, \dots$.

$X_{i,j}$: the lifetime of the component which has been renewed $(i-1)$ times and has been minimally repaired $(j-1)$ times after the time of the $(i-1)^{th}$ renewal, $i = 1, 2, \dots, j = 1, 2, \dots, N_i$. (Note that the distribution of $X_{i,j}$ depends on N_i and

that $\sum_{j=1}^{N_i} X_{i,j} = T$.)

$F_{(r)j}(t)$: the conditional distribution function of $X_{i,j}$ given $N_i = r$.

$\bar{F}_{(r)j}(t)$: $\bar{F}_{(r)j}(t) \equiv 1 - F_{(r)j}(t)$.

$E(X_{i,j} | N_i = r)$: $E(X_{i,j} | N_i = r) \equiv \mu_{(r)j}$, $i = 1, 2, \dots$, $j = 1, 2, \dots, r$, $r = 2, 3, \dots$.

$Y_{i,j}$: the repair time which corresponds to $X_{i,j}$, $i = 1, 2, \dots, j = 1, 2, \dots, N_i$.

$Z_{i,j}$: $Z_{i,j} \equiv \sum_{m=1}^j (X_{i,m} + Y_{i,m})$, $i = 1, 2, \dots$, $j = 1, 2, \dots, N_i - 1$.

Z_i : $Z_i \equiv \sum_{m=1}^{N_i} (X_{i,m} + Y_{i,m}) = T + \sum_{m=1}^{N_i} Y_{i,m}$, $i=1,2,\dots$, i.e., the length of the i^{th}

renewal period. (Obviously, $T_n = \sum_{i=1}^n Z_i = nT + \sum_{i=1}^n \sum_{j=1}^{N_i} Y_{i,j}$, $n=1,2,\dots$.)

$F_{(r)Z_{i,j}}(t)$: the conditional distribution function of $Z_{i,j}$ given $N_i=r$, $i=1,2,\dots$,
 $j=1,2,\dots,r-1$, $r=2,\dots$

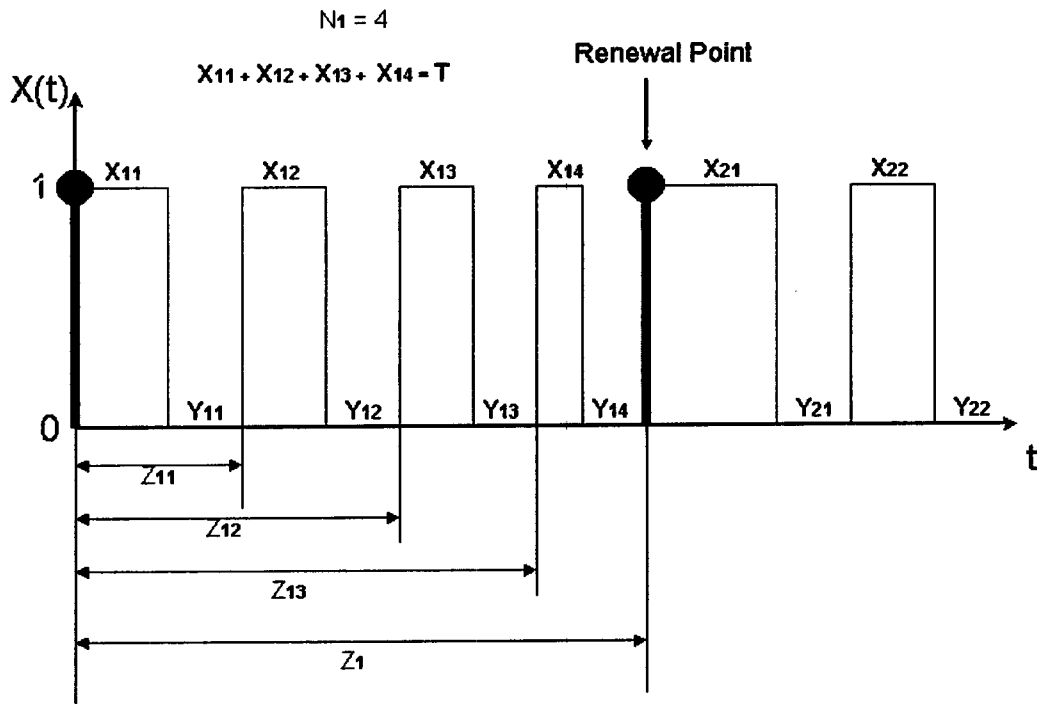
$\bar{F}_{(r)j+1|Z_{i,j}=s}(t)$: the conditional survivor function of $X_{i,j+1}$ given $Z_{i,j}=s$ and
 $N_i=r$, that is, $P\{X_{i,j+1} \geq t | Z_{i,j}=s, N_i=r\}$, $i=1,2,\dots$, $j=1,2,\dots,r-1$,
 $r=2,3,\dots$.

$H(t)$: the distribution of Z_i , $i=1,2,\dots$.

$H^{(n)}(t)$: n -fold convolution of $H(t)$.

$M_H(t)$: $M_H(t) \equiv \sum_{n=1}^{\infty} H^{(n)}(t)$.

$I_T(t)$: $I_T(t) \equiv \begin{cases} 1 & \text{if } t \leq T \\ 0 & \text{otherwise.} \end{cases}$



[Figure 1.] The state diagram for availability model

Now the assumptions are described. We assume that the distribution of $Y_{i,j}$ is (when $N_i \neq 1$), a continuous distribution $G_1(y)$ with mean ν_1 for $i=1,2,\dots$, $j=1,2,\dots,N_i-1$, and is a continuous distribution $G_2(y)$ with mean ν_2 for $i=1,2,\dots$, $j=N_i$. Furthermore, assume that the Z_i 's are mutually independent for $i=1,2,\dots$. Observe that, for the case of $N_i=1$, the corresponding renewal period consists of $X_{i,1}=T$ and $Y_{i,1}$, and the distribution function of $Y_{i,1}$ is $G_2(y)$. Note that $Z_{i,j}$'s are not defined in this case. Also observe that, in this case, $E(Y_{i,1})=\nu_2$ and $Z_i=(T+Y_{i,1})$.

For the sake of a better understanding of many complicated notations and random variables, a state diagram is presented in the above Figure 1.

3. Results on Availability

Under the assumptions described in section 2, the explicit expression for $A(t)$ and the steady state availability is given in the following theorem.

Theorem 1. The explicit expression of the availability at time t , $A(t)$, of the model is given by

$$A(t) = A_0(t) + \int_0^t A_0(t-u) dM_H(u), \quad t \geq 0,$$

where

$$A_0(t) = \exp(-\Lambda(T)) \cdot I_T(t) + \sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \\ \times \left[\bar{F}_{(r)1}(t) + \sum_{j=1}^{r-1} \int_0^t \bar{F}_{(r)j+1|Z_{1,j}=s}(t-s) dF_{(r)Z_{1,j}}(s) \right],$$

and the steady state availability of the model exists and it is given by

$$A = \lim_{t \rightarrow \infty} A(t) \\ = \frac{T}{T + \Lambda(T) \cdot \nu_1 + \nu_2}, \quad (1)$$

where $\Lambda(T) \equiv \int_0^T \lambda(u) du$.

proof.

Observe that

$$A_0(t) \equiv P\{X(t)=1, t \leq T_1\} \\ = \sum_{r=1}^{\infty} P\{X(t)=1, t \leq T_1 | N_1=r\} \cdot P\{N_1=r\} \\ = I_T(t) \cdot \exp(-\Lambda(T)) + \sum_{r=2}^{\infty} \left[P\{X_{1,1} \geq t | N_1=r\} \right]$$

$$\begin{aligned}
& + \sum_{j=1}^{r-1} P\{Z_{1,j} < t \leq Z_{1,j} + X_{1,j+1} \mid N_1 = r\} \Big] \cdot \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \\
& = \exp(-\Lambda(T)) \cdot I_T(t) + \sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \\
& \quad \times \left[\bar{F}_{(r)1}(t) + \sum_{j=1}^{r-1} \int_0^t \bar{F}_{(r)j+1|Z_{1,j}=s}(t-s) dF_{(r)Z_{1,j}}(s) \right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& P\{X(t) = 1, T_n < t \leq T_{n+1}\} \\
& = \sum_{r=1}^{\infty} P\{X(t) = 1, T_n < t \leq T_{n+1}, N_{n+1} = r\} \\
& = \sum_{r=1}^{\infty} P\{X(t) = 1, T_n < t \leq T_{n+1} \mid N_{n+1} = r\} \cdot P\{N_{n+1} = r\} \\
& = P\{T_n < t \leq T_n + T \mid N_{n+1} = 1\} \cdot \exp(-\Lambda(T)) \\
& \quad + \sum_{r=2}^{\infty} \left[P\{T_n < t \leq T_n + X_{n+1,1} \mid N_{n+1} = r\} \right. \\
& \quad \left. + \sum_{j=1}^{r-1} P\{T_n + Z_{n+1,j} < t \leq T_n + Z_{n+1,j} + X_{n+1,j+1} \mid N_{n+1} = r\} \right] \cdot \\
& \quad \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \\
& = \int_0^t \exp(-\Lambda(T)) \cdot I_T(t-u) dH^{(n)}(u) \\
& \quad + \sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \cdot \left[\int_0^t \{ \bar{F}_{(r)1}(t-u) \right. \\
& \quad \left. + \sum_{j=1}^{r-1} \int_0^{t-u} \bar{F}_{(r)j+1|Z_{n+1,j}=s}(t-u-s) dF_{(r)Z_{n+1,j}}(s) \} dH^{(n)}(u) \right] \\
& = \int_0^t \exp(-\Lambda(T)) \cdot I_T(t-u) dH^{(n)}(u) \\
& \quad + \int_0^t \left[\sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \{ \bar{F}_{(r)1}(t-u) \right. \\
& \quad \left. + \sum_{j=1}^{r-1} \int_0^{t-u} \bar{F}_{(r)j+1|Z_{n+1,j}=s}(t-u-s) dF_{(r)Z_{n+1,j}}(s) \} \right] dH^{(n)}(u) \\
& = \int_0^t A_0(t-u) dH^{(n)}(u), \quad \text{by the monotone convergence theorem.}
\end{aligned}$$

Hence,

$$\begin{aligned}
A(t) & = A_0(t) + \sum_{n=1}^{\infty} \int_0^t A_0(t-u) dH^{(n)}(u) \\
& = A_0(t) + \int_0^t A_0(t-u) dM_H(u).
\end{aligned}$$

Note that

$$\lim_{t \rightarrow \infty} A_0(t) \leq \lim_{t \rightarrow \infty} P\{T_1 \geq t\} = 0,$$

and by the Key Renewal Theorem,

$$\begin{aligned}\lim_{t \rightarrow \infty} A(t) &= \lim_{t \rightarrow \infty} \int_0^t A_0(t-u) dM_H(u) \\ &= \frac{1}{E(Z_i)} \int_0^\infty A_0(t) dt.\end{aligned}$$

Here $E(Z_i)$ is calculated. Note that

$$\begin{aligned}E(Z_i) &= \sum_{r=1}^{\infty} E(Z_i | N_i=r) \cdot P(N_i=r) \\ &= \sum_{r=1}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \cdot \{T + (r-1)\nu_1 + \nu_2\} \\ &= T + \Lambda(T) \cdot \nu_1 + \nu_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty A_0(t) dt \\ &= \int_0^\infty \exp(-\Lambda(T)) \cdot I_T(t) dt + \sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \cdot \\ &\quad \left[\int_0^\infty \bar{F}_{(r)1}(t) dt + \sum_{j=1}^{r-1} \int_0^\infty \int_0^t \bar{F}_{(r)j+1|Z_{1,j}=s}(t-s) dF_{(r)Z_{1,j}}(s) dt \right] \\ &= \exp(-\Lambda(T)) \cdot T + \sum_{r=2}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \cdot \{\mu_{(r)1} + \sum_{j=2}^r \mu_{(r)j}\} \\ &= \sum_{r=1}^{\infty} \frac{(\Lambda(T))^{r-1} \cdot \exp(-\Lambda(T))}{(r-1)!} \cdot T \\ &= T,\end{aligned}$$

where, by exchanging the order of integrals, the equality

$$\int_0^\infty \int_0^t \bar{F}_{(r)j+1|Z_{1,j}=s}(t-s) dF_{(r)Z_{1,j}}(s) dt = \mu_{(r)j+1},$$

holds. This completes the proof. ■

Until now, we have considered the case when the distributions of the minimal repair times in a renewal period are identical. However, in many cases, the repair time might be increasing. For example, in view of the ageing and cumulative wear, the repair times increase as the operating time of the component increases. Yeh (1988) proposed a geometric process and considered the optimal replacement problem when successive survival times constitute a non-increasing geometric process and successive repair times constitute a non-decreasing geometric process. The geometric process is introduced in the following definitions.

Definition 1. (Yeh, 1988) Given a sequence of random variables X_1, X_2, \dots if for some $a > 0$, $\{a^{n-1} X_n, n=1, 2, \dots\}$ forms a renewal process, then $\{X_n, n=1, 2, \dots\}$ is a geometric process. a is called the parameter of the geometric process.

Definition 2. (Yeh, 1988) A geometric process is called a non-increasing geometric process if $a \geq 1$ and a non-decreasing geometric process if $a \leq 1$.

Now, we consider the same availability model assuming that minimal repair times in a renewal cycle constitute a strictly increasing geometric process, i.e., a geometric process with

$a < 1$. As in the first model in this section, the component is renewed every time its age reaches at T . For each intervening failure only minimal repair is done. We assume that the sequence $\{Y_{i,j}, j=1, 2, \dots, N_i'\}$ (when $N_i' \geq 1$) forms a geometric process with parameter $a < 1$, $i=1, 2, \dots$. Assume that, for $N_i \neq 1$ ($N_i' \geq 1$), the distribution of $Y_{i,1}$ is $G_1(y)$ and that of the replacement time Y_{i,N_i} is $G_2(y)$, where the mean of $G_1(y)$ is ν_1 and that of $G_2(y)$ is ν_2 . For $N_i = 1$ ($N_i' = 0$), the distribution of $Y_{i,1}$ is $G_2(y)$ and the mean of $Y_{i,1}$ is ν_2 . Observe that, with $a = 1$, the availability model under consideration reduces to the first model. In this situation, by the similar arguments described in the first model, we have the following result.

Corollary 1. When the minimal repair times in a renewal cycle, $\{Y_{i,j}, j=1, 2, \dots, N_i'\}$, form a geometric process with $a < 1$, the steady state availability of the model exists and it is given by

$$A = \lim_{t \rightarrow \infty} A(t) = \frac{T}{T + \left(\exp\left[\left(\frac{1}{a} - 1\right)\Lambda(T)\right] - 1 \right) \frac{a}{1-a} \nu_1 + \nu_2}. \quad (2)$$

proof.

To prove the Corollary 1, it suffices to show that

$$E(Z_i) = T + \left(\exp\left[\left(\frac{1}{a} - 1\right)\Lambda(T)\right] - 1 \right) \frac{a}{1-a} \nu_1 + \nu_2.$$

Since $Z_i = T + \sum_{j=1}^{N_i} Y_{i,j}$, $E(Z_i)$ is given by

$$\begin{aligned} E(Z_i) &= T + E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) + E(Y_{i,N_i}) \\ &= T + E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) + \nu_2, \end{aligned}$$

where $E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right)$ is given by

$$\begin{aligned} E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) &= E\left(E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1) \mid N_i'\right)\right) \\ &= \sum_{n=0}^{\infty} \frac{1 - \left(\frac{1}{a}\right)^n}{1 - \frac{1}{a}} \nu_1 \cdot \frac{(\Lambda(T))^n \cdot \exp(-\Lambda(T))}{n!} \\ &= \left(\exp\left[\left(\frac{1}{a} - 1\right)\Lambda(T)\right] - 1 \right) \frac{a}{1-a} \nu_1. \end{aligned}$$

Therefore,

$$E(Z_i) = T + \left(\exp\left[\left(\frac{1}{a} - 1\right)\Lambda(T)\right] - 1 \right) \frac{a}{1-a} \nu_1 + \nu_2.$$

This completes the proof. ■

Remark 1.

The result obtained in Corollary 1 still holds even though the parameter a satisfies $a > 1$, that is, for the case when the minimal repair times in a renewal cycle, $\{Y_{i,j}, j=1,2,\dots,N_i'\}$, form a strictly decreasing geometric process.

Now, we consider the case when the mean of minimal repair time in a renewal cycle is linearly increasing; that is, we assume that, when $N_i' \geq 1$, $Y_{i,j}$'s have continuous type distribution functions and $E(Y_{i,j}) = \nu_0 + j\nu_1$, for $j=1,2,\dots,N_i'-1$, and $E(Y_{i,N_i'}) = \nu_2$, for $j=N_i'$. Obviously, we also assume that, when $N_i'=0$, $E(Y_{i,1}) = \nu_2$. In this case, we have the following result.

Corollary 2. When the mean of the minimal repair time in a renewal cycle is linearly increasing, the steady state availability of the model exists and it is given by

$$\begin{aligned} A &= \lim_{t \rightarrow \infty} A(t) \\ &= \frac{T}{T + \Lambda(T)(\nu_0 + \nu_1) + (\Lambda(T))^2 \frac{\nu_1}{2} + \nu_2}. \end{aligned} \quad (3)$$

proof.

As in Corollary 1, to prove the Corollary 2, it suffices to show that

$$E(Z_i) = T + \Lambda(T)(\nu_0 + \nu_1) + (\Lambda(T))^2 \frac{\nu_1}{2} + \nu_2$$

Since $Z_i = T + \sum_{j=1}^{N_i'} Y_{i,j}$, $E(Z_i)$ is given by

$$\begin{aligned} E(Z_i) &= T + E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) + E(Y_{i,N_i'}) \\ &= T + E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) + \nu_2, \end{aligned}$$

where $E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right)$ is given by

$$\begin{aligned} E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1)\right) &= E\left(E\left(\sum_{j=1}^{N_i'} Y_{i,j} \cdot I(N_i' \geq 1) \mid N_i'\right)\right) \\ &= \sum_{n=0}^{\infty} \left[\left(\nu_0 + \frac{\nu_1}{2}\right)n + \frac{n^2}{2} \nu_1 \right] \cdot \frac{(\Lambda(T))^n \cdot \exp(-\Lambda(T))}{n!} \\ &= \Lambda(T)(\nu_0 + \nu_1) + (\Lambda(T))^2 \frac{\nu_1}{2} \end{aligned}$$

Therefore, we have obtained the desired result. ■

Remark 2.

If the failure rate function of the component $\lambda(t)$ is strictly IFR(increasing failure rate) and it satisfies $\lim_{t \rightarrow \infty} \lambda(t) = \infty$ then, for the steady state availability given in (1), (2) and (3), it can

be shown that there exists a unique solution T^* satisfying

$$T^* = \text{Arg}[\max_{T \geq 0} A(T)],$$

that is, $A(T^*) = \max_{T \geq 0} A(T)$, where $A(T)$ is the steady state availability when the replacement policy T is used.

The following Example 2 and Remark 3 show and explain the importance and the utility of Theorem 1.

Example 2.

We consider a parallel system consisting of two components. Assume that each component is subject to the operating rule described in section 2(Age Replacement Policy with identically distributed minimal repair times) with replacement policy $T_{(i)}$, $i=1,2$, that is, the i^{th} component is replaced by an identical new component every time its age reaches $T_{(i)}$, $i=1,2$, and for each intervening failure of the component i , the failed component is only minimally repaired. Assume that the two components operate independently of one another. Specifically, while a repair or a replacement of a failed component is occurring in one position, the other components continue to operate. Let the failure rate of the first component be $\lambda_1(t) = \lambda_1 \beta_1 t^{\beta_1 - 1}$ and that of the second component be $\lambda_2(t) = \lambda_2 \beta_2 t^{\beta_2 - 1}$, where $\lambda_1, \lambda_2 > 0$ and $\beta_1, \beta_2 > 1$, which are the Weibull type failure rate functions. Furthermore, we assume that the mean of the identically distributed minimal repair time for the first component is ν_{11} and that of the second component is ν_{12} , and the mean time for renewal(replacement) of the first component is ν_{21} and that of the second component is ν_{22} . Note that, in this situation, it is hard to derive the steady state availability(or limiting efficiency) of the system directly by considering the renewal point of the whole system. However, by considering the availability of each component, we can easily derive the steady state availability of the whole system. Observe that in this case we can express the state $X(t)$ of the coherent system in terms of the component states, $X_1(t), X_2(t)$;

$$X(t) = \phi(X_1(t), X_2(t)) = 1 - \prod_{i=1}^2 (1 - X_i(t)) = \max_{(i=1,2)} X_i(t),$$

where ϕ is the coherent structure of the two components, and thus the availability of the system at time t , $A(t)$, is given by

$$\begin{aligned} A(t) = E(X(t)) &= E\left[1 - \prod_{i=1}^2 (1 - X_i(t))\right] \\ &= 1 - \prod_{i=1}^2 (1 - A_i(t)) \\ &= h(A_1(t), A_2(t)), \end{aligned}$$

where $A_i(t)$ is the availability at time t of the component i , $i=1,2$, and the reliability

function h is defined by $h(p_1, p_2) \equiv 1 - \prod_{i=1}^2 (1 - p_i)$. Then, since we have shown the existence of the steady state availability of each component in Theorem 1, that of the system also exists and is given by

$$\begin{aligned} A &= \lim_{t \rightarrow \infty} A(t) = h(A_1, A_2) \\ &= 1 - \left(1 - \frac{T_{(1)}}{T_{(1)} + \lambda_1 T_{(1)}^{\beta_1} \cdot \nu_{11} + \nu_{21}} \right) \left(1 - \frac{T_{(2)}}{T_{(2)} + \lambda_1 T_{(2)}^{\beta_2} \cdot \nu_{12} + \nu_{22}} \right), \end{aligned} \quad (4)$$

where A_i is the steady state availability of the component i , $i=1,2$. In this situation, the optimal replacement policy, $\mathcal{T}^* = (T_{(1)}^*, T_{(2)}^*)$, maximizing the steady state availability of system in (4) is determined by

$$\mathcal{T}^* = \left(\left[\frac{\nu_{21}}{\lambda_1(\beta_1 - 1)\nu_{11}} \right]^{\frac{1}{\beta_1}}, \left[\frac{\nu_{22}}{\lambda_2(\beta_2 - 1)\nu_{12}} \right]^{\frac{1}{\beta_2}} \right),$$

which is the simple combination of the optimal replacement policies maximizing the steady state availability of each component.

Remark 3.

As briefly remarked in Example 2, when we are interested in deriving the steady state availability $A = \lim_{t \rightarrow \infty} A(t)$ or the limiting efficiency $Eff_{\infty} = \lim_{t \rightarrow \infty} \frac{E(U(t))}{t}$ of the coherent system, we can hardly find the renewal points of the whole system and thus the performance measures (steady state availability or limiting efficiency) could not be obtained directly. However, by considering the availability of each component and by showing the existence of the steady state availability of each component, we can easily obtain the limiting measures of system performance (Note that Eff_{∞} is obtained by deriving $A = \lim_{t \rightarrow \infty} A(t)$). Also the method of proof described in Theorem 1 could be applied to other replacement models by simply modifying some detailed parts.

Acknowledgements

The author thanks the referees for many helpful comments and careful readings of this paper, which have improved the presentation of this paper considerably. This research was supported by a research grant from the Catholic University of Daegu in 2003.

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[Received November 2002, Accepted March 2003]