Commun. Korean Math. Soc. 18 (2003), No. 2, pp. 243–249

THE RADIAL DERIVATIVES ON WEIGHTED BERGMAN SPACES

Si Ho Kang† and Ja Young Kim‡

ABSTRACT. We consider weighted Bergman spaces and radial derivatives on the spaces. We also prove that for each element f in $B^{p,r}$, there is a unique $\widetilde{f}$ in $B^{p,r}$ such that f is the radial derivative of $\widetilde{f}$ and for each $f \in \mathcal{B}^r(i)$, f is the radial derivative of some element of $\mathcal{B}^r(i)$ if and only if $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$.

1. Introduction

Let H be the half-plane in the complex plane $\mathbb{C}$ and let dA denote the usual two-dimensional area measure on H. Let $K(z,w) = -\dfrac{1}{\pi(z-\overline{w})^2}$. Then $K(\cdot,w)$ is the Bergman kernel for B^2([5]), where B^2 is the analytic Bergman space and let $A(H)$ denote the set of all holomorphic functions on H. For $1 \le p < \infty$, and $r \ge 0$, $B^{p,r}$ is defined to be

$$\{f \in A(H) : \|f\|_{p,r} = \Big(\int_H |f(z)|^p K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} < \infty\}.$$

Then $B^{p,r}$ is a closed subspace of $L^{p,r}$. In particular, $B^{2,r}$ is a Hilbert space and hence there is a unique orthogonal projection $P : L^{2,r} \to B^{2,r}$ defined by

$$P(f)(w) = (2r+1)\int_H f(z)\overline{K(z,w)^{1+r}} K(z,z)^{-r} dA(z)$$

Received October 1, 2002.

2000 Mathematics Subject Classification: Primary 31B05, 31B10; Secondary 32A36.

Key words and phrases: weighted Bergman spaces, Bergman kernels, half-plane, radial derivatives.

†This work was supported by Sookmyung Women's University Research Grants 2002.

‡This work was supported by the Post-doctoral Fellowship Program of Korea Science and Engineering Foundation.

for all $f \in L^{2,r}$.

Let $\mathcal{B} = \{f \in A(H) : \|f\| = \sup_{z=x+iy\in H} y|\bigtriangledown f(z)| < \infty\}$. Then $\mathcal{B}$ is a complete semi-normed space and let $\mathcal{B}(i) = \{f \in \mathcal{B} : f(i) = 0\}$. Let $\|f\|_{\mathcal{B}} = \|f\| + |f(i)|$. Then this norm makes $\mathcal{B}(i)$ a Banach space.

Let $\partial^\infty H = \partial H \cup \{\infty\}$ and let $\mathcal{B}_0(i) = \{f \in \mathcal{B}(i) : \lim_{z\to\partial^\infty H} y|\bigtriangledown f(z)| = 0\}$, where $z \to \partial^\infty H$ means $|z| \to \infty$ or $y \to 0$. Then $\mathcal{B}_0(i)$ is also a Banach space. Moreover, $\mathcal{B}(i)$ and $\mathcal{B}_0(i)$ are known to be the dual and predual, respectively, of B^1([5]).

In the following, we let $\mathcal{R}$ denote the radial differentiation of $h \in C^1(H)$ defined by $\mathcal{R}h(z) = xD_1h(z) + yD_2h(z)$, where $z = x + iy \in H$. It is easy to prove that $\mathcal{R}h = 0$ if and only if h is radially constant, that is, $h(z) = h(tz)$ for all $t > 0$ and all $z \in H$. In this paper, we will show that given $f \in B^{p,r}$, there is a unique $\widetilde{f} \in B^{p,r}$ such that $\mathcal{R}\widetilde{f} = f$ and $f \longmapsto \widetilde{f}$ is bounded on $B^{p,r}$. Section 3 is devoted to the relationship between the radial derivative and the function spaces which are introduced in Section 2. Throughout this paper, we use the symbol $A \lesssim B$ for nonnegative constants A, B to indicate that A is dominated by B times some positive constant.

2. Background and preliminary results

The characterizations in [5] are obtained by the following ideas. Let $K(z,w) = -\dfrac{1}{\pi(z-\overline{w})^2}$ and let $K_{\mathbb{D}}(z,w) = \dfrac{1}{\pi(1-z\overline{w})^2}$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Then $(2r+1)K_{\mathbb{D}}(\cdot,w)^{1+r}$ is the reproducing kernel for $B^{2,r}(\mathbb{D})$. Using this fact, we have the reproducing kernel $(2r+1)K(\cdot,w)^{1+r}$ for $B^{2,r}$. For $1 < p < \infty$ and $0 \le r < \infty$, we define $P : L^{p,r} \to B^{p,r}$ by

$$P(f)(w) = \int_H f(z)(2r+1)\overline{K(z,w)^{1+r}}K(z,z)^{-r}dA(z)$$

for all $f \in L^{p,r}$. By Schur's theorem, we see that P is bounded. Then the boundedness of P implies the duality of weighted analytic Bergman spaces : If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\left(B^{p,r}\right)^* \cong B^{q,r}$ under the usual integral pairing

$$\langle f, g\rangle = \int_H f(z)\overline{g(z)}K(z,z)^{-r}dA(z).$$

For $r > 0$, we define

$$\mathcal{B}^r(i) = \{f \in A(H) : f(i) = 0 \text{ and } \|f\|_{\mathcal{B}^r} = \sup_{z=x+iy\in H} y^{1+2r}|\nabla f(z)| < \infty\}.$$

For any z in H and any f in $\mathcal{B}(i)$,

$$\begin{aligned} |f(z)| &= |f(z) - f(i)| = |\nabla f(z_0)||z - i| \text{ for some } z_0 \\ &= (\mathrm{Im} z_0)^{1+2r}|\nabla f(z_0)|\frac{|z-i|}{(\mathrm{Im} z_0)^{1+2r}} \\ &\leq \|f\|_{\mathcal{B}^r}\frac{|z-i|}{(\mathrm{Im} z_0)^{1+2r}}. \end{aligned}$$

Thus for any compact subset K of H, f is uniformly bounded on K. This implies that $\mathcal{B}^r(i)$ is a Banach space. Thus we have several function spaces.

3. The main theorem

LEMMA 3.1. *Let $1 \leq p < \infty$ and $r \geq 0$. If $f \in B^{p,r}$ then f has the vanishing property along the ray, that is, $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$.*

PROOF. Take any $z = x + iy \in H$. Then Jensen's inequality implies that

$$\begin{aligned} |f(z)| &= \left|\frac{1}{V(B(z,\frac{y}{2}))}\int_{B(z,\frac{y}{2})} f(w)dA(w)\right| \\ &\leq \Big(\frac{1}{V(B(z,\frac{y}{2}))}\int_{B(z,\frac{y}{2})} |f(w)|^p\big(\frac{\mathrm{Im} w}{\frac{y}{2}}\big)^{2r}dA(w)\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\frac{1}{y^{1+2r}}\int_{B(z,\frac{y}{2})} |f(w)|^p K(w,w)^{-r}dA(w)\Big)^{\frac{1}{p}} \\ &\leq \frac{\|f\|_{p,r}}{y^{\frac{2+2r}{p}}}. \end{aligned}$$

This completes the proof. □

THEOREM 3.2. *Let $1 \leq p < \infty$ and let $r \geq 0$. Then for each $f \in B^{p,r}$, there is a unique $\widetilde{f} \in B^{p,r}$ such that $f = \mathcal{R}\widetilde{f}$. Moreover, $f \longmapsto \widetilde{f}$ is bounded on $B^{p,r}$.*

PROOF. Suppose that there is $\widetilde{g} \in B^{p,r}$ such that $\mathcal{R}\widetilde{f} = \mathcal{R}\widetilde{g}$. Then $\mathcal{R}(\widetilde{f} - \widetilde{g}) = 0$ and hence $(\widetilde{f} - \widetilde{g})$ is radially constant that is $(\widetilde{f} - \widetilde{g})(z) = (\widetilde{f} - \widetilde{g})(tz)$ for all $t > 0$, and $z \in H$. By Lemma 3.1, $(\widetilde{f} - \widetilde{g})(tz) \to 0$ as

$t \to \infty$. So $\widetilde{f}(z) = \widetilde{g}(z)$ for all $z \in H$. Thus we have the result. Take any f in $B^{p,r}$. Let $\widetilde{f}(z) = -\int_1^\infty \frac{f(tz)}{t}dt$. Note that each $B^{p,r}$-function vanishes along the ray. This property shows that $\widetilde{f}$ is well-defined and $\widetilde{f}$ is analytic. By Minkowski's integral inequality,

$$\begin{aligned}
\|\widetilde{f}\|_{p,r} &= \Big(\int_H |\widetilde{f}(z)|^p K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} \\
&= \Big(\int_H |-\int_1^\infty \frac{f(tz)}{t}dt|^p K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} \\
&\le \int_1^\infty \Big(\int_H \frac{|f(tz)|^p}{t^p} K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} dt \\
&= \int_1^\infty \Big(\int_H \frac{|f(tz)|^p}{t^p} K(tz,tz)^{-r} \frac{1}{t^{2r}} \cdot \frac{1}{t^2} dA(tz)\Big)^{\frac{1}{p}} dt \\
&= \|f\|_{p,r} \int_1^\infty \frac{1}{t^{1+\frac{2r+2}{p}}} dt.
\end{aligned}$$

Since $1 + \frac{2r+2}{p} > 1$, $\widetilde{f} \in B^{p,r}$. Moreover, this implies that $f \longmapsto \widetilde{f}$ is bounded on $B^{p,r}$. For each $j \in \{1,2\}$,

$$D_j\widetilde{f}(z) = D_j\Big(-\int_1^\infty \frac{f(tz)}{t}dt\Big) = -\int_1^\infty D_j f(tz)dt$$

and hence

$$\begin{aligned}
\mathcal{R}\widetilde{f}(z) &= \sum_{j=1}^2 z_j D_j \widetilde{f}(z) = -\int_1^\infty \sum_{j=1}^2 z_j D_j f(tz)dt \\
&= -\int_1^\infty \frac{df(tz)}{dt}dt = f(z) \text{ by Lemma 3.1.}
\end{aligned}$$

□

EXAMPLE 3.3. $\mathcal{B}_0(i)$ is the predual of B^1 and fixed $w = s + it \in H$. Consider $K(z,w) - K(i,w) = \frac{-1}{\pi(z-\overline{w})^2} + \frac{1}{\pi(i-\overline{w})^2}$. Then

$$D_j\Big(K(z,w) - K(i,w)\Big) = \frac{2}{\pi(z-\overline{w})^3}$$

and hence

$$\sup_{z=x+iy\in H} y\frac{4}{\pi|z-\overline{w}|^3} \le \sup_{z=x+iy\in H} \frac{4y}{\pi y t^2} = \frac{2}{\pi t^2}.$$

Since $\lim_{z\to\partial^\infty H} y\frac{4}{\pi|z-\overline{w}|^3} = 0$, $K(\cdot,w) - K(i,w)$ belongs to $\mathcal{B}_0(i)$. Suppose that $-\frac{1}{\pi}\Big(\frac{1}{(z-\overline{w})^2} - \frac{1}{(i-\overline{w})^2}\Big) = \sum_{j=1}^{2} z_j D_j f(z) = \mathcal{R}f(z)$ for some $f \in \mathcal{B}_0(i)$. Put $z = (0,t)$. Then $\frac{1}{\pi(i-\overline{w})^2} = \lim_{t\to\infty}\Big(\frac{-1}{\pi}\big(\frac{1}{(t-\overline{w})^2} - \frac{1}{(i-\overline{w})^2}\big)\Big)$ but $\lim_{t\to\infty} tD_2 f(z) = 0$. Thus we get a contradiction and we know that Theorem 3.2 does not hold on $\mathcal{B}_0(i)$.

THEOREM 3.4. *Suppose that $r > 0$ and $f \in \mathcal{B}^r(i)$. Then there is a unique $\widehat{f} \in \mathcal{B}^r(i)$ such that $\mathcal{R}\widehat{f} = f$ if and only if $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$. Moreover, $f \longmapsto \widehat{f}$ is bounded.*

PROOF. Suppose $r > 0$ and $f \in \mathcal{B}^r(i)$. Let $\widehat{f}(z) = -\int_1^\infty \frac{f(tz) - f(ti)}{t} dt$ Since $D_j\widehat{f}(z) = -\int_1^\infty D_j f(tz)dt$ for all $j = 1, 2$,

$$
\begin{aligned}
|y^{1+2r} \bigtriangledown \widehat{f}(z)| &\leq \sum_{j=1}^{2} y^{1+2r} \int_1^\infty |D_j f(tz)|dt \\
&= \sum_{j=1}^{2} \int_1^\infty \frac{(ty)^{1+2r}|D_j f(tz)|dt}{t^{1+2r}} \\
&\leq 2\|f\|_{\mathcal{B}^r} \int_1^\infty \frac{dt}{t^{1+2r}} = \frac{1}{r}\|f\|_{\mathcal{B}^r}.
\end{aligned}
$$

This implies $\widehat{f} \in \mathcal{B}^r(i)$. Since $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$,

$$
\begin{aligned}
f(z) &= -\int_1^\infty \frac{df}{dt}(tz)dt = -\int_1^\infty \sum_{j=1}^{2} z_j D_j f(tz)dt \\
&= -\sum_{j=1}^{2} z_j \int_1^\infty D_j f(tz)dt = \sum_{j=1}^{2} z_j D_j \widehat{f}(z) = \mathcal{R}\widehat{f}(z).
\end{aligned}
$$

Since $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$, such $\widehat{f}$ is unique. By the above observation, for each $f \in \mathcal{B}^r(i)$, $\|\widehat{f}\|_{\mathcal{B}^r} \lesssim \|f\|_{\mathcal{B}^r}$ and hence $f \longmapsto \widetilde{f}$ is bounded.

Conversely, since $f(z) = \mathcal{R}\widehat{f}(z) = -\int_1^\infty \frac{df}{dt}(tz)dt = -\lim_{s\to\infty} f(tz)\Big]_1^s = f(z) - \lim_{s\to\infty} f(sz)$ for all $z \in H$, $\lim_{s\to\infty} f(sz) = 0$. The proof is complete. □

PROPOSITION 3.5. *There is a function $f \in \mathcal{B}^r(i)$ such that $f \neq \mathcal{R}g$ for all $g \in \mathcal{B}^r(i)$.*

PROOF. Fix $w \in H$ and let $f(z) = K(z,w)^{1+r} - K(i,w)^{1+r}$. Then $f(z) = \Big(-\frac{1}{\pi}\Big)^{1+r}\Big(\frac{1}{(z-\overline{w})^{2+2r}} - \frac{1}{(i-\overline{w})^{2+2r}}\Big)$ and hence

$$|D_j f(z)| = \Big(\frac{1}{\pi}\Big)^{1+r}(2+2r)\Big|\frac{1}{(z-\overline{w})^{3+2r}}\Big| \lesssim \frac{1}{|z-\overline{w}|^{3+2r}}.$$

Thus $\sup_{z=x+iy\in H} y^{1+2r}|D_j f(z)| \lesssim \frac{y^{1+2r}}{|z-\overline{w}|^{3+2r}} \leq \frac{1}{(\mathrm{Im} w)^2}$. Since $\|f\|_{\mathcal{B}^r} \lesssim \frac{1}{(\mathrm{Im} w)^2}$, $f \in \mathcal{B}^r(i)$. Suppose $f = \mathcal{R}g$ for some $g \in \mathcal{B}^r(i)$. Then

$$\Big(\frac{-1}{\pi}\Big)^{1+r}\Big(\frac{1}{(z-\overline{w})^{2+2r}} - \frac{1}{(i-\overline{w})^{2+2r}}\Big) = f(z) = \sum_{j=1}^{2} z_j D_j g(z)$$

for all $z \in H$. Put $z = (0,t)$. Then the right side of the above equality converges to 0 as $t \to \infty$ but the left side of the above equality converges to nonzero which is a contradiction. The proof is complete. □

References

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer-Verlag, New York, 1992.

[2] S. Axler, *Bergman Spaces and Their Operators*, Surveys of Some Recent Results in Operator Theory Vol. 1, Pitman Research Notes in Math. **171** (1988), 1–50.

[3] B. R. Choe, H. Koo, and H. Yi, *Harmonic Bergman Functions as Radial Derivatives of Bergman Functions*, preprint.

[4] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, Inc., New York, 1981.

[5] S. H. Kang and J. Y. Kim, *Toeplitz operators on weighted analytic Bergman spaces of the half-plane*, Bull. Korean Math. Soc. **37** (2000), no. 3, 437–450.

[6] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, INC., New York and Basel, 1990.

Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail: shkang@sookmyung.ac.kr
jykim@sookmyung.ac.kr