Commun. Korean Math. Soc. **18** (2003), No. 2, pp. 375–383

COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM VARIABLES

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Abstract. Under some conditions on an array of rowwise independent random variables, Hu et al.(1998) obtained a complete convergence result for law of large numbers with rate $\{a_n, n \geq 1\}$ which is bounded away from zero. We investigate the general situation for rate $\{a_n, n \geq 1\}$ under similar conditions.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty$$

for all $\epsilon > 0$. We refer to [3] for a survey on results on complete convergence related to strong laws.

Recently, Hu et al. [6] and Hu and Volodin [8] proved the following complete convergence theorem for arrays of rowwise independent random variables.

Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geq 1\}$ a sequence of positive*

Received April 19, 2002.
2000 Mathematics Subject Classification: 60F15, 60G50, 60B12.
Key words and phrases: Arrays, rowwise independence, sums of independent random variables, complete convergence, Rademacher type p Banach space, random elements.

constants bounded away from zero, that is, $\liminf_{n\to\infty} a_n > 0$. *Suppose that for every* $\epsilon > 0$ *and some* $\delta > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$,

(ii) *there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leq \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leq \delta) \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

This result was generalized on Banach space setting in [7].

The proof of Theorem 1 is based on the fact that

$$\sum_{i=1}^{k_n} X_{ni} \to 0 \text{ in probability} \tag{1}$$

as $n \to \infty$. We mention that (1) does not necessarily follow from the conditions of Theorem 1 if $\{a_n, n \geq 1\}$ is not bounded away from zero. To give such an example, we will need the following lemma.

LEMMA 1. *If the random variable* X *is* $N(0,1)$, *then for every* $\epsilon > 0$

$$P(X > \epsilon) \leq e^{-\frac{\epsilon^2}{2}}.$$

PROOF. For any $t > 0$,

$$P(X > \epsilon) = P(tX > t\epsilon) = P(e^{tX} > e^{t\epsilon}) \leq e^{-t\epsilon} E[e^{tX}] = e^{-t\epsilon + \frac{t^2}{2}},$$

since X has moment generating function $e^{\frac{t^2}{2}}$. The result follows by putting $t = \epsilon$. □

REMARK 1. It is well known that $P(X > \epsilon) \leq \frac{1}{\epsilon\sqrt{2\pi}} \exp(-\frac{\epsilon^2}{2})$(see [2], p. 175). Hence, the upper bound of $P(X > \epsilon)$ in Lemma 1 is good when $0 < \epsilon < \frac{1}{\sqrt{2\pi}}$.

EXAMPLE 1. Define a sequence $\{a_n, n \geq 1\}$ by

$$a_n = \begin{cases} 1/n^2, & \text{if } n \text{ is odd,} \\ 1/n, & \text{if } n \text{ is even.} \end{cases}$$

Let $X_1, X_2, \cdots$ be independent and identically distributed $N(0,1)$ random variables. Define an array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ by

$$X_{ni} = \begin{cases} X_i/\sqrt{n}, & \text{if } n \text{ is odd and } 1 \leq i \leq n, \\ X_i/n, & \text{if } n \text{ is even and } 1 \leq i \leq n. \end{cases}$$

Then we have by Lemma 1 that

$$P(|X_{n1}| > \epsilon) \leq \begin{cases} 2\ \exp(-n\epsilon^2/2), & \text{if } n \text{ is odd,} \\ 2\ \exp(-n^2\epsilon^2/2), & \text{if } n \text{ is even.} \end{cases}$$

It follows that

$$\sum_{n=1}^{\infty} a_n \sum_{i=1}^{n} P(|X_{ni}| > \epsilon) = \sum_{n \text{ is odd}} \frac{1}{n} P(|X_{n1}| > \epsilon) + \sum_{n \text{ is even}} P(|X_{n1}| > \epsilon)$$
$$\leq 2\Big[\sum_{n \text{ is odd}} \exp(-\frac{n\epsilon^2}{2})/n + \sum_{n \text{ is even}} \exp(-\frac{n^2\epsilon^2}{2})\Big] < \infty,$$

and so the condition (i) of Theorem 1 holds. Next, we claim that conditions (ii) and (iii) hold. Noting that

$$EX_{n1}^2 = \begin{cases} 1/n, & \text{if } n \text{ is odd,} \\ 1/n^2, & \text{if } n \text{ is even,} \end{cases}$$

we get

$$\sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{n} EX_{ni}^2 I(|X_{ni}| \leq \delta)\Big)^J \leq \sum_{n=1}^{\infty} a_n (nEX_{n1}^2)^J$$
$$= \sum_{n \text{ is odd}} \frac{1}{n^2} + \sum_{n \text{ is even}} \frac{1}{n^{1+J}} < \infty,$$

which implies (ii). Since X_{ni} is symmetric, $EX_{ni}I(|X_{ni}| \leq \delta) = 0$. Thus (iii) holds. But, (1) does not hold, since for odd n

$$\sum_{i=1}^{n} X_{ni} = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \sim N(0,1).$$

However, it is easy to see that $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$.

REMARK 2. For a different example and a general discussion about Theorem 1 we refer to [8].

It is an interesting project to investigate whether Theorem 1 is true or not for general sequences. In this paper, we obtain a complete convergence result without assuming that $\{a_n, n \geq 1\}$ is bounded away from zero, but under slightly modified conditions of Theorem 1. The proof is different from that of Hu et al. [6] and it does not use symmetrization procedure.

2. Main result

To prove the main result, we will need the following lemma which is a version of Hoffmann-Jørgensen [4] inequality for independent, but not necessarily symmetric, random variables.

LEMMA 2. *Let $X_1, \cdots, X_n$ be independent random variables. Let $S_i = \sum_{l=1}^{i} X_l, 1 \leq i \leq n$, and let $S_0 \equiv 0$. Then for every integer $j \geq 1$ and $t > 0$*
(2)
$$P(|S_n| > 6^j t) \leq C_j P(\max_{1\leq i\leq n} |X_i| > \frac{t}{4^{j-1}}) + D_j \max_{1\leq i\leq n}\left[P(|S_i| > \frac{t}{4^j})\right]^{2^j}$$

for some positive constants C_j and D_j depending only on j.

PROOF. From Lemma 1 and Lemma 2 in [1], it follows that

$$P(|S_n| > 6t) \leq P(\max_{1\leq i\leq n} |X_i| > t) + 64 \max_{1\leq i\leq n}\left[P(|S_i| > \frac{t}{4})\right]^2. \tag{3}$$

Thus (2) holds for $j = 1$ with $C_1 = 1$ and $D_1 = 64$. Assume that (2) holds for some j for some positive constants C_j and D_j. Then using (3), we have

$$\begin{aligned}
&P(|S_n| > 6^{j+1} t) \\
&\leq P(\max_{1\leq i\leq n} |X_i| > 6^j t) + 64 \max_{1\leq i\leq n}\left[P(|S_i| > \frac{6^j t}{4})\right]^2 \\
&\leq P(\max_{1<i<n} |X_i| > 6^j t)
\end{aligned}$$

$$+ \ 64 \max_{1\le i\le n}\left[C_j P(\max_{1\le l\le i}|X_l| > \frac{t}{4^j}) + D_j \max_{1\le l\le i}\left[P(|S_l| > \frac{t}{4^{j+1}})\right]^{2^j}\right]^2$$

$$= P(\max_{1\le i\le n}|X_i| > 6^j t) + 64\left[C_j^2\left[P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})\right]^2\right.$$

$$+ \ 2C_jD_jP(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})\max_{1\le i\le n}\left[P(|S_i| > \frac{t}{4^{j+1}})\right]^{2^j}$$

$$\left. + \ D_j^2 \max_{1\le i\le n}\left[P(|S_i| > \frac{t}{4^{j+1}})\right]^{2^{j+1}}\right]$$

$$\le P(\max_{1\le i\le n}|X_i| > 6^j t)$$

$$+ \ 64C_j^2P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j}) + 128C_jD_jP(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})$$

$$+ \ 64D_j^2 \max_{1\le i\le n}\left[P(|S_i| > \frac{t}{4^{j+1}})\right]^{2^{j+1}}$$

$$\le (1 + 64C_j^2 + 128C_jD_j)P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})$$

$$+ \ 64D_j^2 \max_{1\le i\le n}\left[P(|S_i| > \frac{t}{4^{j+1}})\right]^{2^{j+1}}.$$

Hence, we can take $C_{j+1} = 1 + 64C_j^2 + 128C_jD_j$ and $D_{j+1} = 64D_j^2$. □

Now, let $\{a_n, n \ge 1\}$ be a sequence of positive constants without the assumption that it is of bounded away from zero. We state and prove our main result.

THEOREM 2. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Theorem 1 except that* (ii) *and* (iii) *are replaced by* (ii′) *and* (iii′), *respectively:*

(ii′) *there exists $J \ge 2$ such that*

$$\sum_{n=1}^{\infty} a_n\left(\sum_{i=1}^{k_n} Var(X_{ni}I(|X_{ni}| \le \delta))\right)^J < \infty,$$

(iii′) $\max_{1\le i\le k_n}|\sum_{l=1}^{i} EX_{nl}I(|X_{nl}| \le \delta)| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_nP(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

PROOF. Let $X'_{ni} = X_{ni}I(|X_{ni}| \leq \delta), X''_{ni} = X_{ni}I(|X_{ni}| > \delta)$ for $1 \leq i \leq k_n, n \geq 1$. Then

$$P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) \leq P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2}) + P(|\sum_{i=1}^{k_n} X''_{ni}| > \frac{\epsilon}{2})$$
$$\leq P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2}) + \sum_{i=1}^{k_n} P(|X_{ni}| > \delta).$$

By (i), it suffices to estimate $P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2})$. Take j such that $2^j \geq J$. Then we have by Lemma 2 that

$$P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2})$$
$$\leq C_j P(\max_{1\leq i\leq k_n} |X'_{ni}| > \frac{2\epsilon}{24^j}) + D_j \max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^{2^j}$$
$$\leq C_j \sum_{i=1}^{k_n} P(|X_{ni}| > \frac{2\epsilon}{24^j}) + D_j \max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J.$$

Hence by (i) it suffices to estimate $\max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$. On the other hand, condition (iii$'$) implies that there exists an integer N such that

$$\max_{1\leq i\leq k_n} |\sum_{l=1}^{i} EX'_{nl}| < \frac{\epsilon}{4\cdot 24^j} \text{ if } n \geq N.$$

For $n \geq N$, we get by the Markov's inequality that

$$\max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$$
$$\leq \max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})| + |\sum_{l=1}^{i} EX'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$$
$$\leq \max_{1\leq i\leq k_n} P(|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})| > \frac{\epsilon}{4\cdot 24^j})^J$$

$$\leq (\frac{4 \cdot 24^j}{\epsilon})^{2J} \max_{1 \leq i \leq k_n} \left(Var(\sum_{l=1}^{i} X'_{nl}) \right)^J$$

$$= (\frac{4 \cdot 24^j}{\epsilon})^{2J} \left(\sum_{i=1}^{k_n} Var(X'_{ni}) \right)^J .$$

In view of (ii′), the proof is complete. □

REMARK 3. Condition (ii′) in Theorem 2 is a slight modification of condition (ii) in Theorem 1. Although condition (iii′) in Theorem 2 is stronger than condition (iii) in Theorem 1, Corollary 1 and Corollary 2 in [6] can be proved by Theorem 2.

Theorem 2 can be generalized to Banach space setting. Recall that a real separable Banach space $(B, \| \ \|)$ is said to be of (Rademacher) *type* $p, 1 \leq p \leq 2$, if there exists a positive constant C such that

$$E\| \sum_{i=1}^{n} X_i \|^p \leq C \sum_{i=1}^{n} E\|X_i\|^p$$

for all independent mean zero and finite p-th moment random elements $X_1, \cdots, X_n$ with values in B. For discussion of this notion and some equivalent definitions, see [10].

Let us mention that a version of Hoffmann-Jørgensen [4] inequality (Lemma 2) is still valid for independent, but not necessarily symmetric, random elements with values in B. For a random element X with expected value and $p > 0$ denote $\sigma_p(X) = E\|X - EX\|^p$.

THEOREM 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space $(B, \| \ \|)$ of type $p, 1 \leq p \leq 2$, and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$,

(ii) *there exists $J \geq 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} \sigma_p(X_{ni} I(\|X_{ni}\| \leq \delta)) \right)^J < \infty,$$

(iii) $\max_{1 \leq i \leq k_n} \| \sum_{l=1}^{i} EX_{nl} I(\|X_{nl}\| \leq \delta)\| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

PROOF. Let $X'_{ni} = X_{ni} I(\|X_{ni}\| \le \delta)$ for $1 \le i \le k_n, n \ge 1$. If $k_n = \infty$ we have to prove that the series $\sum_{i=1}^{\infty} X_{ni}$ converges a.s. By Corollary 2.2.1 in [9] it is sufficient to prove that for some $\delta > 0$:

(a) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) < \infty$,
(b) $\sum_{i=1}^{\infty} X'_{ni}$ converges a.s.

Condition (a) is satisfied by (i). Since the Banach space is of type p, for any positive integer m we have $\sigma_p(\sum_{i=1}^{m} X'_{ni}) \le C \sum_{i=1}^{m} \sigma_p(X'_{ni})$. By (ii) $\sum_{i=1}^{\infty} \sigma_p(X'_{ni}) < \infty$. This implies that $\sum_{i=1}^{\infty}(X'_{ni} - EX'_{ni})$ converges a.s. Hence (b) is satisfied by (iii). The rest of the proof is the same as that in Theorem 2 except that

$$\max_{1\le i\le k_n} P(\|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})\| > \frac{\epsilon}{4\cdot 24^j})^J$$
$$\le (\frac{4\cdot 24^j}{\epsilon})^{pJ} \max_{1\le i\le k_n} \left(\sigma_p(\sum_{l=1}^{i} X'_{nl})\right)^J$$
$$\le C(\frac{4\cdot 24^j}{\epsilon})^{pJ} \left(\sum_{i=1}^{k_n} \sigma_p(X'_{ni})\right)^J,$$

since B is of type p. □

ACKNOWLEDGMENTS. The authors wish to acknowledge the partial supports: T.-C. Hu to NSC 88-2118-M-007-001, M. Ordóñez Cabrera to BFM 2000-0344-C02-01 and FQM 127, S. H. Sung to the Korea Research Foundation made in the program year of 1998(1998-001-D00144). Part of the research of A. I. Volodin was conducted during his short visit to the Department of Applied Mathematics of Pai Chai University in June, 1999. He wishes to express his gratitude to the Department and especially to Professor S. H. Sung for their exceptionally warm hospitality.

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