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T-SUM OF L-R FUZZY NUMBERS WITH UNBOUNDED SUPPORTS

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Abstract. The sum of $L-R$ fuzzy numbers with unbounded supports based on Archimedean continuous t-norm T is considered. Results are more simple than those of the case for bounded supports.

1. Introduction

Fuzzy arithmetic has grown in importance during recent years as a tool of advance in fuzzy optimization and control theory. Fuzzy arithmetic based on the sup-(t-norm) convolution, with the controllability of the increase of fuzziness, enables us to construct more flexible and adaptable mathematical models in several intelligent technologies based on approximate reasoning and fuzzy logic. Hence a lot of effort is needed in the future to find exact and good approximative computational formulas for fuzzy arithmetic operations. Most of results are restricted in the case of bounded supports. Some results on this topic and applications can be found, e.g. in [1]-[22]. In this paper, we consider the sum of $L-R$ fuzzy numbers with unbounded supports based on a given Archimedean continuous t-norm T.

A function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm [1] if and only if T is symmetric, associative, non-decreasing in each argument, and $T(x,1) = x$ for all $x \in [0,1]$. For arbitrary fuzzy quantities $A_i, i = 1, 2, \cdots, n,\ n \in N$, their T-sum is defined by means of the generalized extension principle [1].

$$(1) \qquad A_1 \oplus_T \cdots \oplus_T A_n(z) = \sup_{\Sigma xi = z} T(A_1(x_1), \cdots, A_n(x_n)), z \in \Re,$$

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where the usual extension of T to an n-ary operation is used.

If $T_1 \leq T_2$(the usual order of t-norms as two-place functions), then for any fuzzy quantity A and B it is $A \oplus_{T_1} B \leq A \oplus_{T_2} B$. Let T_W denote the weakest t-norm defined by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise} \end{cases}$$

and let T_M denote the strongest t-norm defined by $T_M(x,y) = \min(x,y)$ for all $x, y \in [0,1]$. Consequently, for any t-norm T it is $A \oplus_W B \leq A \oplus_T B \leq A \oplus_M B$.

Recall that a t-norm T is Archimedean if and only if T is continuous and $T(x,x) < x$ for all $x \in (0,1)$. In every Archimedean t-norm T is representable by a continuous and decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and $T(x,y) = f^{[-1]}(f(x) + f(y))$, where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) =: \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)], \\ 0 & \text{if } y \in [f(0), \infty]. \end{cases}$$

The function f is the additive generator of T. If f is not bounded, then T is called strict t-norm. If f is bounded, then it can be chosen so that $f(0) = 1$ and the corresponding t-norm T is called a nilpotent t-norm. For two fuzzy quantities A and B, their T-sum defined by (1) can be written as

$$A \oplus_T B(z) = \sup_{x+y=z} f^{[-1]}(f(A(x)) + f(B(y))). \tag{2}$$

A function h is subadditive if for all $s, t \geq 0$ $h(x+t) \leq h(x) + h(t)$.

A function, denoted L or R, is a reference function of fuzzy numbers if and only if $L(0) = 1$ and L is strictly decreasing continuous function on $[0, \infty)$. For instance, $L(x) = \max(0, 1 - x^p), p > 0$ (which has bounded support and can be normalized as $L(1) = 0$); $L(x) = e^{-x^p}, p > 0$ or $L(x) = 1/(1 + x^p), p > 0$(which has an unbounded support).

A fuzzy A is said to be an $L - R$ fuzzy number if and only if the corresponding membership function satisfies for all $x \in R$

$$A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & \text{for } x \leq a, \\ R(\frac{x-a}{\beta}) & \text{for } x \geq a, \end{cases}$$

where $a \in R$ is the peak of A, $\alpha > 0$ and $\beta > 0$ is the left and right spreads, respectively. Symbolically, we write

$$A = (a, \alpha, \beta)_{LR}.$$

2. T-sum of L-R fuzzy numbers-bounded cases

In this section, we summarize recent known results of T-sums of $L-R$ fuzzy numbers with bounded supports.

The next result on the sum of $L-R$ fuzzy numbers based on an Archimedian continuous t-norm T is due to Fullér and Keresztfalvi [4].

THEOREM 1. *Let $A_i = (a_i, \alpha, \beta)_{LR}$, $i = 1, \cdots, n$, be $L-R$ fuzzy numbers with twice differentiable concave shapes L and R. Let T be a t-norm with twice differentiable strictly convex additive generator f. Let $B_n = A_1 \oplus_T \cdots \oplus_T A_n$. Then*

$$(3) \quad B_n(z) = \begin{cases} f^{[-1]}\left(nf\left(L\left(\dfrac{S_n - z}{n\alpha}\right)\right)\right) & \text{for } S_n - n\alpha \le z \le S_n, \\ f^{[-1]}\left(nf\left(R\left(\dfrac{z - S_n}{n\beta}\right)\right)\right) & \text{for } S_n \le z \le S_n + n\beta, \\ 0 & \text{else,} \end{cases}$$

where $S_n = a_1 + \cdots + a_n$.

For a general strict t-norm T Marková [17] have the following results.

THEOREM 2. *Let T be a strict t-norm with additive generator f. The conclusion of Theorems 1 is true if and only if the composite functions $f \circ L$ and $f \circ R$ are convex. If T is nilpotent, Marková also got the following sufficient and necessary condition in which she relax the convexity of $f \circ L$ and $f \circ R$.*

THEOREM 3. *Let T be a nilpotent t-norm with normed additive generator f. The conclusions of Theorems 1 are true if and only if the composite functions $K_1 = f \circ L$ and $K_2 = f \circ R$ fulfill the next two conditions (for $i = 1, 2$):*

(1) K_i is convex on $[0, K_i^{-1}(\frac{1}{2})]$

(2) $K_i(x) \ge 1 - K_i(2K_i^{-1}(\frac{1}{2}) - x)$, for all $x \in [K_i^{-1}(\frac{1}{2}), 1]$.

For the case of different spreads, Hong and Hwang [8] provided an upper bound of the membership function.

THEOREM 4. *Let T be an Archimedean t-norm with additive generator f and let $A_i = (a_i, \alpha_i, \beta_i)_{LR}, i = 1, 2$, be LR-fuzzy numbers. If*

$f \circ L$ and $f \circ R$ are convex functions, then the membership function of their T-sum $A_1 \oplus_T A_2$ is less than or equal to

$$(4) \qquad A_2^*(z) = \begin{cases} f^{[-1]}\left(2f\left(L\left(\frac{1}{2} + \frac{(a-z)-\alpha^*}{2\alpha_*}\right)\right)\right) \\ \quad \text{if } a-(\alpha_1+\alpha_2) \le z \le a-\alpha^*, \\ f^{[-1]}\left(2f\left(L\left(\frac{(a-z)}{2\alpha^*}\right)\right)\right) \\ \quad \text{if } a-\alpha^* \le z \le a, \\ f^{[-1]}\left(2f\left(R\left(\frac{(z-a)}{2\beta^*}\right)\right)\right) \\ \quad \text{if } a \le z \le a+\beta^*, \\ f^{[-1]}\left(2f\left(R\left(\frac{1}{2} + \frac{(z-a)-\beta^*}{2\beta_*}\right)\right)\right) \\ \quad \text{if } a-\beta^* \le z \le a+\beta_1+\beta_2, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\beta^* = max\{\beta_1, \beta_2\}$, $\beta_* = min\{\beta_1, \beta_2\}$, $\alpha^* = max\{\alpha_1, \alpha_2\}$, $\alpha_* = min\{\alpha_1, \alpha_2\}$ and $a = a_1 + a_2$. In particular if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $A_2(z) = A_2(z)$.

The following result gives a sufficient and necessary condition that the membership function of T-sum achieves the upper bound due to Hong and Hwang [13].

THEOREM 5. *Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$, be LR-fuzzy numbers with different spreads $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. If $f \circ L$, $f \circ R$ are convex functions, then the membership function of the T-sum $A_1 \oplus_T A_2$ is exactly same as A_2^* in Theorem 4 if and only if $f \circ R(x) = f \circ L(x) = cx$ for some $c > 0$, in fact,*

$$A_2^*(z) = A_1 \oplus_T A_2 = (a_1 + a_2, max(\alpha_1, \alpha_2), max(\beta_1, \beta_2))_{LR}.$$

3. T-sum of L-R fuzzy numbers-unbounded cases

We now consider T-sum of $L - R$ fuzzy numbers with unbounded supports. These cases are simpler. We assume L and R have unbounded supports.

The following result is due to Hong and Ro [14]

THEOREM 6. *Let $A_i = (z_i, \alpha, \beta)_{LR}, i = 1, 2, \cdots, n$ be $L-R$ fuzzy numbers and let T be an Archimedean t-norm with additive generator f. If $f \circ L$ and $f \circ R$ are convex functions, then the membership function of T-sum $B_n = A_1 \oplus_T \cdots \oplus_T A_n$ is*

$$B_n(z) = \begin{cases} f^{[-1]}\left(nf\left(L\left(\frac{s_n - z}{n\alpha}\right)\right)\right) & \text{if } z \leq S_n, \\ f^{[-1]}\left(nf\left(R\left(\frac{z - s_n}{n\beta}\right)\right)\right) & \text{if } S_n \leq z, \end{cases}$$

where $s_n = a_1 + \cdots + a_n$.

THEOREM 7. *Let T be a t-norm with additive generator f. The conclusion of Theorem 6 is true if and only if T is a strict t-norm and the composite functions $f \circ L$ and $f \circ R$ are convex.*

PROOF. The if part is Theorem 6. For the sufficient condition, let $A_1 = A_2 = (0, 1, 1)_{LR}$. Then for $z \in [0, \infty)$ it is from (2) and (3)

$$(5)\ A_1 \oplus_T A_2(z) = f^{[-1]}(2f \circ R(\frac{z}{2})) = f^{[-1]}(\inf_{\substack{x+y=z \\ z,y \in [0,\infty)}} (f \circ R(x) + f \circ R(y))).$$

Suppose that $f(0) = 1$, i.e., T is a nilpotent t-norm. Then for z such that $f \circ R(\frac{z}{2}) \geq \frac{1}{2}$, $\inf_{\substack{x+y=z \\ x,y \in [0,\infty)}} (f \circ R(x) + f \circ R(y)) \geq 1$. Hence $f \circ R(z) = f \circ R(z) + f \circ R(0) \geq 1$, which implies $R(z) = 0$. It contradicts that R has unbounded support. Therefore T is a strict t-norm and $f^{[-1]} = f^{-1}$. Now, from (5) $f \circ R(\frac{x+y}{2}) \leq \frac{f \circ R(x) + f \circ R(y)}{2}$ for all $x, y \in [0, \infty)$, and hence $f \circ R$ is convex. The case of $f \circ L$ can be proved similarly.

From above result, we see that there is no unbounded support version of Theorem 3. Next we consider unbounded support version of Theorem 4. We also note that $f \circ R$ is convex then f should be strict.

THEOREM 8. *Let T be an Archimedean t-norm with additive generator f and let $A_i = (a_i, \alpha_i, \beta_i)_{LR}, i = 1, 2,$ be LR-fuzzy numbers. If $f \circ L$ and $f \circ R$ are convex functions, then the membership function of their T-sum $A_1 \oplus_T A_2$ is less than or equal to*

$$A_2^*(z) = \begin{cases} f^{-1}\left(2f \circ L\left(\frac{a - z}{2\alpha^*}\right)\right) & \text{if } z \leq a, \\ f^{-1}\left(2f \circ R\left(\frac{z - a}{2\beta^*}\right)\right) & \text{if } z \geq a, \end{cases}$$

where $\beta^* = max\{\beta_1, \beta_2\}$, $\beta_* = min\{\beta_1, \beta_2\}$, $\alpha^* = max\{\alpha_1, \alpha_2\}$, $\alpha_* = min\{\alpha_1, \alpha_2\}$ and $a = a_1 + a_2$. In particular, if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $A_2(z) = A_2(z)$.

PROOF. We first consider the case for $z \geq a$. As mentioned in Section 1, the investigated membership function is

$$A_1 \oplus_T A_2(z) = f^{-1}(\inf_{x_1+x_2=z} f(A_1(x_1)) + f(A_2(x_2))).$$

We put $B_i = (0, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$. Since $A_i(x) = B_i(x - ai)$ for all x, we have for each z,

$$\begin{aligned} A_1 \oplus_T A_2(z) &= f^{-1}(\inf_{x_1+x_2=z} f(A_1(x_1)) + f(A_2(x_2))) \\ &= f^{-1}(\inf_{y_1+y_2=z-a} (f(B_1(y_1)) + f(B_2(y_2)))) \\ &= B_1 \oplus_T B_2(z - a). \end{aligned}$$

It is well-known by [1] that right-hand side of $B_1 \oplus_T B_2$ depends only on right-hand sides of B_1 and B_2 and hence we have that

$$A_1 \oplus_T A_2 = f^{-1}(\inf_{\substack{x_1+x_2=z \\ x_i \geq 0, i=1,2}} f(B_1(x_1)) + f(B_2(x_2))).$$

Without loss of generality we assume $\beta_1 \leq \beta_2$ and suppose that $x_1 + x_2 = z$, $x_i \geq 0$, $i = 1, 2$. Then we have

$$\begin{aligned} \frac{1}{2}(f(B_1(x_1)) + f(B_2(x_2))) &= \frac{1}{2}\left(f \circ R\left(\frac{x_1}{\beta_1} + f \circ R\left(\frac{x_2}{\beta_2}\right)\right)\right) \\ &\geq f \circ R\left(\frac{1}{2}\left(\frac{x_1}{\beta_1} + \frac{x_2}{\beta_2}\right)\right) \\ &\geq f \circ R\left(\frac{z}{2\beta_2}\right), \end{aligned}$$

where the first inequality comes from the convexity of $f \circ R$. So we are done and the case for $z \leq a$ is similar.

Finally, we consider unbounded supports version of Theorem 5. But, we have exactly same type of result as Theorem 5 and the proof is also same as the case 1 of the proof of Theorem 6 (Hong and Hwang [13]), so we write down the result without proof.

THEOREM 9. *Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$, be LR-fuzzy numbers with different spreads $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. If $f \circ L$, $f \circ R$ are convex functions, then the membership function of the T-sum $A_1 \oplus_T A_2$ is exactly same as A_2^* in Theorem 8 if and only if $f \circ R(x) = f \circ L(x) = cx$ for some $c > 0$, in fact,*

$$A_2^*(z) = A_1 \oplus_T A_2 = (a_1 + a_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))_{LR}.$$

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