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L^p ESTIMATES WITH WEIGHTS FOR THE $\bar{\partial}$-EQUATION ON REAL ELLIPSOIDS IN $\mathbb{C}^n$

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ABSTRACT. We prove weighted L^p estimates with respect to the non-isotropic norm for the $\bar{\partial}$-equation on real ellipsoids, where weights are powers of the distance to the boundary. The non-isotropic norm is smaller than the usual norm, by a factor which is equal to the distance to the boundary in the complex tangential component and which is equal to the m-th root of the distance to the boundary in the complex normal component. Here m is the maximal order of contact of the boundary of the real ellipsoid with complex analytic curves.

1. Introduction

In this paper we will study weighted L^p estimates for the $\bar{\partial}$-equation on real ellipsoids $\Omega = \{z = x + \sqrt{-1}y \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^n (x_j^{2\mu_j} + y_j^{2\nu_j}) - 1 < 0,\ \mu_j, \nu_j \in \mathbb{N}\}$. Let $m = \max_{1\le j\le n} \min\{2\mu_j, 2\nu_j\}$. Then Ω is of finite type m in the sense that m is equal to the maximal order of contact of the boundary of Ω with complex analytic curves. For a differential form of type (p,q), $f = \sum_{|I|=p,|J|=q} f_{IJ}dz^I \wedge d\bar{z}^J$ and $z \in \Omega$ we define the isotropic and non-isotropic norms of f at z

$$|f(z)| = \sum\nolimits_{|I|=p,|J|=q} |f_{IJ}(z)|,$$
$$|f(z)|_\Omega = |\rho(z)||f(z)| + |\rho(z)|^{\frac{1}{m}} |\bar{\partial}\rho(z) \wedge f(z)|,$$

respectively. If $1 \le p \le \infty$ and $0 < \alpha < \infty$, we define a non-isotropic L^p space $L^p_\alpha(\Omega, |\cdot|_\Omega)$ with weight α to be the set of all $(0,1)$-forms f

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satisfying

$$||f||_{p,\alpha,\Omega} = \begin{cases} \sup_{z\in\Omega} |\rho(z)|^{\alpha-1}|f(z)|_\Omega < \infty & \text{if } p = \infty \\ \left(\int_\Omega |f(z)|_\Omega^p |\rho(z)|^{\alpha-1}\, dV\right)^{\frac{1}{p}} < \infty & \text{if } 1 \le p < \infty \end{cases}$$

and L^p space $L^p_\alpha(\Omega)$ with weight α to be the set of all measurable functions g satisfying

$$||g||_{p,\alpha} = \begin{cases} \sup_{z\in\Omega} |\rho(z)|^{\alpha-1}|g(z)| < \infty & \text{if } p = \infty \\ \left(\int_\Omega |g(z)|^p |\rho(z)|^{\alpha-1}\, dV\right)^{\frac{1}{p}} < \infty & \text{if } 1 \le p < \infty. \end{cases}$$

Now we can state our main theorem.

Main Theorem. *Let Ω and m be as above. Let $f \in L^p_\alpha(\Omega, |\cdot|_\Omega)$ be a $\bar\partial$-closed form of type $(0,1)$. If $1 \le p < \infty$ and $\alpha > 0$, or if $p = \infty$ and $\alpha > 1$ then there exists a solution $u \in L^p_\alpha(\Omega)$ for $\bar\partial u = f$ in Ω such that*

$$||u||_{p,\alpha} \le C_{p,\alpha}||f||_{p,\alpha,\Omega}.$$

For strongly pseudoconvex domains, similar results were studied by Henkin-Dautov [6] and Ahn-Cho [1]. If the domain is strongly pseudoconvex, then the non-isotropic norm at z is similarly defined except that we replace the weight $|\rho(z)|^{\frac{1}{m}}$ by $|\rho(z)|^{\frac{1}{2}}$. The L^p estimates without weights for the $\bar\partial$ are well-known [8].

Bonami and Charpentier [3] proved non-isotropic L^1 estimates for the $\bar\partial$-equation on complex ellipsoids. Complex ellipsoids are of finite strict type, whereas real ellipsoids are only of finite type but not of strict type.

For real ellipsoids the sharp Hölder estimates for the $\bar\partial$ were obtained by Diederich-Fornæss-Wiegerinck [5] and optimal L^p and Hölder estimates for the $\bar\partial_b$ were proved by Shaw [9] using the integral kernel. Our theorem is in spirit similar to Shaw's one but there are two differences: First we obtain weighted L^p estimates with respect to the non-isotropic norm $|\cdot|_\Omega$ where weights are powers of the distance to the boundary. Second to solve the $\bar\partial$-equation we use weighted kernels which were constructed by Berndtsson and Andersson [2].

Recently Cumenge [4] proved non-isotropic L^1 estimates on convex domains of finite type.

2. Construction of a solution operator for the $\bar\partial$

By a linear holomorphic change of coordinates we can always assume that $\nu_j \le \mu_j$. In this section we introduce the "right" support function constructed by Diederich-Fornæss-Wiegerinck [5] and with this support function we obtain a solution operator for the $\bar\partial$-equation.

To define a support function we introduce new notations

$$\zeta_j = \xi_j + \sqrt{-1}\eta_j, \quad \rho_j = \frac{\partial\rho}{\partial\zeta_j}, \quad \rho_{\bar j} = \frac{\partial\rho}{\partial\bar\zeta_j}, \quad 1 \le j \le n.$$

Following Diederich-Fornæss-Wiegerinck [5], for $z, \zeta \in \Omega$, we let

$$\begin{aligned} F(z,\zeta) = &\sum_{j=1}^{n} \rho_j(\zeta)(\zeta_j - z_j) \\ &+ \gamma\sum_{j=1}^{n} \left[(\eta_j^{2\nu_j-2} - \xi_j^{2\mu_j-2})(\zeta_j - z_j)^2 + (\zeta_j - z_j)^{2\nu_j}\right], \end{aligned}$$

where $\gamma > 0$ is a sufficiently small constant. Then we have the following lemma.

Lemma 2.1. (i) *$F(z,\zeta)$ is holomorphic in z.*
(ii) *$F(z,\zeta) = 0$ if and only if $z = \zeta$. Moreover $d_\zeta F(z,\zeta)|_{\zeta=z} = \partial\rho(\zeta) \ne 0$.*
(iii) *For γ chosen small enough, there exists $c > 0$ such that*

$$2\mathrm{Re}\, F(z,\zeta) \ge \rho(\zeta) - \rho(z) + c\sum_{j=1}^{n} \left[(\xi_j^{2\mu_j-2} + \eta_j^{2\nu_j-2})|\zeta_j - z_j|^2 + |\zeta_j - z_j|^{2\nu_j}\right].$$

Proof. See [5] or [9]. □

We decompose $F(z,\zeta) = \sum_{j=1}^n P_j(z,\zeta)(z_j - \zeta_j)$, where

$$P_j(z,\zeta) = -\rho_j(\zeta) + \gamma\left[(\eta_j^{2\nu_j-2} - \xi_j^{2\mu_j-2})(z_j - \zeta_j) + (z_j - \zeta_j)^{2\nu_j-1}\right]$$

and set

$$S(z,\zeta) = F(z,\zeta) - \rho(\zeta), \quad Q(z,\zeta) = \frac{1}{\rho(\zeta)}\sum_{j=1}^{n} P_j(z,\zeta)d\zeta_j.$$

Then the Berndtsson and Andersson kernel is

$$K^r(z,\zeta) = \sum_{j=0}^{n-1} c_{r,j} \frac{|\rho(\zeta)|^{r+j}\, b \wedge (\bar\partial_\zeta Q)^j \wedge (\bar\partial_\zeta b)^{n-1-j}}{S(z,\zeta)^{r+j}|\zeta - z|^{2(n-j)}},$$

where $r > 0$ is suitably large, $c_{r,j}$'s are some constants and $b = \sum_{j=1}^n (\bar z_j - \bar\zeta_j)d\zeta_j$. Since due to Lemma 2.1 (iii) Re $S(z,\zeta) > 0$ for $z, \zeta \in \Omega$ we can define an integral operator

$$T^r f(z) = \int_{\zeta\in\Omega} K^r(z,\zeta) \wedge f(\zeta), \quad z \in \Omega.$$

Since the kernel $K^r(z,\zeta)$ vanishes for $\zeta \in b\Omega$ the integral operators T^r are indeed a solution operator in Ω (see [2] for details). From the definition of $Q = Q(z,\zeta)$ we can easily obtain

$$(\bar\partial Q)^j = \frac{1}{\rho^j}\Big(\sum_{i=1}^n \bar\partial P_i d\zeta_i\Big)^j - j\frac{\bar\partial\rho}{\rho^{j+1}} \wedge \Big(\sum_{i=1}^n \bar\partial P_i d\zeta_i\Big)^{j-1} \wedge \Big(\sum_{i=1}^n P_i d\zeta_i\Big).$$

After canceling $|\rho(\zeta)|^j$ in fraction we can write

$$K^r(z,\zeta) =: |\rho(\zeta)|^r \sum_{j=0}^{n-1} \widetilde{K}_j^r(z,\zeta) + |\rho(\zeta)|^{r-1} \sum_{j=1}^{n-1} K_j^r(z,\zeta) \wedge \bar\partial\rho(\zeta),$$

where $\widetilde{K}_j^r(z,\zeta)$ and $K_j^r(z,\zeta)$ satisfy the following estimates:

$$|\widetilde{K}_0^r(z,\zeta)| \lesssim \frac{1}{|S(z,\zeta)|^r|\zeta - z|^{2n-1}}.$$

$$(2.1) \qquad |\widetilde{K}_j^r(z,\zeta)| \lesssim \frac{|\big(\sum_{i=1}^n \bar\partial_\zeta P_i(z,\zeta)d\zeta_i\big)^j|}{|S(z,\zeta)|^{r+j}|\zeta - z|^{2n-2j-1}}, \quad 1 \le j \le n-1.$$

$$|K_j^r(z,\zeta)| \lesssim \frac{|\big(\sum_{i=1}^n \bar\partial_\zeta P_i(z,\zeta)d\zeta_i\big)^{j-1}|}{|S(z,\zeta)|^{r+j}|\zeta - z|^{2n-2j-1}}, \quad 1 \le j \le n-1.$$

Fix $\zeta_0 \in \Omega$ sufficiently close to the boundary of Ω and choose a small ball $B(\zeta_0, \lambda)$ with radius $\lambda > 0$. Further we may assume that $d\rho(\zeta_0) = (1, 0, \cdots, 0)$ and $|\rho_{\bar 1}(\zeta)| \ge 1/2$ on $B(\zeta_0, \lambda)$. To improve the estimates $|K_j^r(z,\zeta)|$, $1 \le j \le n-1$ for $z,\zeta \in B(\zeta_0,\lambda)\cap\Omega$ we introduce the following frame of $(0,1)$-vector fields

$$Y_1 = \frac{\sqrt{-1}}{\rho_{\bar 1}}\frac{\partial}{\partial\bar\zeta_1}, \quad Y_i = \frac{\partial}{\partial\bar\zeta_i} - \frac{\rho_{\bar i}}{\rho_{\bar 1}}\frac{\partial}{\partial\bar\zeta_1}, \quad i = 2,\cdots,n \tag{2.2}$$

with the dual co-frame

$$\omega_1 = -\sqrt{-1}\bar\partial\rho, \quad \omega_i = d\bar\zeta_i, \quad i = 2,\cdots,n.$$

Then for $g \in C^1(B(\zeta_0,\lambda)\cap\Omega)$ we have $\bar\partial g = \sum_{j=1}^n (Y_j g)\omega_j$. We define

$$\bar\partial^t g = \sum_{j=2}^n (Y_j g)\omega_j. \tag{2.3}$$

Since $\bar\partial\rho\wedge\bar\partial g = \bar\partial\rho\wedge\bar\partial^t g$ the third inequality in (2.1) can be improved as the following form

$$|K_j^r(z,\zeta)| \lesssim \frac{|(\sum_{i=1}^n \bar\partial_\zeta^t P_i(z,\zeta)d\zeta_i)^{j-1}|}{|S(z,\zeta)|^{r+j}|\zeta - z|^{2n-2j-1}}$$

for $1 \le j \le n-1$ and $z,\zeta \in B(\zeta_0,\lambda)\cap\Omega$.

3. Reduction of integral kernels

In this section we will reduce the integral of $|K^r(z,\zeta)|$ over Ω to the integration over small parts of $\mathbb{C}^2$ or $\mathbb{R}^2$. To do this we need the following local coordinate systems.

LEMMA 3.1. (i) *There exist a positive constant λ and an open neighborhood U of $b\Omega$ such that for every $z \in U$ one can find a smooth change of coordinates $t = t(z,\zeta) = (t_1,\cdots,t_{2n})$ in the ball $B(z,\lambda)$ satisfying*

$$\begin{gathered} t_1(z,\zeta) = \rho(\zeta), \quad t_2(z,\zeta) = \operatorname{Im} F(z,\zeta), \\ t_{2j-1}(z,\zeta) + \sqrt{-1}t_{2j}(z,\zeta) = (\zeta_j - z_j), \quad j = 2,\cdots,n, \\ |\zeta - z| \approx |t_1 - \rho(z)| + |t_2| + |t'|, \quad t' = (t_3,\cdots,t_{2n}), \quad \textit{for } \zeta \in B(z,\lambda) \\ |t(z,\zeta)| < 1 \quad \textit{and } |\det J_{\mathbb{R}}(t(z,\cdot))| \approx 1 \quad \textit{for } \zeta \in B(z,\lambda). \end{gathered} \tag{3.1}$$

(ii) *For every $\zeta \in \Omega\cap U$ there is a coordinate system $u = (u_1,\cdots,u_{2n})$ for $z \in B(\zeta,\tilde\lambda)$, where $\tilde\lambda$ is independent of ζ satisfying*

$$u_1(z) = \rho(z), \quad u_2 = \operatorname{Im} F(z,\zeta) \quad \textit{and } u(\zeta) = (\rho(\zeta),0,\cdots,0),$$

which has properties analogous to those of the coordinate system t on $B(z,\lambda)$.

PROOF. By Lemma 2.1 (ii) and the compactness of $b\Omega$ everything is obvious. For details see Lemma V.3.4 of Range's book [8]. □

The following lemma is the key step for the proof of Main Theorem.

LEMMA 3.2. *For all $z \in U \cap \Omega$, $1 \le j \le n-1$ and $s > -1$ to be decided later there exists a constant C independent of z satisfying the following estimates:*

$$\text{(i)} \quad \int_{\Omega\cap B(z,\lambda)} |\rho(\zeta)|^s \left|\widetilde{K}_0^r(z,\zeta)\right| dV(\zeta) \le C \iint_{|(t_1,t_2)|<1} \frac{|t_1|^s \, dt_1 dt_2}{(|t_1|+|t_2|+|\rho(z)|)^r(|t_1-|\rho(z)||+|t_2|)}.$$

$$\text{(ii)} \quad \int_{\Omega\cap B(z,\lambda)} |\rho(\zeta)|^s \left|K_j^r(z,\zeta)\right| dV(\zeta) \le C \iint_{\substack{|(t_1,t_2)|<1\\ |w|<1\\ w=(t_3,t_4)}} \frac{|t_1|^s \, dt_1 dt_2 dV(w)}{(|t_1|+|t_2|+|w|^m+|\rho(z)|)^{r+1}|w|}.$$

$$\text{(iii)} \quad \int_{\Omega\cap B(z,\lambda)} |\rho(\zeta)|^s \left|\widetilde{K}_j^r(z,\zeta)\right| dV(\zeta) \le C \iint_{\substack{|(t_1,t_2)|<1\\ |w|<1\\ w=(t_3,t_4)}} \frac{|t_1|^s \, dt_1 dt_2 dV(w)}{(|t_1|+|t_2|+|w|^m+|\rho(z)|)^{r+1}|w|}.$$

An analog statement to the integration of $|\rho(\cdot)|^s\left|\widetilde{K}_j^r(\cdot,\zeta)\right|$ and $|\rho(\cdot)|^s \left|K_j^r(\cdot,\zeta)\right|$ over $\Omega \cap B(\zeta,\tilde{\lambda})$ is also true for every $\zeta \in U \cap \Omega$.

REMARK. In section 4 we will show that the right hand sides of (i)–(iii) are bounded for each $z \in U \cap \Omega$ and suitably chosen s.

PROOF. Fix $z \in U \cap \Omega$. From the definition of $S(z,\zeta)$ and Lemma 2.1 (iii) it is easy to see that for every $\zeta \in \Omega$

$$\begin{aligned} |S(z,\zeta)| \gtrsim & |\operatorname{Im} F(z,\zeta)| + |\rho(z)| + |\rho(\zeta)| \\ & + \sum_{j=1}^{n} \left[(\xi_j^{2m-2} + \eta_j^{2\nu_j-2})|\zeta_j - z_j|^2 + |\zeta_j - z_j|^{2\nu_j} \right]. \end{aligned} \tag{3.2}$$

Part (i). By the inequality (2.1), change of coordinates introduced in Lemma 3.1 and Fubini's theorem we easily see that

$$\begin{aligned}
&\int_{\Omega\cap B(z,\lambda)} |\rho(\zeta)|^s \left|\widetilde{K}_0^r(z,\zeta)\right| dV(\zeta) \\
&\lesssim \int_{|t|<1} \frac{|t_1|^s\, dt}{(|t_1|+|t_2|+|\rho(z)|)^r(|t_1-|\rho(z)||+|t_2|+|t'|)^{2n-1}} \\
&\lesssim \int_{|(t_1,t_2)|<1} \frac{|t_1|^s\, dt_1 dt_2}{(|t_1|+|t_2|+|\rho(z)|)^r} \\
&\quad\times \int_{|t'|<1} \frac{dt'}{(|t_1-|\rho(z)||+|t_2|+|t'|)^{2n-1}}.
\end{aligned}$$

Introducing polar coordinates in $t'\in\mathbb{R}^{2n-2}$ with $r=|t'|$ and integrating with respect to r we obtain the inequality (i).

Part (ii). First for $j=1$ we note that the following inequalities are satisfied:

$$|S(z,\zeta)| \gtrsim |t_1|+|t_2|+|\rho(z)|+|w|^m, \quad |\zeta-z|\gtrsim |w|+|t''|,$$

where $w=(t_3,t_4)$ and $t''=(t_5,\cdots,t_{2n})\in\mathbb{R}^{2n-4}$. As Part (i) if we change coordinates of Lemma 3.1 (i) and introduce polar coordinates in t'' with $r=|t''|$ then the inequality (ii) is also true. Now we may assume that $2\le j\le n-1$. To reduce the integral we have to precisely estimate $\left|\left(\sum_{k=1}^n \bar\partial_\zeta^t P_k(z,\zeta)d\zeta_k\right)^{j-1}\right|$. From the definition of $P_k(z,\zeta)$ and (2.2) we immediately obtain that

$$\begin{aligned}
\left|Y_i P_k(z,\zeta)\right| \le \delta_{ik}\Big[&|\xi_k|^{2\mu_k-2}+|\eta_k|^{2\nu_k-2} \\
&+|\zeta_k-z_k|\left(\sigma(\mu_k)|\xi_k|^{2\mu_k-3}+\sigma(\nu_k)|\eta_k|^{2\nu_k-3}\right)\Big]
\end{aligned}$$

for $i,k=2,\cdots,n$ and

$$\left|Y_i P_1(z,\zeta)\right| \lesssim |\xi_i|^{2\mu_i-1}+|\eta_i|^{2\nu_i-1}, \quad i=2,\cdots,n,$$

where $\sigma(\ell)=0$ if $\ell=1$ and $\sigma(\ell)=1$ for $\ell=2,3,\ldots$. Calculating the $(j-1)$-th exterior product, from (2.3), we see that the term $\left|\left(\sum_{k=1}^n \bar\partial_\zeta^t P_k(z,\zeta)d\zeta_k\right)^{j-1}\right|$ is bounded by some constant times a sum of the following form

$$\begin{aligned}
P(k_1,\cdots,k_{j-1}) := \prod_{\alpha=1}^{j-1} \Big[&|\xi_{k_\alpha}|^{2\mu_{k_\alpha}-2}+|\eta_{k_\alpha}|^{2\nu_{k_\alpha}-2} \\
&+|\zeta_{k_\alpha}-z_{k_\alpha}|\left(\sigma(\mu_{k_\alpha})|\xi_{k_\alpha}|^{2\mu_{k_\alpha}-3}+\sigma(\nu_{k_\alpha})|\eta_{k_\alpha}|^{2\nu_{k_\alpha}-3}\right)\Big],
\end{aligned}$$

where $2 \leq k_1 < \cdots < k_{j-1} \leq n$. Rearranging the indices, we can always assume that $k_1 = n-j+2, k_2 = n-j+3, \cdots, k_{j-1} = n$. For the simplification of notation we denote $P(n-j+2, \cdots, n)$ by P_0. Then if we express P_0 in t coordinates we have

$$P_0 = \prod_{k=n-j+2}^{n} \Big[|t_{2k-1} + x_k|^{2\mu_k - 2} + |t_{2k} + y_k|^{2\nu_k - 2}$$

$$+ |t_{2k-1} + \sqrt{-1} t_{2k}| \big(\sigma(\mu_k) |t_{2k-1} + x_k|^{2\mu_k - 3} + \sigma(\nu_k) |t_{2k} + y_k|^{2\nu_k - 3} \big) \Big].$$

Here we may assume that the dimension of ambient space is greater than 2, since if $n = 2$, the inequality (ii) is obvious. We claim that

$$(3.3) \quad \int_{\Omega \cap B(z,\lambda)} \frac{|\rho(\zeta)|^s P_0}{|S(z,\zeta)|^{r+j} |\zeta - z|^{2n-2j-1}} \, dV(\zeta)$$

$$\lesssim \int_{\substack{|w|<1 \\ w=(t_3,t_4)}} dV(w) \int_{|(t_1,t_2)|<1} \frac{|t_1|^s \, dt_1 dt_2}{(|t_1| + |t_2| + |w|^m + |\rho(z)|)^{r+1} |w|}.$$

By (3.1) and inequality (3.2) we have $|\zeta - z| \gtrsim |w| + |\tilde{t}|$ and

(3.4)

$$|S(z,\zeta)|$$

$$\gtrsim |t_1| + |t_2| + |w|^m + |\rho(z)| + \sum_{k=n-j+2}^{n} \Big[\big(|t_{2k-1} + x_k|^{2\mu_k - 2}$$

$$+ |t_{2k} + y_k|^{2\nu_k - 2} \big) \big(|t_{2k-1}|^2 + |t_{2k}|^2 \big) + \big(\sqrt{|t_{2k-1}|^2 + |t_{2k}|^2} \big)^{\nu_k} \Big],$$

where we write $\tilde{t} = 0$ if $n = 3$ or $j = n-1$ and otherwise $\tilde{t} = (t_5, \cdots, t_{2n-2j+2})$. For the simplification of integrals we introduce another notation $\hat{t} = (t_{2n-2j+3}, \cdots, t_{2n})$. Then by the above inequalities and the Fubini theorem we see that

$$\int_{\Omega \cap B(z,\lambda)} \frac{|\rho(\zeta)|^s P_0}{|S(z,\zeta)|^{r+j} |\zeta - z|^{2n-2j-1}} \, dV(\zeta)$$

$$\lesssim \int_{|w|<1} \int_{|(t_1,t_2)|<1} |t_1|^s \, dt_1 dt_2 dV(w)$$

$$\times \int_{|\hat{t}|<1} \frac{P_0 \, d\hat{t}}{(|t_1| + |t_2| + |w|^m + |\rho(z)| + \cdots)^{r+j}}$$

$$\times \int_{|\tilde{t}|<1} \frac{d\tilde{t}}{(|w| + |\tilde{t}|)^{2n-2j-1}},$$

where $(\cdots)$ in $d\hat{t}$ integral is the summation term from the right hand side of (3.4). It is clear that

$$\int_{|\tilde{t}|<1} \frac{d\tilde{t}}{(|w|+|\tilde{t}|)^{2n-2j-1}} \lesssim \frac{1}{|w|} \tag{3.5}$$

if we change polar coordinates in $\tilde{t}$. Note that if $n=3$ or $j=n-1$, i.e., $\tilde{t}=0$, then (3.5) is autonomous. To obtain the inequality (3.3), by (3.5), it suffices to show that

$$\begin{aligned} &\int_{|\hat{t}|<1} \frac{P_0\, d\hat{t}}{(|t_1|+|t_2|+|w|^m+|\rho(z)|+\cdots)^{r+j}} \\ &\lesssim \frac{1}{(|t_1|+|t_2|+|w|^m+|\rho(z)|)^{r+1}}. \end{aligned} \tag{3.6}$$

To do this we need the following lemma.

LEMMA 3.3. *For $q>1, k\geq 1, I=\{(x,y): |x|^2+|y|^2<1\}$ and A positive, close to 0 there exists a constant C satisfying the following estimates*

$$\int_I \frac{|x+s|^k\, dxdy}{(A+|x+s|^k(x^2+y^2))^q} \leq \frac{C}{A^{q-1}}, \tag{i}$$

$$\int_I \frac{|x+s|^{k-1}|x|\, dxdy}{(A+|x+s|^k(x^2+y^2))^q} \leq \frac{C}{A^{q-1}}, \tag{ii}$$

$$\int_I \frac{|x+s|^{k-1}|y|\, dxdy}{\left(A+|x+s|^k(x^2+y^2)+(\sqrt{x^2+y^2})^{k+2}\right)^q} \leq \frac{C}{A^{q-1}}, \tag{iii}$$

$$\int_I \frac{|\xi|^k\, dxdy}{(A+|\xi|^k(x^2+y^2))^q} \leq \frac{C}{A^{q-1}}, \tag{iv}$$

$$\int_I \frac{|\xi|^{k-1}|x|\, dxdy}{\left(A+|\xi|^k(x^2+y^2)+(\sqrt{x^2+y^2})^{k+2}\right)^q} \leq \frac{C}{A^{q-1}}, \tag{v}$$

independent of $s\in[-1,1]$ and $\xi\in\mathbb{C}$ with $|\xi|\leq 1$.

PROOF OF LEMMA 3.3. See Lemma (3.13) of [9]. □

We integrate the left hand side of (3.6) with respect to $dt_{2n-2j+3}$, $dt_{2n-2j+4}$, respectively using Lemma 3.3 and deleting $\left(|t_{2n-2j+3}|^2+|t_{2n-2j+4}|^2\right)^{\nu_{n-j+2}}$ if convenient. It is easy to see that we have almost the same integral except that the exponent $(r+j)$ in the denominator becomes $(r+j-1)$ and $t_{2n-2j+3}, t_{2n-2j+4}$ vanishes. We repeat this for $t_{2n-2j+5}, \cdots, t_{2n}$. After $2(j-1)$-times integration, we have the inequality (3.6).

Part (iii). By the straightforward computation we have

$$\begin{aligned}
&\left|\left(\sum_{k=1}^{n} \bar{\partial}_\zeta P_k(z,\zeta) d\zeta_k\right)^j\right| \\
&\leq {\sum}' \prod_{\beta=1}^{j} \Big[|\xi_{\ell_\beta}|^{2\mu_{\ell_\beta}-2} + |\eta_{\ell_\beta}|^{2\nu_{k_\ell}-2} + |\zeta_{\ell_\beta} - z_{\ell_\beta}| \\
&\qquad\qquad \times \left(\sigma(\mu_{\ell_\beta})|\xi_{\ell_\beta}|^{2\mu_{\ell_\beta}-3} + \sigma(\nu_{\ell_\beta})|\eta_{\ell_\beta}|^{2\nu_{\ell_\beta}-3}\right)\Big] \\
&\lesssim {\sum}' \prod_{\alpha=1}^{j-1} \Big[|\xi_{k_\alpha}|^{2\mu_{k_\alpha}-2} + |\eta_{k_\alpha}|^{2\nu_{k_\alpha}-2} + |\zeta_{k_\alpha} - z_{k_\alpha}| \\
&\qquad\qquad \times \left(\sigma(\mu_{k_\alpha})|\xi_{k_\alpha}|^{2\mu_{k_\alpha}-3} + \sigma(\nu_{k_\alpha})|\eta_{k_\alpha}|^{2\nu_{k_\alpha}-3}\right)\Big],
\end{aligned}$$

where $\sum'$ is summed over all j-tuples, $1 \leq \ell_1 < \cdots < \ell_j \leq n$ and $(j-1)$-tuples $2 \leq k_1 < \cdots < k_{j-1} \leq n$, respectively. Thus the inequality (iii) follows from the Part (ii). The proof of an analog statement is not exactly the same as the one of (i)–(iii). But by virtue of (ii) of Lemma 3.1 and (iv), (v) of Lemma 3.3 they can be proved in the same way and the proof is even much easier. So we omit the proof. □

4. Estimates of integral kernels

We will prove two propositions which are essential part of this paper.

PROPOSITION 4.1. *Let r be a suitably large number depending on given α, ε.*

(i) *For $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\int_\Omega |\rho(\zeta)|^{r-1-\varepsilon} \left|\widetilde{K}_0^r(z,\zeta)\right| dV(\zeta) \leq C_\varepsilon |\rho(z)|^{-\varepsilon} \quad \textit{for } z \in \Omega.$$

(ii) *For $\alpha > 0$ and $\alpha - \varepsilon > -1$ there exists a constant $C_{\alpha,\varepsilon}$ such that*

$$\int_\Omega |\rho(z)|^{\alpha-\varepsilon} \left|\widetilde{K}_0^r(z,\zeta)\right| dV(z) \leq C_{\alpha,\varepsilon} |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \quad \textit{for } \zeta \in \Omega.$$

PROOF. *Part* (i). Since the only singularity of the integral occurs for $\zeta = z$, it is clear that if $|\zeta - z| \geq \lambda$ the integral is bounded and so the inequality (i) holds. If $z \in \Omega \setminus U$, where U is an open set appeared in Lemma 3.1, then $|S(z,\zeta)| \geq c > 0$ for every $\zeta \in \Omega$ and so $|\widetilde{K}_0^r(z,\zeta)|$ is integrable. Again in this case the inequality (i) holds. Therefore we may assume $z \in \Omega \cap U$ and it suffices to show that

$$I_1 = \int_{\Omega \cap B(z,\lambda)} |\rho(\zeta)|^{r-1-\varepsilon} \left|\widetilde{K}_0^r(\zeta, z)\right| \, dV(\zeta) \lesssim |\rho(z)|^{-\varepsilon}.$$

It follows from Lemma 3.2 (i) that

$$\begin{aligned} I_1 &\lesssim \int_{-1}^{1}\int_{-1}^{1} \frac{|t_1|^{r-1-\varepsilon}\, dt_1 dt_2}{(|t_1| + |t_2| + |\rho(z)|)^r(|t_1 - |\rho(z)|| + |t_2|)} \\ &\lesssim \int_{-1}^{1}\int_{-1}^{1} \frac{dt_1 dt_2}{(|t_1| + |t_2| + |\rho(z)|)^{1+\varepsilon}(|t_1 - |\rho(z)|| + |t_2|)}. \end{aligned}$$

If we make the change of variables $t_1 = |\rho(z)|t_1'$ and $t_2 = |\rho(z)|t_2'$ and omit the primes, this becomes

$$\begin{aligned} I_1 &\lesssim |\rho(z)|^{-\varepsilon} \int_0^\infty \int_1^\infty \frac{dt_1 dt_2}{(t_1 + t_2 + 1)^{1+\varepsilon}((t_1 - 1) + t_2)} \\ &\lesssim |\rho(z)|^{-\varepsilon} \int_0^\infty \int_0^\infty \frac{dt_1 dt_2}{(t_1 + t_2 + 2)^{1+\varepsilon}(t_1 + t_2)} \\ &\lesssim |\rho(z)|^{-\varepsilon} \int_0^\infty \frac{\delta \, d\delta}{(\delta + 2)^{1+\varepsilon}\delta} \\ &\lesssim |\rho(z)|^{-\varepsilon}, \quad \text{if } \varepsilon > 0. \end{aligned}$$

Part (ii). As Part (i) it is enough to show that for $\zeta \in \Omega \cap U$ fixed

$$I_2 = \int_{\Omega \cap B(\zeta,\tilde\lambda)} |\rho(\zeta)|^{\alpha-\varepsilon} \left|\widetilde{K}_0^r(\zeta, z)\right| \, dV(z) \lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1}.$$

From the coordinate system u introduced in (ii) of Lemma 3.1 and an analog statement of Lemma 3.2 it follows that

$$I_2 \lesssim \int_{-1}^{1}\int_{-1}^{1} \frac{|u_1|^{\alpha-\varepsilon}\, du_1 du_2}{(|u_1| + |u_2| + |\rho(\zeta)|)^r(|u_1 - |\rho(\zeta)|| + |u_2|)}.$$

If we make the change of variables $u_1 = |\rho(z)|u_1'$ and $u_2 = |\rho(z)|u_2'$ and omit the primes, we have

$$I_2 \lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u_1|^{\alpha-\varepsilon}\, du_1 du_2}{(|u_1|+|u_2|+1)^r(|u_1-1|+|u_2|)}.$$

If $\alpha - \varepsilon \geq 0$, then

$$\begin{aligned}
I_2 &\lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du_1 du_2}{(|u_1|+|u_2|+1)^{r-\alpha+\varepsilon}(|u_1-1|+|u_2|)} \\
&\lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \int_0^{\infty}\int_0^{\infty} \frac{du_1 du_2}{(u_1+u_2+2)^{r-\alpha+\varepsilon}(u_1+u_2)} \\
&\lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \int_0^{\infty} \frac{\delta\, d\delta}{(\delta+2)^{r-\alpha+\varepsilon}\delta} \\
&\lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1} \quad \text{if } r-\alpha+\varepsilon > 1.
\end{aligned}$$

If $-1 < \alpha - \varepsilon < 0$, then for $0 < \eta < 1$ we have

$$\begin{aligned}
I_3 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du_1 du_2}{|u_1|^{\varepsilon-\alpha}(|u_1|+|u_2|+1)^r(|u_1-1|+|u_2|)} \\
&\lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du_1 du_2}{|u_1|^{\varepsilon-\alpha}|u_1-1|^{\eta}|u_2|^{1-\eta}(|u_1|+|u_2|+1)^r} \\
&\lesssim \int_{-\infty}^{\infty} \frac{du_1}{|u_1|^{\varepsilon-\alpha}|u_1-1|^{\eta}(|u_1|+1)^{\frac{r}{2}}} \int_{-\infty}^{\infty} \frac{du_2}{|u_2|^{1-\eta}(|u_2|+1)^{\frac{r}{2}}}.
\end{aligned}$$

But since $0 < \eta < 1$ and $0 < \varepsilon - \alpha < 1$ it is easy to see that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{du_1}{|u_1|^{\varepsilon-\alpha}|u_1-1|^{\eta}(|u_1|+1)^{\frac{r}{2}}} \\
&\lesssim \int_0^{\infty} \frac{du_1}{u_1^{\varepsilon-\alpha}(u_1+1)^{\frac{r}{2}}} + \int_0^{\infty} \frac{du_1}{u_1^{\eta}(u_1+2)^{\frac{r}{2}}} < \infty.
\end{aligned}$$

Therefore I_3 is bounded even though $-1 < \alpha - \varepsilon < 0$. Hence $I_2 \lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1}$ if $\alpha - \varepsilon > -1$. □

PROPOSITION 4.2. *Let $1 \leq j \leq n-1$ and r be a suitably large number depending on given α, ε.*

(i) *For $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\int_\Omega |\rho(\zeta)|^{r-1-\frac{1}{m}-\varepsilon} \left|K_j^r(z,\zeta)\right| dV(\zeta) \leq C_\varepsilon |\rho(z)|^{-\varepsilon} \quad \text{for } z \in \Omega.$$

(ii) *For $\alpha > 0$ and $\alpha - \varepsilon > -1$ there exists a constant $C_{\alpha,\varepsilon}$ such that*

$$\int_\Omega |\rho(z)|^{\alpha-\varepsilon}\left|K_j^r(z,\zeta)\right| \, dV(z) \le C_{\alpha,\varepsilon}|\rho(\zeta)|^{\alpha-\varepsilon-r+1+\frac{1}{m}} \quad \text{for } \zeta \in \Omega.$$

(iii) *In the above integral if we replace $\left|K_j^r(z,\zeta)\right|$ by $\left|\widetilde{K}_j^r(z,\zeta)\right|$ then we can also obtain the same results.*

PROOF. *Part* (i). Fix $z \in \Omega \cap U$. As the proof of (i) of Proposition 4.1 we have

$$\begin{aligned} I_1 &= \int_{\Omega\cap B(z,\lambda)} |\rho(\zeta)|^{r-1-\frac{1}{m}-\varepsilon}\left|K_j^r(z,\zeta)\right| \, dV(\zeta) \\ &\lesssim \int_{\substack{|w|<1 \\ w=(t_3,t_4)}} dV(w) \int_{-1}^{1}\int_{-1}^{1} \frac{|t_1|^{r-1-\frac{1}{m}-\varepsilon}\, dt_1 dt_2}{(|t_1|+|t_2|+|w|^m+|\rho(z)|)^{r+1}|w|}. \end{aligned}$$

If we make the change of variables $w = |\rho(z)|^{\frac{1}{m}}w', t_1 = |\rho(z)|t_1'$ and $t_2 = |\rho(z)|t_2'$ and omit the primes, this becomes

$$I_1 \lesssim |\rho(z)|^{-\varepsilon}\int_{w\in\mathbb{R}^2}\int_0^\infty\int_0^\infty \frac{t_1^{r-1-\frac{1}{m}-\varepsilon}\, dt_1 dt_2 dV(w)}{(t_1+t_2+|w|^m+|\rho(z)|)^{r+1}|w|}.$$

If we choose r so that $r - 1 - 1/m - \varepsilon > 0$, then

$$\begin{aligned} I_1 &\lesssim |\rho(z)|^{-\varepsilon}\int_{w\in\mathbb{R}^2}\int_0^\infty\int_0^\infty \frac{dt_1 dt_2 dV(w)}{(t_1+t_2+|w|^m+|\rho(z)|)^{2+\frac{1}{m}+\varepsilon}|w|} \\ &\lesssim |\rho(z)|^{-\varepsilon}\int_{w\in\mathbb{R}^2} \frac{dV(w)}{|w|(|w|^m+1)^{\frac{1}{m}+\varepsilon}} \lesssim |\rho(z)|^{-\varepsilon} \quad \text{if } \varepsilon > 0. \end{aligned}$$

Part (ii). As the proof of (ii) of Proposition 4.1, by the analog statement of Lemma 3.2, we have

$$\begin{aligned} I_2 &= \int_{\Omega\cap B(\zeta,\tilde\lambda)} |\rho(z)|^{\alpha-\varepsilon}\left|K_j^r(z,\zeta)\right| dV(z) \\ &\lesssim \int_{\substack{|w|<1 \\ w=(u_3,u_4)}} \int_{-1}^{1}\int_{-1}^{1} \frac{|u_1|^{\alpha-\varepsilon}\, du_1 du_2 dV(w)}{(|u_1|+|u_2|+|w|^m+|\rho(\zeta)|)^{r+1}|w|}. \end{aligned}$$

If we make the change of variables $w = |\rho(\zeta)|^{1/m}w', u_1 = |\rho(\zeta)|u_1'$ and $u_2 = |\rho(\zeta)|u_2'$ and omit the primes, then we have

$$I_2 \lesssim |\rho(\zeta)|^{\alpha-\varepsilon-r+1+\frac{1}{m}} \int_{w\in\mathbb{R}^2}\int_0^\infty\int_0^\infty \frac{u_1^{\alpha-\varepsilon}\, du_1du_2dV(w)}{(u_1+u_2+|w|^m+1)^{r+1}|w|}.$$

If $\alpha-\varepsilon \geq 0$, then

$$\begin{aligned} I_3 &= \int_{w\in\mathbb{R}^2}\int_0^\infty\int_0^\infty \frac{u_1^{\alpha-\varepsilon}\, du_1du_2dV(w)}{(u_1+u_2+|w|^m+1)^{r+1}|w|} \\ &\lesssim \int_{w\in\mathbb{R}^2} \frac{dV(w)}{(|w|^m+1)^{r-\alpha+\varepsilon-1}|w|} \\ &\lesssim \int_0^\infty \frac{\delta\, d\delta}{(\delta^m+1)^{r-\alpha+\varepsilon-1}\delta} \lesssim 1 \quad \text{if } r-\alpha+\varepsilon-\frac{1}{m} > 1. \end{aligned}$$

If $-1 < \alpha-\varepsilon < 0$ and r is sufficiently large, then we also obtain that

$$I_3 = \int_{w\in\mathbb{R}^2}\int_0^\infty\int_0^\infty \frac{du_1du_2dV(w)}{u_1^{\varepsilon-\alpha}(u_1+u_2+|w|^m+1)^{r+1}|w|} \lesssim 1.$$

□

5. Proof of Main Theorem

For $z \in \Omega$ we define operators

$$\widetilde{T}_j^r f(z) = \int_{\zeta\in\Omega} |\rho(\zeta)|^r \widetilde{K}_j^r(z,\zeta)\wedge f(\zeta), \quad 0 \leq j \leq n-1,$$

$$T_j^r f(z) = \int_{\zeta\in\Omega} |\rho(\zeta)|^{r-1} K_j^r(z,\zeta)\wedge \bar{\partial}\rho(\zeta)\wedge f(\zeta), \quad 1 \leq j \leq n-1.$$

Then a solution u of $\bar{\partial}u = f$ becomes

$$u(z) = \big(T^r f\big)(z) = \sum_{j=0}^{n-1}\big(\widetilde{T}_j^r f\big)(z) + \sum_{j=1}^{n-1}\big(T_j^r f\big)(z).$$

To prove Main Theorem we divide estimates into two parts.

5.1. The case $1 < \alpha < \infty$. It is enough to show that

$$\sum_{j=0}^{n-1} ||\widetilde{T}_j^r f||_{p,\alpha} + \sum_{j=1}^{n-1} ||T_j^r f||_{p,\alpha} \leq C_{p,\alpha} ||f||_{p,\alpha,\Omega}, \tag{5.1}$$

when $p = 1$ and $p = \infty$. Then by the Marcinkiewicz Interpolation Theorem (see Theorem 6.28 of [7]) it is easy to see that the inequality (5.1) holds for $1 < p < \infty$. We first consider $\widetilde{T}_j^r f$, $0 \leq j \leq n-1$. Recall that $|f(\zeta)|_\Omega = |\rho(\zeta)||f(\zeta)| + |\rho(\zeta)|^{\frac{1}{m}} |f(\zeta)|$ for $\zeta \in \Omega$. For the case $p = 1$ we have

$$\begin{aligned}
&\int_\Omega |\widetilde{T}_j^r f(z)||\rho(z)|^{\alpha-1}\, dV(z) \\
&\lesssim \int\limits_{\zeta\in\Omega} |f(\zeta)|_\Omega |\rho(\zeta)|^{r-1} \int\limits_{z\in\Omega} |\rho(z)|^{\alpha-1} \big|\widetilde{K}_j^r(z,\zeta)\big|.
\end{aligned}$$

By Proposition 4.1 (ii) and Proposition 4.2 (iii) we have

$$\int_\Omega |\rho(z)|^{\alpha-1} \big|\widetilde{K}_0^r(z,\zeta)\big|\, dV(z) \lesssim |\rho(\zeta)|^{\alpha-r},$$

$$\int_\Omega |\rho(z)|^{\alpha-1} \big|\widetilde{K}_j^r(z,\zeta)\big|\, dV(z) \lesssim |\rho(\zeta)|^{\alpha-r+\frac{1}{m}}, \quad 1 \leq j \leq n-1,$$

respectively. Therefore for any $0 \leq j \leq n-1$, we have

$$\int_\Omega \big|\widetilde{T}_j^r f(z)\big||\rho(z)|^{\alpha-1}\, dV(z) \lesssim \int_\Omega |f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}\, dV(\zeta).$$

Here we use the fact that $|\rho(z)| \lesssim 1$ for every $z \in \Omega$. For the case $p = \infty$ it follows from Proposition 4.1 (i) and Proposition 4.2 (iii) that

$$\begin{aligned}
&\big|\widetilde{T}_j^r f(z)\big||\rho(z)|^{\alpha-1} \\
&\lesssim \sup_{\zeta\in\Omega} \big[|f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}\big] |\rho(z)|^{\alpha-1} \int\limits_{\zeta\in\Omega} |\rho(\zeta)|^{r-\alpha} \big|\widetilde{K}_j^r(z,\zeta)\big| \\
&\lesssim \sup_{\zeta\in\Omega} \big[|f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}\big], \quad \text{if } \alpha > 1.
\end{aligned}$$

Next we consider $T_j^r f$, $1 \le j \le n-1$. If $p = 1$, then we obtain

$$\begin{aligned}
&\int_\Omega \left|T_j^r f(z)\right| |\rho(z)|^{\alpha-1}\, dV(z) \\
&\lesssim \int\limits_{\zeta\in\Omega} |\bar\partial\rho(\zeta)\wedge f(\zeta)| |\rho(\zeta)|^{r-1} \int\limits_{z\in\Omega} |\rho(z)|^{\alpha-1} \left|K_j^r(z,\zeta)\right| \\
&\lesssim \int\limits_{\zeta\in\Omega} |f(\zeta)|_\Omega |\rho(\zeta)|^{r-1-\frac{1}{m}} |\rho(\zeta)|^{\alpha-r+\frac{1}{m}} \\
&\lesssim \int\limits_{\zeta\in\Omega} |f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}
\end{aligned}$$

by Proposition 4.2 (ii). Here in using Proposition 4.2 (ii) we put $\varepsilon = 1 > 0$. If $p = \infty$, similarly by Proposition 4.2 (i), we see that

$$\begin{aligned}
&\left|T_j^r f(z)\right| |\rho(z)|^{\alpha-1} \\
&\lesssim \sup_{\zeta\in\Omega} \left[|f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}\right] |\rho(z)|^{\alpha-1} \int\limits_{\zeta\in\Omega} |\rho(\zeta)|^{r-\alpha-\frac{1}{m}} \left|K_j^r(z,\zeta)\right| \\
&\lesssim \sup_{\zeta\in\Omega} \left[|f(\zeta)|_\Omega |\rho(\zeta)|^{\alpha-1}\right], \quad \text{if } \alpha > 1.
\end{aligned}$$

5.2. The case $0 < \alpha \le 1$. We can not use the interpolation argument, since in this case one can not prove the estimate (5.1) for $p = \infty$. For $p = 1$ the estimate (5.1) is proved as the case 5.1, since the proof does not depend on α. Now let $1 < p < \infty$ and q satisfy the equality $1/q + 1/p = 1$. Then by Hölder's inequality and Proposition 4.1 (i) and 4.2 (iii) we have for sufficiently small $\varepsilon > 0$ and $0 \le j \le n-1$

$$\begin{aligned}
\left|\widetilde{T}_j^r f(z)\right| &\le \int |\rho|^{r-1} \left|\widetilde{K}_j^r\right| |f|_\Omega \\
&= \int \left(|\rho|^{r-1}\left|\widetilde{K}_j^r\right|\right)^{\frac{1}{q}} |\rho|^{-\varepsilon} \left(|\rho|^{r-1}\left|\widetilde{K}_j^r\right|\right)^{\frac{1}{p}} |\rho|^{\varepsilon} |f|_\Omega \\
&\lesssim \left(\int_{\zeta\in\Omega} |\rho|^{r-1-\varepsilon q} \left|\widetilde{K}_j^r\right|\right)^{\frac{1}{q}} \left(\int_{\zeta\in\Omega} |\rho|^{r-1+\varepsilon p} \left|\widetilde{K}_j^r\right| |f|_\Omega^p\right)^{\frac{1}{p}} \\
&\lesssim |\rho(z)|^{-\varepsilon} \left(\int_{\zeta\in\Omega} |\rho|^{r-1+\varepsilon p} \left|\widetilde{K}_j^r\right| |f|_\Omega^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus it follows from Proposition 4.1 (ii) and 4.2 (iii) that

(5.2)

$$\int_\Omega |\widetilde{T}_j^r f(z)|^p |\rho(z)|^{\alpha-1}\, dV(z) \lesssim \int_{z\in\Omega} |\rho|^{\alpha-1-\varepsilon p} |\widetilde{K}_j^r| \int_{\zeta\in\Omega} |\rho|^{r-1+\varepsilon p} |f|_\Omega^p$$
$$\lesssim \int_{\zeta\in\Omega} |f|_\Omega^p |\rho|^{\alpha-1},$$

if we choose $\varepsilon > 0$ so small that $\alpha - 1 - \varepsilon p > -1$. Again by Hölder's inequality and Proposition 4.2 (i) we have for sufficiently small $\varepsilon > 0$ and $1 \le j \le n-1$

$$|T_j^r f(z)| \le \int |\rho|^{r-1-\frac{1}{m}} |K_j^r| |\rho|^{\frac{1}{m}} |\bar\partial\rho \wedge f| = \int |\rho|^{r-1-\frac{1}{m}} |K_j^r| |f|_\Omega$$
$$\lesssim \left(\int_{\zeta\in\Omega} |\rho|^{r-1-\frac{1}{m}-\varepsilon q} |K_j^r| \right)^{\frac{1}{q}} \left(\int_{\zeta\in\Omega} |\rho|^{r-1-\frac{1}{m}+\varepsilon p} |K_j^r| |f|_\Omega^p \right)^{\frac{1}{p}}$$
$$\lesssim |\rho(z)|^{-\varepsilon} \left(\int_{\zeta\in\Omega} |\rho|^{r-1-\frac{1}{m}+\varepsilon p} |K_j^r| |f|_\Omega^p \right)^{\frac{1}{p}}.$$

Thus it follows from Proposition 4.2 (ii) that

$$\int_\Omega |T_j^r f(z)|^p |\rho(z)|^{\alpha-1}\, dV(z)$$
$$\lesssim \int_{z\in\Omega} |\rho|^{\alpha-1-\varepsilon p} |\widetilde{K}_j^r| \int_{\zeta\in\Omega} |\rho|^{r-1-\frac{1}{m}+\varepsilon p} |f|_\Omega^p \tag{5.3}$$
$$\lesssim \int_{\zeta\in\Omega} |f|_\Omega^p |\rho|^{\alpha-1},$$

if we choose $\varepsilon > 0$ so small that $\alpha - 1 - \varepsilon p > -1$. Combining (5.2) and (5.3) we see that the inequality (5.1) holds for $1 < p < \infty$ and $0 < \alpha \le 1$.

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