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ON SOME SEMI-INVARIANT SUBMANIFOLDS OF CODIMENSION 3 IN A COMPLEX PROJECTIVE SPACE

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Abstract. In this paper, We characterize a semi-invariant submanifold of codimension 3 satisfying $\nabla_\xi A = 0$ in a complex projective space $\mathbb{C}P^{n+1}$, where $\nabla_\xi A$ is the covariant derivative of the shape operator A in the direction of the distinguished normal with respect to the structure vector field ξ.

0. Introduction

A CR submanifold M is called a *semi-invariant submanifold* of a Kaehlerian manifold with complex structure J if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(\Delta, \Delta^\perp)$ such that dim $\Delta^\perp = 1$ and the unit normal in $J\Delta^\perp$ is called a *distinguished normal* ([1], [2], [17]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, g). A typical example of a semi-invariant submanifold is real hypersurfaces. But, new examples of nontrivial semi-invariant submanifold with higher codimension in a complex projective space are constructed in [9] and [14].

For the real hypersurface of a complex space form, many results are known ([3], [8], [10], [11], [15], [16] etc.). One of them Takagi ([15]) classified homogeneous real hypersurfaces of a complex projective space by means of six model spaces of type A_1, A_2, B, C, D and E, further he explicitly write down their principal curvatures and multiplicities in the table in [16]. Cecil and Ryan ([3]) extensively studied a real hypersurface

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which is realized a tube of constant radius r over a complex submanifold of a complex projective space $\mathbb{C}P^n$ on which ξ is principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the corresponding focal map φ_r has constant rank. From this point of view, Okumura ([11]) characterized real hypersurface of type A_1 and A_2 in $\mathbb{C}P^n$ by the property that the shape operator A and structure tensor field ϕ commute. From the different point of view, Ki, Kim and one of the present authors give another characterization of real hypersurfaces of type A_1 and A_2 of $\mathbb{C}P^n$ satisfying $\nabla_\xi A = 0$, where ∇_ξ denotes the covariant derivative with respect to the structure vector field ξ. Namely, they proved the following:

THEOREM A ([8]). *Let M be a connected real hypersurface of $\mathbb{C}P^n$ satisfying $\nabla_\xi A = 0$. Then M is a Hopf real hypersurface. Further if $\alpha \neq 0$, then M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$, where $0 < r < \frac{\pi}{2}$ and $r \neq \pi/4$),*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ have been investigated in [6],[7],[9],[18] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold. One of them, Ki, Song and Takagi ([9]) assert the following:

THEOREM B ([9]). *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ such that the third fundamental form satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< c/2)$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M. If the structure vector field ξ is an eigenvector for the shape operator A in the direction of the distinguished normal, then M is a Hopf real hypersurface in a complex projective space $\mathbb{C}P^n$.*

The main purpose of the present paper is to extend Theorem A under certain conditions on a semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$, that is, we prove

THEOREM. *Let M be a connected real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ such that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< c/2)$, where 2-form ω is defined by $\omega(X,Y) = g(X, \phi Y)$. If M satisfies $\nabla_\xi A =$*

0, then M is a Hopf real hypersurface of $\mathbb{C}P^n$. Further, if $g(A\xi,\xi)\neq 0$, then M is locally congruent to one of the following spaces:

(A_1) a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$, where $0<r<\pi/2$ and $r\neq\pi/4$),

(A_2) a tube of radius r over a totally geodesic $\mathbb{C}P^k(0\le k\le n-2)$, where $0<r<\pi/2$ and $r\neq\pi/4$.

All manifolds in this paper are assumed to be connected and of class C^∞, and the dimension of semi-invariant submanifold is greater then 2.

1. Preliminaries

In the following, we review fundamental properties of a submanifold of codimension 3 in a complex projective space ([9]).

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold equipped with parallel almost complex structure J and a Riemannian metric tensor G and covered by a system of coordinate neighborhoods $\{\tilde{V};y^A\}$.

Let M be a real (2n-1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{\tilde{V};x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i:M\to\tilde{M}$.

Throughout this paper the following convention on the range of indices are used, unless otherwise stated:

$$A,B,C,\cdots=1,2,\cdots,2(n+1);\quad i,j,\cdots=1,2,\cdots,2n-1.$$

The summation convention will be used with respect to those system of indices. In the sequel we identify $i(M)$ with M itself and represent the immersion by $y^A=y^A(x^h)$.

We put

$$B_i{}^A=\partial_i y^A,\quad \partial_i=\partial/\partial x^i$$

and denote by C, D and E are three mutually orthogonal unit normals to M. Then denoting by g the fundamental metric tensor with components g_{ji} on M, we have $g_{ji}=G(B_j,B_i)$ since the immersion is isometric, where we have put $B_j=(B_j{}^A)$.

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a CR *submanifold* ([1], [19]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(\Delta,\Delta^\perp)$ such that for any $p\in M$ we have $J\Delta_p=\Delta_p$, $\quad J\Delta_p{}^\perp\subset M_p{}^\perp$, where $M_p{}^\perp$ denotes the normal space of M at p. In particular, M is said to be *a semi-invariant submanifold* ([2], [17]) provided that $dim\Delta^{\perp} = 1$. In this case the unit vector field in $J\Delta^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denoted this by C ([17]). Then we have

$$(1.1) \qquad JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, JE = D,$$

where we have put $\phi_{ji} = G(JB_j, B_i), \xi_i = G(JB_i, C), \xi^h$ being associated components of ξ_h (see [9]). A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By the Hermitian property of J, it is seen that ϕ_{ji} is skew-symmetric, and that

$$\phi_i{}^r \phi_r{}^h = -\delta_i^h + \xi_i \xi^h, \quad \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0,$$
$$g_{rs} \phi_j{}^r \phi_i{}^s = g_{ji} - \xi_j \xi_i, \quad \xi_r \xi^r = 1,$$

namely, the aggregate (ϕ, ξ, g) defines *almost contact metric structure.*

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric tensor g, the equation of Gauss for M of $\tilde{M}$ is obtained:

$$(1.2) \qquad \nabla_j B_i = A_{ji} C + K_{ji} D + L_{ji} E,$$

where A_{ji}, K_{ji} and L_{ji} are components of the second fundamental forms in the direction of normals C, D, E respectively. Equations of Weingarten are also given

$$(1.3) \qquad \begin{aligned} \nabla_j C &= -A_j{}^h B_h + l_j D + m_j E, \\ \nabla_j D &= -K_j{}^h B_h - l_j C + n_j E, \\ \nabla_j E &= -L_j{}^h B_h - m_j C - n_j D, \end{aligned}$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and l_j, m_j and n_j being components of the third fundamental forms.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla^{\perp} C$. The distinguished normal C is said to *parallel* in the normal bundle if we have $\nabla^{\perp} C = 0$, that is, l_j and m_j vanish identically.

Since J is parallel, by differentiating (1.1) covariantly along M and using (1.1), (1.2) and (1.3), and by comparing the tangential and normal parts, we find (see [18])

$$(1.4) \qquad \nabla_j \phi_i{}^h = -A_{ji} \xi^h + A_j{}^h \xi_i,$$

$$\nabla_j \xi_i = -A_{jr}\phi_i{}^r, \tag{1.5}$$

$$K_{ji} = -L_{jr}\phi_i{}^r - m_j\xi_i, \tag{1.6}$$

$$L_{ji} = K_{jr}\phi_i{}^r + l_j\xi_i. \tag{1.7}$$

There is no loss of generality such that we may assume $T_rA_{(3)} = 0$ (see [9]).

Now we put $U_j = \xi^r\nabla_r\xi_j$. Then U is orthogonal to the structure vector ξ. Because of (1.5) and properties of the almost contact metric structure, it follows that

$$\phi_{ji}U^r = A_{jr}\xi^r - \alpha\xi_j, \tag{1.8}$$

$$U^r\nabla_j\xi_r = A_{jr}^2\xi^r - \alpha A_{jr}\xi^r, \tag{1.9}$$

where we have put $\alpha = A_{ji}\xi^j\xi^i$.

REMARK. In what follows, to write our formulas in convention forms, we denote by $\beta = A_{ji}^2\xi^j\xi^i, T_rA_{(2)} = k$ and $\nu = (\nabla_t k)\xi^t$.

From (1.8), we get $g(U,U) = \beta - \alpha^2$. Thus we easily see that $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

Differentiating (1.8) covariantly along M and making use of (1.4) and (1.5), we find

$$\xi_j(A_{kr}U^r + \nabla_k\alpha) + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} + \alpha A_{kr}\phi_j{}^r, \tag{1.10}$$

which shows that

$$(\nabla_k A_{rs})\xi^r\xi^s = 2A_{kr}U^r + \nabla_k\alpha. \tag{1.11}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature c, which is called a *complex space form.* Then equations of Gauss and Codazzi are given by

$$\begin{aligned} R_{kjih} = {} & \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ & + A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} \\ & + L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned} \tag{1.12}$$

$$\begin{aligned}(1.13)\qquad &\nabla_k A_{ji} - \nabla_j A_{ki} - l_k K_{ji} + l_j K_{ki} - m_k L_{ji} + m_j L_{ki}\\ &= \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}),\end{aligned}$$

$$(1.14)\qquad \nabla_k K_{ji} - \nabla_j K_{ki} = l_j A_{ki} - l_k A_{ji} + n_k L_{ji} - n_j L_{ki},$$

$$(1.15)\qquad \nabla_k L_{ji} - \nabla_j L_{ki} = m_j A_{ki} - m_k A_{ji} - n_k K_{ji} + n_j K_{ki},$$

where R_{kjih} are covariant components of the Riemann-Christoffel curvature tensor of M, and those of the Ricci by

$$(1.16)\qquad \nabla_k l_j - \nabla_j l_k = A_{jr} K_k{}^r - A_{kr} K_j{}^r + m_j n_k - m_k n_j,$$

$$(1.17)\qquad \nabla_k m_j - \nabla_j m_k = A_{jr} L_k{}^r - A_{kr} L_j{}^r + n_j l_k - n_k l_j,$$

$$(1.18)\qquad \nabla_k n_j - \nabla_j n_k = K_{jr} L_k{}^r - K_{kr} L_j{}^r + l_j m_k - l_k m_j + \frac{c}{2}\phi_{kj}.$$

The normal connection of a semi-invariant submanifold M of codimension 3 in a complex projective space CP^{n+1} is said to be L-$flat$ if it satisfies $dn = \frac{c}{2}\omega$, namely, $\nabla_j n_i - \nabla_i n_j = \frac{c}{2}\phi_{ji}$, where d denotes the exterior differential operator and the 2-form ω is defined by $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M ([12]). For a semi-invariant submanifold with L-$flat$ normal connection, it is known that

THEOREM K ([7]). *Let M be a semi-invariant submanifold of codimension 3 with L-flat normal connection in $\mathbb{C}P^{n+1}$. If $A\xi = \alpha\xi$, then we have $A_{(2)} = A_{(3)} = 0$.*

From (1.6) and (1.7), we have

$$(1.19)\qquad K_{jr}\xi^r = -m_j, \quad L_{jr}\xi^r = l_j,$$

$$(1.20)\qquad m_r\xi^r = -k, \quad l_r\xi^r = 0$$

because of $T_r A_{(3)} = 0$. Further we obtain

$$(1.21)\qquad \phi_{ir} l^r = m_i + k\xi_i, \quad \phi_{ir} m^r = -l_i,$$

$$(1.22)\qquad K_{jr} L_i{}^r + K_{ir} L_j{}^r + l_j m_i + l_i m_j = 0.$$

2. Semi-invariant submanifolds satisfying $dn = 2\theta\omega$

In this section we shall suppose that M is a semi-invariant submanifold of dodimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ and that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar θ on M, namely,

$$\nabla_j n_i - \nabla_i n_j = 2\theta\phi_{ji}. \tag{2.1}$$

Then from (1.18) we have

$$K_{jr}L_i{}^r - K_{ir}L_j{}^r + l_j m_i - l_i m_j = 2(\theta - \frac{c}{4})\phi_{ij},$$

which together with (1.22) yields

$$K_{jr}L_i{}^r + l_j m_i = (\theta - \frac{c}{4})\phi_{ij}. \tag{2.2}$$

We notice here that θ is constant if $n > 2$ (see [9]).

Further, Ki, Song and Takagi proved the following:

LEMMA 2.1 ([9]). *Let M be a semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ satisfying (2.1). If $\theta \neq \frac{c}{2}$, then we have $\nabla_j{}^\perp C = -k\xi_j E$ on M. Futher if $A\xi = \alpha\xi$, then the distinguished normal is parallel in the normal bundle.*

In what follows, we assume that M satisfies (2.1) with $\theta \neq \frac{c}{2}$ and $n > 2$. Then by Lemma 2.1 and (1.3), we have

$$l_j = 0, \quad m_j = -k\xi_j. \tag{2.3}$$

Thus (1.6), (1.7) and (2.2) turn out respectively to

$$L_{jr}\phi_i{}^r = -K_{ji} + k\xi_j\xi_i, \tag{2.4}$$

$$K_{jr}\phi_i{}^r = L_{ji}, \tag{2.5}$$

$$K_{jr}L_i{}^r = (\theta - \frac{c}{4})\phi_{ij}. \tag{2.6}$$

From the last two equations, it follows that

$$L_{ji}^2 = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{2.7}$$

Furthermore, if we make use of (2.3), then the other structure equa tions (1.13) $\sim$ (1.17) are reduced respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = k(\xi_j L_{ki} - \xi_k L_{ji}) + \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \tag{2.8}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = n_k L_{ji} - n_j L_{ki}, \tag{2.9}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = k(\xi_k A_{ji} - \xi_j A_{ki}) - n_k K_{ji} + n_j K_{ki}, \tag{2.10}$$

$$A_{jr}K_k{}^r - A_{kr}K_j{}^r = k(n_k\xi_j - n_j\xi_k), \tag{2.11}$$

$$A_{jr}L_k{}^r - A_{kr}L_j{}^r = \xi_k\nabla_j k - \xi_j\nabla_k k + k(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r), \tag{2.12}$$

where we have used (1.5). Because of (1.19) and (2.3), it is clear that

$$K_{jr}\xi^r = k\xi_j, \quad L_{jr}\xi^r = 0. \tag{2.13}$$

Multiplying (2.11) and (2.12) with ξ^k and summing for the index k, we have respectively

$$\xi^s A_{sr}K_j{}^r = kA_{jr}\xi^r + k(n_j - \mu\xi_j), \tag{2.14}$$

$$K_{jr}U^r = \nu\xi_j - \nabla_j k + kU_j \tag{2.15}$$

by virtue of (1.8), (2.4) and (2.13), where $\mu = n^t\xi_t$.

Transforming (2.14) by $\phi_k{}^j$ and taking account of (1.8), (2.5), (2.6) and (2.13), we find

$$K_{jr}U^r = k(\phi_{kr}n^r - U_k), \tag{2.16}$$

which together with (2.15) implies that

$$\nabla_j k = \nu\xi_j - k(\phi_{jr}n^r - 2U_j). \tag{2.17}$$

If we transform (2.12) by ${\phi_i}^k$ and make use of (2.4) and (2.17), then we obtain

$$A_{sr}{L_j}^r{\phi_i}^s + A_{jr}{K_i}^r = k\{(n_i - \mu\xi_i)\xi_j + 2\xi_j(A_{ir}\xi^r - \alpha\xi_i) + 2\xi_i A_{jr}\xi^r - A_{ji} - A_{sr}{\phi_j}^r{\phi_i}^s\},$$

or, use (2.11)

$$A_{sr}{L_j}^r{\phi_i}^s = A_{sr}{L_i}^r{\phi_j}^s. \tag{2.18}$$

Since θ is constant if $n > 2$, by differentiation (2.7) covariantly gives

$$L_{jr}\nabla_k {L_i}^r + L_{ir}\nabla_k {L_j}^r = (\theta - \frac{c}{4})(\xi_j A_{kr}{\phi_i}^r + \xi_i A_{kr}{\phi_j}^r),$$

or using (2.6), (2.10), (2.13), (2.16) and the last equation, it is verified that ([6])

$$\begin{aligned}(\theta - \frac{c}{4})(A_{ir}{\phi_k}^r + A_{kr}{\phi_i}^r) + (k^2 + \theta - \frac{c}{4})(U_k\xi_i + U_i\xi_k) \\ + k\{A_{kr}{L_i}^r + A_{ir}{L_k}^r - k(\xi_i\phi_{kr}n^r + \xi_k\phi_{ir}n^r)\} = 0.\end{aligned} \tag{2.19}$$

3. Semi-invariant submanifolds satisfying $\nabla_\xi A = 0$

We continue now, our arguments under the same hypotheses $dn = 2\theta\omega$ for a scalar $\theta(\neq \frac{c}{2})$ as in Section 2. Furthermore, suppose, throughout this paper, that $\nabla_\xi A = 0$. Then by (2.8) we have

$$\xi^r\nabla_j A_{ir} = kL_{ji} - \frac{c}{4}\phi_{ji} \tag{3.1}$$

because of (2.3).

REMARK. Let H denote by the second fundamental form in the direction of the distinguished normal C. Then by definition, the Lie derivative of H with respect to the structure vector field ξ is given by

$$L_\xi A_{ji} = \xi^r\nabla_r A_{ji} + (\nabla_j\xi^r)A_{ir} + (\nabla_i\xi^r)A_{jr},$$

which together with (1.5) implies that $L_\xi A_{ji} = \xi^r\nabla_r A_{ji}$. Thus, the condition $\nabla_\xi A = 0$ is equivalent to $L_\xi H = 0$.

Because of (2.13) and (3.1), it follows that $(\nabla_j A_{rs})\xi^r\xi^s = 0$. Differentiating this covariantly, and using (3.1), we find

$$(\nabla_k \nabla_j A_{rs})\xi^r\xi^s + 2\nabla_k\xi^r(kL_{jr} - \frac{c}{4}\phi_{jr}) = 0,$$

which together with (1.5) and (2.4) yields

$$(\nabla_k \nabla_j A_{rs})\xi^r\xi^s = 2kA_k{}^r K_{jr} - 2(k^2 + \frac{c}{4})\xi_j A_{kr}\xi^r,$$

from which, taking the skew-symmetric part and using the Ricci identity for A,

$$R_{kjis}(A_r{}^s\xi^r)\xi^i = (k^2 + \frac{c}{4})(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r) + k^2(n_k\xi_j - n_j\xi_k),$$

or, using (1.12), (2.13) and (2.14),

$$(A_{jr}\xi^r)(A^2_{ks}\xi^s) - (A_{kr}\xi^r)(A^2_{js}\xi^s) = 0.$$

Hence we have

$$\alpha A^2_{jr}\xi^r = \beta A_{jr}\xi^r. \tag{3.2}$$

We set $\Omega = \{p \in M : \beta(p) - \alpha^2(p) \neq 0\}$, and suppose that Ω is nonempty. In the sequel, we discuss our arguments on the open set Ω of M. Then from (3.2) we have

$$A^2_{jr}\xi^r = \lambda A_{jr}\xi^r, \tag{3.3}$$

where the function λ given by $\alpha\lambda = \beta$ is defined.

Now, we put

$$A_{jr}\xi^r = \alpha\xi_j + \rho W_j, \tag{3.4}$$

where ρ is a function on M which does not vanish on Ω and W is a unit vector field orthogonal to the structure vector field ξ. Then we have $\phi U = -\rho W$ and hence

$$\rho\phi_{jr}W^r = -U_j, \tag{3.5}$$

where $\rho^2 = \beta - \alpha^2$ because of (1.8). Further with (3.3) we get

$$A_{jr}W^r = \rho\xi_j + (\lambda - \alpha)W_j \tag{3.6}$$

by virtue of $\rho \neq 0$ on Ω. Hence we have

$$A_{jr}^2 W^r = \lambda A_{jr}W^r. \tag{3.7}$$

Because of (1.8), (2.4) and (3.4) we have

$$K_{jr}U^r = \rho L_{jr}W^r, \tag{3.8}$$

where we have used (3.4) and (3.6).

Differentiating (3.3) covariantly and making use of (1.5), (3.1) and (3.4), we find

(3.9)
$$\rho(\nabla_k A_{jr})W^r = (\nabla_k\lambda)A_{jr}\xi^r + (\lambda - \alpha)(kL_{jk} + \frac{c}{4}\phi_{jk}) + \frac{c}{4}A_{jr}\phi_k{}^r - kL_{kr}A_j{}^r + A_{jr}^2 A_{ks}\phi^{rs} - \lambda A_{jr}A_{ks}\phi^{rs}.$$

Multiplying (3.9) with ξ^k and summing for k, and taking account of (2.13) and the hypotheses $\nabla_\xi A = 0$, we obtain

$$A_{jr}^2 U^r - \lambda A_{jr}U^r + d\lambda(\xi)A_{jr}\xi^r = 0,$$

where we have put $d\lambda(\xi) = \xi^t\nabla_t\lambda$, which unable us to obtain $\alpha d\lambda(\xi) = 0$ and hence $d\lambda(\xi) = 0$. Therefore it follows that

$$A_{jr}^2 U^r = \lambda A_{jr}U^r. \tag{3.10}$$

Applying (3.9) by W^j and making use of (2.13), (3.6) and (3.7), we find

$(\nabla_k A_{rs})W^r W^s = \nabla_k\lambda$ because of $\rho \neq 0$ on Ω. From this and (2.8), we see that

$$W^s(\nabla_s A_{jr})W^r = \nabla_j\lambda + k(L_{rs}W^r W^s)\xi_j.$$

Multiplying (3.9) with ρW^k and summing for k, and using (3.4), (3.8), (3.10) and the last equation, we get

$$\begin{aligned} &\rho^2\{\nabla_j\lambda + k(L_{sr}W^r W^s)\xi_j\} + \{A_j{}^r K_{rs}U^s - (\lambda - \alpha)K_{jr}U^r\} \\ &= \rho d\lambda(W)A_{jr}\xi^r + \frac{c}{4}\{A_{jr}U^r - (\lambda - \alpha)U_j\}. \end{aligned} \tag{3.11}$$

If we take the skew-symmetric part of (3.9) and use (2.8), (2.12) and (3.8), then we get

$$\begin{aligned}&(k\nabla_j k - kK_{jr}U^r - \frac{c}{4}U_j)\xi_k - (k\nabla_k k - kK_{kr}U^r - \frac{c}{4}U_k)\xi_j\\&= (\nabla_k\lambda)A_{jr}\xi^r - (\nabla_j\lambda)A_{kr}\xi^r + \frac{c}{2}(\lambda-\alpha)\phi_{jk}\\&\quad - (k^2+\frac{c}{4})(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r)\\&\quad + A_{jr}^2A_{ks}\phi^{rs} - A_{kr}^2A_{js}\phi^{rs} - 2\lambda A_{jr}A_{ks}\phi^{rs},\end{aligned}$$

which, applying ξ^j and making use of (2.13), (2.15), (3.4) and(3.10),

$$\alpha\nabla_j\lambda = 2kK_{jr}U^r + \frac{c}{2}U_j. \tag{3.12}$$

Since we have $\rho^2 = \alpha(\lambda-\alpha)$, (3.11) turns out to be

$$\begin{aligned}k\{A_j{}^rK_{rs}U^s + (\lambda-\alpha)K_{jr}U^r\} &= \rho d\lambda(W)A_{jr}\xi^r - \rho k(K_{rs}U^rW^s)\xi_j\\&\quad + \frac{c}{4}A_{jr}U^r - \frac{3}{4}c(\lambda-\alpha)U_j,\end{aligned}$$

where we have used (3.8) and (3.12).

On the other hand, transforming (2.19) by ρW^k and taking account of (3.5), (3.6) and (3.8), we find

$$\begin{aligned}&k\{A_i{}^rK_{rs}U^s + (\lambda-\alpha)K_{ir}U^r\}\\&\quad + (\theta - \frac{c}{4})\{A_{ir}U^r - (\lambda-\alpha)U_i\} - k^2(n_tU^t)\xi_i = 0.\end{aligned}$$

Combining the last two equations, it follows that

$$\begin{aligned}&\rho d\lambda(W)A_{jr}\xi^r - k\{k(n_tU^t) + \rho K_{rs}U^rW^s\}\xi_j\\&\qquad + \theta A_{jr}U^r - (\theta + \frac{c}{2})(\lambda-\alpha)U_j = 0,\end{aligned}$$

which implies $d\lambda(W) = 0$ and hence $k\{k(n_tU^t) + \rho K_{rs}U^rW^s\} = 0$ on Ω. Therefore we have

$$\theta A_{jr}U^r = (\theta + \frac{c}{2})(\lambda-\alpha)U_j. \tag{3.13}$$

Thus $\theta = 0$ is not impossible on Ω and thus we can put

$$A_{jr}U^r = \tau U_j, \tag{3.14}$$

where the function τ given by $\theta\tau = (\theta + \frac{c}{2})(\lambda - \alpha)$ is defined. From (3.10) and (3.16), it is seen that $(\lambda - \tau)AU = 0$.

Now, suppose that $\lambda - \tau \neq 0$ on Ω. Then we have $AU = 0$ on this set. Thus, (3.13) tell us that $\theta + \frac{c}{2} = 0$. However, we see, using (2.7), that $\theta - \frac{c}{4} \geq 0$. It is contradictory. Thus, we have $\lambda = \tau$ on Ω. Hence we obtain $\frac{c}{2}\lambda + (\theta + \frac{c}{2})\alpha = 0$. Differentiating this covariantly and taking account of (3.12), we find

$$2kK_{jr}U^r + \frac{c}{2}U_j - \lambda\nabla_j\alpha = 0. \tag{3.15}$$

On the other hand, it is, using (1.11) and (3.1), clear that $2A_{jr}U^r + \nabla_j\alpha = 0$, which connected with (3.14) implies that

$$\nabla_j\alpha = -2\lambda U_j \tag{3.16}$$

because of the fact that $\tau = \lambda$.

From (3.15) and (3.16), it follows that

$$kK_{jr}U^r + (\lambda^2 + \frac{c}{4})U_j = 0.$$

Therefore, $k = 0$ is not impossible on Ω and hence it is seen that

$$K_{jr}U^r = xU_j, \tag{3.17}$$

where we have put

$$kx + \lambda^2 + \frac{c}{4} = 0. \tag{3.18}$$

Transforming (3.17) by $\phi_i{}^j$ and taking account of (2.5), we find

$$L_{ir}U^r = x\phi_{ir}U^r. \tag{3.19}$$

Because of (2.6), (3.17) and (3.19), it is verified that

$$x^2 = \theta - \frac{c}{4}. \tag{3.20}$$

If we take account of (3.17), then (2.15) is reduced to $\nabla_j k = \nu\xi_j + (k - x)U_j$. Differentiating (3.18) covariantly and using this, we obtain

$$x\{\nu\xi_j + (k - x)U_j\} + 2\lambda\nabla_j\lambda = 0,$$

which together with $d\lambda(\xi) = 0$ gives $\nu = 0$. Thus, we have

$$\nabla_j k = (k - x)U_j. \tag{3.21}$$

If we apply (2.19) by U^i and make use of (3.3), (3.14) with $\tau = \lambda$, (3.19) and (3.20), then we find

$$x\{\alpha x + (2\lambda - \alpha)k\}(A_{kr}\xi^r - \alpha\xi_k) = 0,$$

which connected with (3.18) implies that

$$\alpha x + (2k - \alpha)k = 0. \tag{3.22}$$

Differentiating (3.18) covariantly, and using (3.12), (3.17),(3.21) and (3.22), we get $(3kx + \frac{c}{2})U_j = 0$, which together with (3.18) implies that $\lambda^2 + \frac{c}{12} = 0$, a contradiction. Thus, Ω is empty. So we see, using Lemma 2.1, that the distinguished normal is parallel in the normal bundle. Summing up we have

LEMMA 3.1. *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ satisfying $dn = 2\theta\omega$ for a certain scalar $\theta \neq \frac{c}{2}$. If M satisfies $\nabla_\xi A = 0$. Then the distinguished normal is parallel in the normal bundle.*

4. Proof of theorem

Let M be a connected real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ for a certain scalar $\theta < \frac{c}{2}$ in $\mathbb{C}P^{n+1}$. Suppose that $\nabla_\xi A = 0$. Then by Lemma 3.1 we have $k = 0$ on M. Thus (2.3) tells us that the distinguished normal C is parallel in the normal bundle. Hence, by Lemma 4.1 of [9], we have $A_{(2)} = A_{(3)} = 0$. Therefore, by the reduction theorem in [5], [13], M is a real hypersurface in a complex projective space $\mathbb{C}P^n$. Since we have $\nabla^\perp C = 0$, equations (1.13) and (3.1) are reduced respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}),$$

$$\xi^r\nabla_j A_{ir} = -\frac{c}{4}\phi_{ji}.$$

Using (1.4), (1.5) and above two equations, it is proved in [8] that $g(U,U) = 0$. Hence we have $A\phi = \phi A$. Thus, by Theorem A we have our Theorem.

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