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FUZZY G-CLOSURE OPERATORS

YONG CHAN KIM AND JUNG MI KO

ABSTRACT. We introduce a fuzzy g-closure operator induced by a fuzzy topological space in view of the definition of Šostak [13]. We show that it is a fuzzy closure operator. Furthermore, it induces a fuzzy topology which is finer than a given fuzzy topology. We investigate some properties of fuzzy g-closure operators.

1. Introduction and preliminaries

Šostak [13] introduced the fuzzy topology as an extension of Chang's fuzzy topology [2]. It has been developed in many directions [3, 4, 7-10]. Balasubramanian and Sundaram [1] gave the concept of generalized fuzzy closed sets in a Chang's fuzzy topology as an extension of generalized closed sets of Levine [11] in topological spaces.

In this paper, we introduce a fuzzy g-closure operator induced by Šostak's fuzzy topological space. We show that it is a fuzzy closure operator. Furthermore, it induces a fuzzy topology which is finer than a given fuzzy topology. We investigate some properties of (generalized) fuzzy continuous maps and fuzzy generalized irresolute maps. Moreover, we study the relationship between (resp. strongly) r-closed graphs and r-FT_2 (resp. r-$FT_{2\frac{1}{2}}$) spaces.

Throughout this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$ and I^X be the family of all fuzzy sets. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

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Let $Pt(X)$ be the family of all fuzzy points in X. For $\mu, \lambda \in I^X$, μ is called *quasi-coincident* with λ, denoted by $\mu\ q\ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu\ \overline{q}\ \lambda$. Let χ_A be a characteristic function for A.

DEFINITION 1.1 ([13]). A function $\tau : I^X \to I$ is called a *fuzzy topology* on X if it satisfies the following conditions:
(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,
(O3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$.
The pair (X, τ) is called a *fuzzy topological space* (for short, fts).

DEFINITION 1.2 ([3, 10]). A function $C : I^X \times I_0 \to I^X$ is called a *fuzzy closure operator* if it satisfies the following conditions: for $\lambda, \mu \in I^X$ and $r, s \in I_0$,
(C1) $C(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C(\lambda, r)$,
(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,
(C4) $C(\lambda, r) \leq C(\lambda, s)$, if $r \leq s$,
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

THEOREM 1.3 ([3, 10]). *Let C be a fuzzy closure operator on X. Define a function $\tau_C : I^X \to I$ on X by*

$$\tau_C(\lambda) = \bigvee \{r \in I \mid C(\overline{1} - \lambda, r) = \overline{1} - \lambda\}.$$

Then τ_C is a fuzzy topology on X.

THEOREM 1.4 ([3, 8, 10]). *Let (X, τ) be a fts. We define operators $C_\tau, I_\tau : I^X \times I_0 \to I^X$ as follows:*

$$C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq \mu, \tau(\overline{1} - \mu) \geq r\},$$

$$I_\tau(\lambda, r) = \bigvee \{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r\}.$$

Then:
(1) *C_τ is a fuzzy closure operator.*
(2) $\tau_{C_\tau} = \tau$.
(3) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$, *for each $r \in I_0, \lambda \in I^X$.*

DEFINITION 1.5 [9, 10]. Let $\overline{0} \notin \Theta$ be a subset of I^X. A function $\beta : \Theta \to I$ is called a *fuzzy basis* on X if it satisfies the following conditions:

(B1) $\beta(\overline{1}) = 1$,

(B2) $\beta(\mu_1 \wedge \mu_2) \geq \beta(\mu_1) \wedge \beta(\mu_2)$, for all $\mu_1, \mu_2 \in \Theta$.

THEOREM 1.6 [9, 10]. *Let $\beta : \Theta \to I$ be a fuzzy basis on X. For each $\mu \in I^X$, we define the function $\tau_\beta : I^X \to I$ as follows:*

$$\tau_\beta(\mu) = \begin{cases} \bigvee\{\bigwedge_{j\in\Lambda} \beta(\mu_j)\} & \text{if } \mu = \bigvee_{j\in\Lambda} \mu_j, \ \text{ for } \{\mu_j\}_{j\in\Lambda} \subset \Theta, \\ 1 & \text{if } \mu = \overline{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) (X, τ_β) is a fuzzy topological space.

(2) A map $f : (Y, \tau') \to (X, \tau_\beta)$ is fuzzy continuous if and only if for each $\mu \in \Theta$, $\tau'(f^{-1}(\mu)) \geq \beta(\mu)$.

THEOREM 1.7 [9, 10]. *Let $\{(X_i, \tau_i)\}_{i\in\Gamma}$ be a family of fuzzy topological spaces, X a set and for each $i \in \Gamma$, $f_i : X \to X_i$ a function. Let*

$$\Theta = \{\overline{0} \neq \mu = \wedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j}) \mid \tau_{k_j}(\nu_{k_j}) > 0 \text{ for all } k_j \in K\}$$

for every finite index sets $K = \{k_1, \cdots, k_n\} \subset \Gamma$. Define the function $\beta : \Theta \to I$ on X by

$$\beta(\mu) = \bigvee\{\wedge_{j=1}^{n} \tau_{k_j}(\nu_{k_j}) \mid \mu = \wedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\}$$

for every finite index sets $K = \{k_1, \cdots, k_n\} \subset \Gamma$. Then:

(1) β is a fuzzy basis on X.

(2) The fuzzy topology τ_β generated by β is the coarsest fuzzy topology on X which for each $i \in \Gamma$, f_i is fuzzy continuous.

(3) A function $f : (Y, \tau') \to (X, \tau_\beta)$ is fuzzy continuous if and only if for each $i \in \Gamma$, $f_i \circ f : (Y, \tau') \to (X_i, \tau_i)$ is fuzzy continuous.

Let (X, τ) be a fuzzy topological space and A be a subset of X. The pair (A, τ_A) is said to be a *subspace* of (X, τ) if τ_A is endowed with the coarsest fuzzy topology on A for which the inclusion map i is fuzzy continuous.

Let X be the product $\prod_{i\in\Gamma} X_i$ of the family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of fuzzy topological spaces. The coarsest fuzzy topology $\tau = \otimes_{i\in\Gamma}\tau_i$ on X for which each the projections $\pi_i : X \to X_i$ is fuzzy continuous is called the *product fuzzy topology* of $\{\tau_i \mid i \in \Gamma\}$, and (X, τ) is called the *product fuzzy topology space.*

2. Fuzzy g-closure operators

DEFINITION 2.1. Let (X, τ) be a fts , $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) A fuzzy set λ is called *r-generalized fuzzy closed* (for short, r-gfc) if $C_\tau(\lambda, s) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq s$ for all $0 < s \leq r$.

(2) A fuzzy set λ is called *r-generalized fuzzy open* (for short, r-gfo) if $\overline{1} - \lambda$ is r-gfc.

THEOREM 2.2. *Let (X, τ) be a fts.*

(1) *If λ_1 and λ_2 are r-gfc sets, then $\lambda_1 \vee \lambda_2$ is a r-gfc set.*

(2) *If λ is r-gfc set and $\lambda \leq \mu \leq C_\tau(\lambda, r)$, then μ is a r-gfc set.*

(3) *If $\tau(\overline{1} - \lambda) \geq r$ and $r \in I_0$, then λ is a r-gfc set.*

(4) *λ is r-gfo if and only if $\mu \leq I_\tau(\lambda, r)$ whenever $\mu \leq \lambda$ and $\tau(\overline{1} - \mu) \geq r$.*

(5) *If λ_1 and λ_2 are r-gfo sets, then $\lambda_1 \wedge \lambda_2$ is a r-gfo set.*

(6) *If $I_\tau(\lambda, r) \leq \mu \leq \lambda$ and λ is r-gfo, then μ is r-gfo.*

(7) *If $\tau(\lambda) \geq r$ and $r \in I_0$, then λ a r-gfo set.*

PROOF. (1) Let λ_1 and λ_2 be r-gfc sets and $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau(\mu) \geq s$, for $0 < s \leq r$. For $i \in \{1, 2\}$, $\lambda_i \leq \mu$ such that $\tau(\mu) \geq s$, for $0 < s \leq r$, we have $C_\tau(\lambda_i, s) \leq \mu$. By (C3) of Definition 1.2, it implies, for $0 < s \leq r$,

$$C_\tau(\lambda_1 \vee \lambda_2, s) = C_\tau(\lambda_1, s) \vee C_\tau(\lambda_2, s) \leq \mu.$$

Hence $\lambda_1 \vee \lambda_2$ is r-gfc.

(2) For $\mu \leq \rho$ such that $\tau(\rho) \geq s$, for $0 < s \leq r$, since λ is r-gfc set and $\lambda \leq \mu$, $\lambda \leq \rho$ implies $C_\tau(\lambda, s) \leq \rho$. Also, $\mu \leq C_\tau(\lambda, s)$ implies

$$C_\tau(\mu, s) \leq C_\tau(C_\tau(\lambda, s), s) = C_\tau(\lambda, s) \leq \rho.$$

Hence μ is r-gfc. Others are easily proved. □

DEFINITION 2.3. Let (X, τ) be a fts. A *fuzzy g-closure operator induced by (X, τ)* is a map $GC_\tau : I^X \times I_0 \to I^X$ as follows:

$$GC_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \lambda \leq \mu, \ \mu \text{ is r-gfc}\}.$$

THEOREM 2.4. *Let (X, τ) be a fts. Then it holds the following properties.*

(1) *GC_τ is a fuzzy closure operator such that $GC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$ for $\lambda \in I^X$ and $r \in I_0$.*

(2) *Define a function $\tau_G : I^X \to I$ on X by*

$$\tau_G(\lambda) = \bigvee\{r \in I \mid GC_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda\}.$$

Then τ_G is a fuzzy topology on X such that $\tau(\lambda) \leq \tau_G(\lambda)$ for all $\lambda \in I^X$.

PROOF. (1) (C1), (C2) and (C4) are easily proved from the definition of GC_τ.

(C3) Since $\lambda, \mu \leq \lambda \vee \mu$, we have

$$GC_\tau(\lambda, r) \vee GC_\tau(\mu, r) \leq GC_\tau(\lambda \vee \mu, r).$$

Suppose $GC_\tau(\lambda, r) \vee GC_\tau(\mu, r) \not\geq GC_\tau(\lambda \vee \mu, r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$(A) \qquad GC_\tau(\lambda, r)(x) \vee GC_\tau(\mu, r)(x) < t < GC_\tau(\lambda \vee \mu, r)(x).$$

Since $GC_\tau(\lambda, r)(x) < t$ and $GC_\tau(\mu, r)(x) < t$, there exist r-gfc sets λ_1, μ_1 with $\lambda \leq \lambda_1$ and $\mu \leq \mu_1$ such that

$$\lambda_1(x) < t, \mu_1(x) < t.$$

Since $\lambda \vee \mu \leq \lambda_1 \vee \mu_1$ and $\lambda_1 \vee \mu_1$ is r-gfc from Theorem 2.2 (1), we have $GC_\tau(\lambda \vee \mu, r)(x) \leq (\lambda_1 \vee \mu_1)(x) < t$. It is a contradiction for (A).

(C5) From (C2) and (C3), we have $GC_\tau(\lambda, r) \leq GC_\tau(GC_\tau(\lambda, r), r)$. Suppose

$$GC_\tau(\lambda, r) \not\geq GC_\tau(GC_\tau(\lambda, r), r).$$

There exist $x \in X$ and $t \in (0, 1)$ such that

$$(B) \qquad GC_\tau(\lambda, r)(x) < t < GC_\tau(GC_\tau(\lambda, r), r)(x).$$

Since $GC_\tau(\lambda, r)(x) < t$, there exists r-gfc set λ_1 with $\lambda \leq \lambda_1$ such that

$$GC_\tau(\lambda, r)(x) \leq \lambda_1(x) < t.$$

Since $\lambda \leq \lambda_1$, we have $GC_\tau(\lambda, r) \leq \lambda_1$. Again, $GC_\tau(GC_\tau(\lambda, r), r) \leq \lambda_1$. Hence $GC_\tau(GC_\tau(\lambda, r), r)(x) \leq \lambda_1(x) < t$. It is a contradiction for (B). Thus,

$$GC_\tau(\lambda, r) \geq GC_\tau(GC_\tau(\lambda, r), r).$$

Thus, GC_τ is a fuzzy closure operator. Since $C_\tau(\lambda, r)$ is r-gfc, then $GC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.

(2) By Theorem 1.3, τ_G is a fuzzy topology on X. By (1), $C_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda$ implies $GC_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda$. Thus, $\tau(\lambda) \leq \tau_G(\lambda)$ for all $\lambda \in I^X$. □

EXAMPLE 2.5. Let X be a nonempty set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{3} & \text{if } \lambda = \overline{0.3}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.5}, \\ \frac{1}{4} & \text{if } \lambda = \overline{0.6}, \\ 0 & \text{otherwise.} \end{cases}$$

(1) If $\overline{0.4} \leq \lambda < \overline{0.5}$ and $0 < r \leq \frac{1}{4}$, λ is r-gfc.
(2) If $\lambda = \overline{0.5}$ and $0 < r \leq \frac{1}{2}$, λ is r-gfc.
(3) If $\lambda > \overline{0.6}$ and $r \in I_0$, λ is r-gfc.
We can obtain a fuzzy topology $\tau_G : I^X \to I$ as follows:

$$\tau_G(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{1}, \\ \frac{1}{4} & \text{if } \overline{0.5} < \lambda \leq \overline{0.6}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.5}, \\ 1 & \text{if } \lambda < \overline{0.4}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\tau(\lambda) \leq \tau_G(\lambda)$ for all $\lambda \in I^X$.

NOTATION 2.6. Let (X, τ) be a fts and $x_t \in Pt(X)$. We denote

$$Q_\tau(x_t, r) = \{\mu \in I^X \mid x_t \, q \, \mu, \;\; \tau(\mu) \geq r\},$$

$$\mathcal{G}_\tau(x_t, r) = \{\mu \in I^X \mid x_t \, q \, \mu, \;\; \mu \text{ is r-gfo}\}.$$

DEFINITION 2.7. Let (X, τ) be a fts , $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) x_t is called a *r-cluster point* of λ if for each $\mu \in Q_\tau(x_t, r)$, we have $\mu \, q \, \lambda$.

(2) x_t is called a *rg-cluster point* of λ if for each $\mu \in \mathcal{G}_\tau(x_t, r)$, we have $\mu \, q \, \lambda$.

THEOREM 2.8. *Let (X, τ) be a fts.*
(1) $C_\tau(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid$ *x_t is a r-cluster point of λ* $\}$.
(2) $GC_\tau(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid$ *x_t is a rg-cluster point of λ*$\}$.
(3) *x_t is a r-cluster point of λ if and only if $x_t \in C_\tau(\lambda, r)$.*
(4) *x_t is a rg-cluster point of λ if and only if $x_t \in GC_\tau(\lambda, r)$.*

PROOF. (1) and (3) are similarly proved as following (2) and (4).

(2) Put $\rho = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a rg-cluster point of } \lambda\}$.

Suppose $GC_\tau(\lambda, r) \not\leq \rho$. Then there exists $x \in X$ and $t \in (0,1)$ such that

$$(C) \qquad GC_\tau(\lambda, r)(x) > t > \rho(x).$$

Since $\rho(x) < t$, then x_t is not a rg- cluster point of λ. There exists $\mu \in \mathcal{G}(x_t, r)$ with $\lambda \overline{q} \mu$. So, $\lambda \leq \overline{1} - \mu$ with r-gfo $\overline{1} - \mu$ implies $GC_\tau(\lambda, r)(x) \leq (\overline{1} - \mu)(x) < t$. It is a contradiction for (C). Hence $GC_\tau(\lambda, r) \leq \rho$.

Suppose $GC_\tau(\lambda, r) \not\geq \rho$. Then there exists $x \in X$ and $s \in (0,1)$ such that

$$(D) \qquad GC_\tau(\lambda, r)(y) < s < \rho(y).$$

Since $GC_\tau(\lambda, r)(y) < s$, by the definition of GC_τ, there exists r-gfc $\mu \in I^X$ with $\lambda \leq \mu$ such that

$$GC_\tau(\lambda, r)(y) \leq \mu(y) < s < \rho(y).$$

There exists $\overline{1} - \mu \in \mathcal{G}(y_s, r)$ with $\lambda \ \overline{q} \ (\overline{1} - \mu)$. Hence y_s is not a rg-cluster point of λ. It is a contradiction for (D). So, $GC_\tau(\lambda, r) \geq \rho$.

(4) ($\Rightarrow$) It is trivial.

($\Leftarrow$) Let x_t be not a rg- cluster point of λ. There exists $\mu \in \mathcal{G}_\tau(x_t, r)$ such that $\mu \ \overline{q} \ \lambda$, that is, $\lambda \leq \overline{1} - \mu$. It implies

$$GC_\tau(\lambda, r)(x) \leq (\overline{1} - \mu)(x) < t.$$

Thus, $x_t \notin GC_\tau(\lambda, r)$. □

3. Strongly r-closed graphs and r-closed graphs

DEFINITION 3.1. Let (X, τ) and (Y, η) be fts's. Let $f : (X, \tau) \to (Y, \eta)$ be a function.

(1) f is called *fuzzy continuous* if $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) f is called *generalized fuzzy continuous* (for short, gf-continuous) if $f^{-1}(\mu)$ is r-gfc for $\eta(\overline{1} - \mu) \geq r$.

(3) f is called *generalized fuzzy irresolute* (for short, gf-irresolute) if $f^{-1}(\mu)$ is r-gfc for each r-gfc set $\mu \in I^Y$.

THEOREM 3.2. *Let (X, τ) and (Y, η) be fts's satisfying the condition:*
(T) $\tau_G(\overline{1} - \lambda) \geq r$ *implies* $GC_\tau(\lambda, r) = \lambda$.
Then the following statements are equivalent.
(1) $f : (X, \tau_G) \to (Y, \eta_G)$ *is fuzzy continuous.*
(2) $f(GC_\tau(\lambda, r)) \leq GC_\eta(f(\lambda), r)$, *for each* $\lambda \in I^X$ *and* $r \in I_0$.
(3) $GC_\tau(f^{-1}(\mu), r) \leq f^{-1}(GC_\eta(\mu, r))$, *for each* $\mu \in I^Y$ *and* $r \in I_0$.

PROOF. (1) $\Rightarrow$ (2). Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that

$$f(GC_\tau(\lambda, r)) \not\leq GC_\eta(f(\lambda), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f(GC_\tau(\lambda, r))(y) > t > GC_\eta(f(\lambda), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it is a contradiction since $f(GC_\tau(\lambda, r))(y) = 0$.

If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$(E) \qquad f(GC_\tau(\lambda, r))(y) \geq GC_\tau(\lambda, r)(x) > t > GC_\eta(f(\lambda), r)(f(x)).$$

Since $GC_\eta(f(\lambda), r)(f(x)) < t$, by the definition of GC_η, there exists r-gfc $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that

$$(F) \qquad GC_\eta(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.$$

Since $\lambda \leq f^{-1}(\mu)$, $GC_\tau(f^{-1}(\mu), r) \geq GC_\tau(\lambda, r)$. By (E) and (F),

$$GC_\tau(f^{-1}(\mu), r)(x) \geq GC_\tau(\lambda, r)(x) > t > \mu(f(x)) = f^{-1}(\mu)(x).$$

By **(T)**, $GC_\tau(f^{-1}(\mu), r) \neq f^{-1}(\mu)$ implies $\tau_G(\overline{1} - f^{-1}(\mu)) < r$. Moreover, $\eta_G(\overline{1} - \mu) \geq r$ because $GC_\tau(\mu, r) = \mu$. So, $\eta_G(\overline{1} - \mu) \geq r > \tau_G(f^{-1}(\overline{1} - \mu))$. Hence $f : (X, \tau_G) \to (Y, \eta_G)$ is not fuzzy continuous.

(2) $\Rightarrow$ (3). By (2), put $\lambda = f^{-1}(\mu)$. Since $f(f^{-1}(\mu)) \leq \mu$, then

$$GC_\tau(f^{-1}(\mu), r) \leq f^{-1}(f(GC_\tau(f^{-1}(\mu), r))) \leq f^{-1}(GC_\eta(\mu, r)).$$

(3) $\Rightarrow$ (1). Since $GC_\eta(\mu, r) = \mu$ implies $GC_\tau(f^{-1}(\mu), r) = f^{-1}(\mu)$, we have $\tau_G(\overline{1} - f^{-1}(\mu)) = \tau_G(f^{-1}(\overline{1} - \mu)) \geq \eta_G(\overline{1} - \mu)$ for all $\mu \in I^Y$. □

THEOREM 3.3. *Let (X, τ) and (Y, η) be fts's. If $f : (X, \tau) \to (Y, \eta)$ is gf-irresolute, then $f : (X, \tau_G) \to (Y, \eta_G)$ is fuzzy continuous.*

PROOF. Suppose there exist $\mu \in I^Y$ such that $\tau_G(f^{-1}(\mu)) \not\geq \eta_G(\mu)$. Then there exists $r \in I_0$ with $GC_\eta(\overline{1}-\mu, r) = \overline{1}-\mu$ such that

$$(G) \qquad \tau_G(f^{-1}(\mu)) < r \leq \eta_G(\mu).$$

Since $GC_\eta(\overline{1}-\mu, r) = \overline{1}-\mu$ and f is gf-irresolute,

$$\begin{aligned} f^{-1}(\overline{1}-\mu) &= f^{-1}(GC_\eta(\overline{1}-\mu, r)) \\ &= f^{-1}\Big(\bigwedge\{\rho \in I^Y \mid \overline{1}-\mu \leq \rho, \rho \text{ is r-gfc}\}\Big) \\ &= \bigwedge\{f^{-1}(\rho) \in I^Y \mid \overline{1}-\mu \leq \rho, \rho \text{ is r-gfc}\} \\ &\geq \bigwedge\{f^{-1}(\rho) \in I^Y \mid f^{-1}(\overline{1}-\mu) \leq f^{-1}(\rho), f^{-1}(\rho) \text{ is r-gfc}\} \\ &\geq GC_\tau(\overline{1}-f^{-1}(\mu), r). \end{aligned}$$

It implies $GC_\tau(\overline{1}-f^{-1}(\mu), r) = \overline{1}-f^{-1}(\mu)$ from Theorem 2.4(1). Hence $\tau_G(f^{-1}(\mu)) \geq r$. It is a contradiction for (G). □

EXAMPLE 3.4. The converse of Theorem 3.3 is not true. Let $X = \{a, b\}$ be a set. We define fuzzy topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.8}, \\ 0 & \text{otherwise,} \end{cases} \quad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.6}, \\ 0 & \text{otherwise.} \end{cases}$$

Then an identity function $id_X : (X, \tau) \to (X, \eta)$ is not fuzzy continuous. A fuzzy point $a_{0.7}$ is a $\frac{1}{2}$- gfc set on (X, η) but $a_{0.7}$ is not a $\frac{1}{2}$- gfc set on (X, τ) because

$$a_{0.7} \leq a_{0.8}, \tau(a_{0.8}) \geq s, 0 < s \leq \frac{1}{2}, C_\tau(a_{0.7}, s) = \overline{1} \not\leq a_{0.8}.$$

Thus, $id_X : (X, \tau) \to (X, \eta)$ is not a gf-irresolute map.

(1) For a_t with $0 < t \leq 0.8$, $a_t \vee b_s$ is 1-gfc on (X, τ). Thus,

$$GC_\tau(a_t, 1) = \bigwedge_{s \in I_0} (a_t \vee b_s) = a_t \vee \bigwedge_{s \in I_0} b_s = a_t.$$

(2) For $\lambda \in I^X - \{a_t \mid 0 < t \leq 0.8\}$, λ is a 1-gfc set. So, $GC_\tau(\lambda, 1) = \lambda$. By (1) and (2), $GC_\tau(\lambda, r) = \lambda$ for all $\lambda \in I^X$ and $r \in I_0$. Similarly, $GC_\eta(\lambda, r) = \lambda$ for all $\lambda \in I^X$ and $r \in I_0$. We can obtain fuzzy topologies

$$\tau_G(\lambda) = \eta_G(\lambda) = 1, \ \forall \lambda \in I^X.$$

The identity function $id_X : (X, \tau_G) \to (X, \eta_G)$ is fuzzy continuous.

EXAMPLE 3.5. We define fuzzy topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.7}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.4}, \\ 0 & \text{otherwise}, \end{cases} \quad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.4}, \\ 0 & \text{otherwise}. \end{cases}$$

We can obtain fuzzy topologies $\tau_G, \eta_G : I^X \to I$ as follows:

$$\tau_G(\lambda) = \begin{cases} 1 & \text{if } \overline{0} \le \lambda < \overline{0.3}, \\ 1 & \text{if } \overline{0.4} \le \lambda < \overline{0.6}, \\ 1 & \text{if } \overline{0.7} \le \lambda, \\ 0 & \text{otherwise}, \end{cases} \quad \eta_G(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ 1 & \text{if } \overline{0} < \lambda < \overline{0.6}, \\ 0 & \text{otherwise}. \end{cases}$$

Then an identity function $id_X : (X, \tau) \to (X, \eta)$ is fuzzy continuous. But $id_X : (X, \tau_G) \to (X, \eta_G)$ is not fuzzy continuous because

$$0 = \tau_G(\overline{0.35}) < \eta_G(\overline{0.35}) = 1.$$

DEFINITION 3.6. A fts (X, τ) is called:

(1) r-FT_2 if for each $x_t \neq y_s$, there exist $\mu_1 \in Q_\tau(x_t, r)$ and $\mu_2 \in Q_\tau(y_s, r)$ such that $\mu_1 \wedge \mu_2 = \overline{0}$.

(2) r-$FT_{2\frac{1}{2}}$ if for each $x_t \neq y_s$, there exist $\mu_1 \in Q_\tau(x_t, r)$ and $\mu_2 \in Q_\tau(y_s, r)$ such that $C_\tau(\mu_1, r) \wedge C_\tau(\mu_2, r) = \overline{0}$.

THEOREM 3.7. *Let (A, τ_A) be a subspace of (X, τ) and $R : (X, \tau) \to (A, \tau_A)$ be a gf-continuous retraction ,that is, $R(a) = a$ for all $a \in A$. If (X, τ) is r-FT_2, then $GC_\tau(\chi_A, r) = \chi_A$.*

PROOF. Suppose $GC_\tau(\chi_A, r) \not\leq \chi_A$. Then there exist $x \in X$ and $t \in (0, 1)$ such that

$$GC_\tau(\chi_A, r)(x) \geq t > \chi_A(x).$$

Since $\chi_A(x) < t$, then $x \notin A$. So, $R(x) \neq x$ implies $R(x)_t \neq x_t$. Since (X, τ) is r-FT_2, there exist $\mu_1 \in Q_\tau(R(x)_t, r)$ and $\mu_2 \in Q_\tau(x_t, r)$ such that $\mu_1 \wedge \mu_2 = \overline{0}$. Since $i : (A, \tau_A) \to (X, \tau)$ is an inclusion map and

$$\Big(i^{-1}(\mu_1)(R(x)) = \mu_1(R(x))\Big) + t > 1,$$

then $i^{-1}(\mu_1) \in Q_{\tau_A}(R(x)_t, r)$. Since $R : X \to A$ is a gf-continuous function, $R^{-1}(i^{-1}(\mu_1))$ is r-gfo. Hence $R^{-1}(i^{-1}(\mu_1)) \in \mathcal{G}_\tau(x_t, r)$. Furthermore, since $\mu_2 \in \mathcal{G}_\tau(x_t, r)$, $\Big(R^{-1}(i^{-1}(\mu_1)) \wedge \mu_2\Big) \in \mathcal{G}_\tau(x_t, r)$. Since $GC_\tau(\chi_A, r)(x) \geq t$, that is, $x_t \in GC_\tau(\chi_A, r)$, by Theorem 2.8 (4), x_t is a rg- cluster point of χ_A. Thus,

$$\chi_A \, q \, \Big(R^{-1}(i^{-1}(\mu_1)) \wedge \mu_2\Big).$$

So, there exists $y \in X$ such that

$$\chi_A(y) + \Big(R^{-1}(i^{-1}(\mu_1)) \wedge \mu_2\Big)(y) > 1.$$

It implies $y \in A$ and $R(y) = y$ because R is a retraction. Hence

$$\Big(R^{-1}(i^{-1}(\mu_1)) \wedge \mu_2\Big)(y) = \mu_1(R(y)) \wedge \mu_2(y) = \mu_1(y) \wedge \mu_2(y) > 0.$$

It is a contradiction from $\mu_1 \wedge \mu_2 = \overline{0}$. Thus, $GC_\tau(\chi_A, r) = \chi_A$. □

THEOREM 3.8. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a gf-continuous function which (Y, η) is r-FT_2 and $C_{\tau\otimes\eta}(\mu, r) = \mu$ for $\mu \in I^{X\times Y}$. Then $GC_\tau(\pi_1(\mu \wedge \chi_{G(f)}), r) = \pi_1(\mu \wedge \chi_{G(f)})$, where $G(f) = \{(x, f(x)) \mid x \in X\}$ and π_1 is the projection of $X \times Y$ onto X.*

PROOF. Suppose

$$GC_\tau(\pi_1(\mu \wedge \chi_{G(f)}), r) \not\leq \pi_1(\mu \wedge \chi_{G(f)}).$$

Then there exist $x \in X$ and $t \in (0, 1)$ such that

$$(G) \qquad GC_\tau(\pi_1(\mu \wedge \chi_{G(f)}), r)(x) \geq t > \pi_1(\mu \wedge \chi_{G(f)})(x).$$

Let $\lambda \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(f(x)_t, r)$ such that $\Big(\pi_1^{-1}(\lambda) \wedge \pi_2^{-1}(\rho)\Big) \in Q_{\tau\otimes\eta}((x, f(x))_t, r)$. Since $f : X \to Y$ is gf-continuous, then $f^{-1}(\rho) \in \mathcal{G}_\tau(x_t, r)$. So, $(\lambda \wedge f^{-1}(\rho)) \in \mathcal{G}_\tau(x_t, r)$. Since $x_t \in GC_\tau(\pi_1(\mu \wedge \chi_{G(f)}), r)$, we have

$$(\lambda \wedge f^{-1}(\rho)) \, q \, \pi_1(\mu \wedge \chi_{G(f)}).$$

So, there exists $z \in X$ such that

$$(\lambda \wedge f^{-1}(\rho))(z) + \pi_1(\mu \wedge \chi_{G(f)})(z) > 1.$$

Since $\pi_1(\mu \wedge \chi_{G(f)})(z) > 0$, we have

$$(\lambda \wedge f^{-1}(\rho))(z) + \mu(z, f(z)) > 1.$$

It implies

$$\left(\pi_1^{-1}(\lambda) \wedge \pi_2^{-1}(\rho)\right)(z, f(z)) + \mu(z, f(z)) > 1.$$

Thus,

$$\left(\pi_1^{-1}(\lambda) \wedge \pi_2^{-1}(\rho)\right) \, q \, \mu.$$

Moreover, $\left(\pi_1^{-1}(\lambda) \wedge \pi_2^{-1}(\rho)\right) \in Q_{\tau\otimes\eta}((x, f(x))_t, r)$. Hence

$$(x, f(x))_t \in C_{\tau\otimes\eta}(\mu, r) = \mu, \quad (x, f(x))_t \in \chi_{G(f)}.$$

It implies $x_t \in \pi_1(\mu \wedge \chi_{G(f)})$. It is a contradiction for (G). □

THEOREM 3.9. *Let (X, τ) and (Y, η) be fts's. If $f : X \to Y$ has a r-closed graph, that is, $C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$ and $g : X \to Y$ be a gf-continuous, then $GC_\tau(\chi_A, r) = \chi_A$ where $A = \{x \in X \mid f(x) = g(x)\}$.*

PROOF. Since $\chi_A = \pi_1(\chi_{G(f)} \wedge \chi_{G(g)})$ and $C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$, by Theorem 3.8, we have $GC_\tau(\chi_A, r) = \chi_A$. □

THEOREM 3.10. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a fuzzy continuous function which (Y, τ) is r-FT_2. Then:*

(1) $C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$ where $G(f) = \{(x, f(x)) \mid x \in X\}$.

(2) If $g : X \to Y$ is a gf-continuous function, then $GC_\tau(\chi_A, r) = \chi_A$ where $A = \{x \in X \mid f(x) = g(x)\}$.

PROOF. (1) Suppose $C_{\tau\otimes\eta}(\chi_{G(f)}, r) \not\leq \chi_{G(f)}$. Then there exist $(x, y) \in X \times Y$ and $t \in (0, 1)$ such that

$$C_{\tau\otimes\eta}(\chi_{G(f)}, r)(x, y) \geq t > \chi_{G(f)}(x, y).$$

Since $\chi_{G(f)}(x, y) < t$, $(x, y) \notin G(f)$, that is, $f(x) \neq y$. Since (Y, τ) is r-FT_2, for $f(x)_t \neq y_t$, there exist $\lambda \in Q_\eta(y_t, r)$ and $\rho \in Q_\eta(f(x)_t, r)$ such that $\lambda \wedge \mu = \overline{0}$. Since f is fuzzy continuous, then $f^{-1}(\rho) \in Q_\tau(x_t, r)$.

On the other hand, since $(x, y)_t \in C_{\tau\otimes\eta}(\chi_{G(f)}, r)$, then $(x, y)_t$ is r-cluster point of $\chi_{G(f)}$. For $\pi_1^{-1}(f^{-1}(\rho)) \wedge \pi_2^{-1}(\lambda) \in Q_{\tau\otimes\eta}((x, y)_t, r)$, we have

$$\left(\pi_1^{-1}(f^{-1}(\rho)) \wedge \pi_2^{-1}(\lambda)\right) \, q \, \chi_{G(f)}.$$

There exists $(a, b) \in X \times Y$ such that

$$\left(\pi_1^{-1}(f^{-1}(\rho)) \wedge \pi_2^{-1}(\lambda)\right)(a, b) + \chi_{G(f)}(a, b) > 1.$$

Since $\chi_{G(f)}(a, b) = 1$, then $b = f(a)$. So,

$$\left(\pi_1^{-1}(f^{-1}(\rho)) \wedge \pi_2^{-1}(\lambda)\right)(a, b) = \rho(f(a)) \wedge \lambda(f(a)) \neq \overline{0}.$$

It is a contradiction for $\lambda \wedge \mu = \overline{0}$.

(2) It is easily proved from (1) and Theorem 3.9. □

LEMMA 3.11. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a function. For $\mu \in I^X$ and $\rho \in I^Y$, we have the following properties:*

(1) $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \overline{q}\, \chi_{G(f)}$ if and only if $f(\mu) \wedge \rho = \overline{0}$.

(2) $C_{\tau \otimes \eta}(\chi_{G(f)}, r) = \chi_{G(f)}$ if and only if for each $x_t \in Pt(X), y_s \in Pt(Y)$ with $f(x) \neq y$, there exist $\mu \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(y_s, r)$ such that $f(\mu) \wedge \rho = \overline{0}$.

PROOF. (1) ($\Rightarrow$) Let $f(\mu) \wedge \rho \neq \overline{0}$. There exists $y \in Y$ such that $f(\mu)(y) \wedge \rho(y) > 0$. By the definition of $f(\mu)$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(\mu)(y) \wedge \rho(y) \geq \mu(x) \wedge \rho(f(x)) > 0.$$

It implies

$$\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right)(x, f(x)) + \chi_{G(f)}(x, f(x)) > 1.$$

Thus, $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) q\, \chi_{G(f)}$.

($\Leftarrow$) Let $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) q\, \chi_{G(f)}$. Then there exists $(x, y) \in X \times Y$ such that

$$\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right)(x, y) + \chi_{G(f)}(x, y) > 1.$$

Thus, $y = f(x)$ with $\mu(x) \wedge \rho(f(x)) > 0$. It implies $f(\mu)(f(x)) \wedge \rho(f(x)) > 0$. Thus, $f(\mu) \wedge \rho \neq \overline{0}$.

(2) ($\Rightarrow$) Let $x_t \in Pt(X), y_s \in Pt(Y)$ with $f(x) \neq y$. Put $p = t \wedge s$. Since $(x, y)_p \notin \chi_{G(f)} = C_{\tau \otimes \eta}(\chi_{G(f)}, r)$, by Theorem 2.8 (3), $(x, y)_p$ is not r-cluster point of $\chi_{G(f)}$. So, there exist $\mu \in Q_\tau(x_p, r)$ and $\rho \in Q_\eta(y_p, r)$ with $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \in Q_{\tau \otimes \eta}((x, y)_t, r)$ such that

$$\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \overline{q}\, \chi_{G(f)}.$$

By (1), it implies $f(\mu) \wedge \rho = \overline{0}$. Furthermore, $\mu \in Q_\tau(x_p, r)$ and $\rho \in Q_\eta(y_p, r)$ imply $\mu \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(y_s, r)$.

($\Leftarrow$) Suppose $C_{\tau\otimes\eta}(\chi_{G(f)}, r) \not\leq \chi_{G(f)}$. Then there exist $(x, y) \in X \times Y$ and $t \in (0, 1)$ such that

$$(H) \qquad C_{\tau\otimes\eta}(\chi_{G(f)}, r)(x, y) \geq t > \chi_{G(f)}(x, y).$$

Since $\chi_{G(f)}(x, y) < t$, then $(x, y) \notin G(f)$, that is, $f(x)_t \neq y_t$. There exist $\mu \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(y_t, r)$ such that $f(\mu) \wedge \rho = \overline{0}$. By (1),

$$\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \overline{q} \chi_{G(f)}.$$

Hence $(x, y)_t$ is not a r-cluster point of $\chi_{G(f)}$. By Theorem 2.8 (3), $C_{\tau\otimes\eta}(\chi_{G(f)}, r)(x, y) < t$. It is a contradiction for (H). Hence we have
$C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$. □

THEOREM 3.12. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a fuzzy continuous injective function with $C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$. Then (X, τ) is r-FT_2.*

PROOF. (1) ($\Rightarrow$) Let $a, b \in X$ with $a \neq b$. Then $f(a) \neq f(b)$. So, $\chi_{G(f)}(a, f(b)) = 0$. Since $(a, f(b))_t \notin \chi_{G(f)} = C_{\tau\otimes\eta}(\chi_{G(f)}, r)$, then $(a, f(b))_t$ is not r-cluster point of $\chi_{G(f)}$. Then there exist $\mu \in Q_\tau(a_t, r)$ and $\rho \in Q_\eta(f(b)_t, r)$ with $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \in Q_{\tau\otimes\eta}((a, f(b))_t, r)$ such that

$$\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(\rho)\right) \overline{q}\, \chi_{G(f)}.$$

By Lemma 3.11(1), $f(\mu) \wedge \rho = \overline{0}$. Since f is fuzzy continuous, $f^{-1}(\rho) \in Q_\tau(b_t, r)$ such that $f(\mu) \wedge f(f^{-1}(\rho)) = \overline{0}$. It implies $f(\mu \wedge f^{-1}(\rho)) = \overline{0}$ because f is injective. Therefore $\mu \wedge f^{-1}(\rho) = \overline{0}$. □

DEFINITION 3.13. Let (X, τ) and (Y, η) be fts's. A function $f : X \to Y$ has a *strongly r-closed graph* if for each $\chi_{G(f)}(x, y) = 0$, there exist $\mu \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(y_s, r)$ such that $\left(\pi_1^{-1}(\mu) \wedge \pi_2^{-1}(GC_\eta(\rho, r))\right) \wedge \chi_{G(f)} = \overline{0}$.

The following corollaries is easily proved from Lemma 3.11.

COROLLARY 3.14. *Let (X, τ) and (Y, η) be fts's. A function $f : X \to Y$ has a strongly r-closed graph if for each $\chi_{G(f)}(x, y) = 0$, there exist $\mu \in Q_\tau(x_t, r)$ and $\rho \in Q_\eta(y_s, r)$ such that $f(\mu) \wedge GC_\eta(\rho, r) = \overline{0}$.*

COROLLARY 3.15. *Let (X,τ) and (Y,η) be fts's. If $f : X \to Y$ has a strongly r-closed graph, then $C_{\tau\otimes\eta}(\chi_{G(f)}, r) = \chi_{G(f)}$, that is, f has a r-closed graph.*

THEOREM 3.16. *Let (X,τ) and (Y,η) be fts's. Let $f : X \to Y$ be fuzzy continuous which (Y,η) is r-$FT_{2\frac{1}{2}}$. Then f has a strongly r-closed graph.*

PROOF. Since $\chi_{G(f)}(x,y) = 0$, $(x,y) \notin G(f)$, that is, $f(x) \neq y$. Since (Y,τ) is r-$FT_{2\frac{1}{2}}$, for $f(x)_t \neq y_s$, there exist $\lambda \in Q_\eta(y_s, r)$ and $\rho \in Q_\eta(f(x)_t, r)$ such that $C_\eta(\lambda, r) \wedge C_\tau(\rho, r) = \overline{0}$. Since $f : X \to Y$ is fuzzy continuous, $f^{-1}(\rho) \in Q_\tau(x_t, r)$ such that

$$f(f^{-1}(\rho)) \leq \rho \leq C_\tau(\rho, r).$$

It implies $f(f^{-1}(\rho)) \wedge C_\eta(\lambda, r) = \overline{0}$. Since $GC_\eta(\lambda, r) \leq C_\eta(\lambda, r)$, we have $f(f^{-1}(\rho)) \wedge GC_\eta(\lambda, r) = \overline{0}$. Hence f has a strongly r-closed graph. □

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Department of Mathematics
Kangnung National University
Gangwondo 210-702, Korea
E-mail: yck@kangnung.ac.kr
jmko@kangnung.ac.kr