Commun. Korean Math. Soc. **18** (2003), No. 2, pp. 207–214

A NOTE ON SYMMETRIC DIFFERENCES OF ORTHOMODULAR LATTICES

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ABSTRACT. There exist two distinct symmetric differences in a non Boolean orthomodular lattice. Let L be an orthomodular lattice. Then L is a Boolean algebra if and only if one symmetric difference is equal to the other. An orthomodular lattice L is Boolean if and only if one of two symmetric differences of L is associative.

1. Introduction

An *orthocomplementation* on a bounded poset P is a *unary operation* $'$ on P which satisfies the following properties: (1) if $x \leq y$ then $y' \leq x'$; (2) $x'' = x$; (3) $x \vee x' = 1$ and $x \wedge x' = 0$. We call a bounded poset P with an orthocomplementation an *orthoposet*. Two elements x, y of an orthoposet are *orthogonal*, written $x \perp y$, in case $x \leq y'$. An *ortholattice* (abbreviated by OL) is an orthoposet which is also a lattice.

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [5]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. The *commutator of* a *and* b of an OML L is denoted by $a*b$, and is defined by $a*b = (a\vee b)\wedge(a\vee b')\wedge(a'\vee b)\wedge(a'\vee b')$.

For elements a, b of an ortholattice, we say a *commutes with* b, in symbols $a\,\mathbf{C}\,b$, if $a = (a \wedge b) \vee (a \wedge b')$. If L is an OML, then the relation $\mathbf{C}$ is symmetric [p.22, 5].

One of our most important computational tools is the Foulis-Holland Theorem: *Let a, b, c be elements in an OML L such that one of them commutes with the other two. Then the sublattice generated by $\{a, b, c\}$ is distributive* [p.25, 5].

Received May 24, 2002.
2000 Mathematics Subject Classification: 06C15.
Key words and phrases: orthomodular lattice, symmetric difference, commutator, Boolean algebra.

A *ring* is called *Boolean* if all of its elements are *idempotent* (i.e., $a^2 = a$). For any x and y in a Boolean algebra B, the *symmetric difference* of x and y (in symbols $x+y$) is defined by $x+y = (x \wedge y') \vee (x' \wedge y)$. We can prove that the equivalence of a Boolean algebra and a Boolean ring using the symmetric difference in a Boolean algebra [4]. In this paper, we will study symmetric differences of orthomodular lattices.

The symmetric difference of a Boolean algebra has the following properties.

PROPOSITION 1.1. *The symmetric difference of x and y, $x+y$, of a Boolean algebra B has the following properties for all x, y, z and w in B.*

(1) $x + x = 0$
(2) $x + 1 = x'$
(3) $x + y = y + x$
(4) $x + 0 = x$
(5) $(x + y) + z = x + (y + z)$
(6) $(x + y) \wedge z = (x \wedge z) + (y \wedge z)$
(7) $x \vee y = (x + y) + (x \wedge y)$
(8) $x + y = x' + y'$
(9) $x \wedge y' \leq x + y \leq x \vee y$
(10) $x \perp y \Longleftrightarrow x + y = x \vee y$
(11) $x \leq y \Longleftrightarrow x + y = x' \wedge y$
(12) $(x + y) \perp (x \wedge y)$
(13) $x + y = (x \vee y) \wedge (x' \vee y')$
(14) $x = y \Longleftrightarrow x + y = 0$
(15) $x + y = y + z \Longleftrightarrow x = z$
(16) $(x + y) \vee (y + z) = (x \vee y \vee z) \wedge (x \wedge y \wedge z)'$
(17) $(x + y) \vee (y + z) \vee (z + w) = (x \vee y \vee z \vee w) \wedge (x \wedge y \wedge z \wedge w)'$
(18) $x \wedge y = (x + y') \wedge x = x + (x \wedge y')$.

PROOF. Each simple calculation is omitted. □

2. Symmetric differences of orthomodular lattices

DEFINITION 2.1. Let L be an OML, and $+$ be a binary operation on L. We call $+$ a *symmetric difference* on L if $+$ satisfies the following properties for all x, y in L :

(1) $x + x = 0$
(2) $x + 1 = x'$

(3) $x + y = y + x$.

It is known that the OML F_2 generated by two distinct elements is isomorphic to $2^4 \times MO2$ [2]. Therefore we have the following theorem.

THEOREM 2.2. *There exist two distinct symmetric differences in a non Boolean OML.*

PROOF. It is sufficient to show that there are only two polynomials on the OML F_2 which satisfy (1), (2), and (3) of Definition 2.1. Indeed, there are exactly three polynomials $(x \wedge y') \vee (x' \wedge y)$, $(x \vee y) \wedge (x' \vee y')$ and $(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$ in F_2 which satisfy (1) and (3) [pp.81–85, 1]. Finally, only two polynomials $(x \wedge y') \vee (x' \wedge y)$ and $(x \vee y) \wedge (x' \vee y')$ satisfy (2). □

We will denote $(x' \wedge y) \vee (x \wedge y')$ and $(x \vee y) \wedge (x' \vee y')$ with $x +_1 y$ and $x +_2 y$, respectively. That is,

$$x +_1 y = (x' \wedge y) \vee (x \wedge y')$$

$$x +_2 y = (x \vee y) \wedge (x' \vee y').$$

The two symmetric differences $x +_1 y$ and $x +_2 y$ of x, y in an OML L have the following common properties.

PROPOSITION 2.3. *Two symmetric differences $x +_1 y$ and $x +_2 y$ have the following properties for all x, y in an OML and each $i = 1, 2$.*

(1) $x +_i y = x' +_i y'$
(2) $x +_i 0 = x$
(3) $x \perp y \Longleftrightarrow x +_i y = x \vee y$
(4) $x \leq y \Longleftrightarrow x +_i y = x' \wedge y$
(5) $(x +_i y) \perp (x \wedge y)$
(6) $(x \wedge y') \leq (x +_i y) \leq (x \vee y)$
(7) $x +_i y = x' +_i y'$
(8) $x +_i (x \wedge y') = x \wedge (x' \vee y)$

PROOF. Each simple calculation is omitted. □

REMARK. For x, y in an ortholattice L, if we define $x +_1 y$ and $x +_2 y$ to be the same as in an OML, then $x +_1 y$ and $x +_2 y$ have all properties in Definition 2.1 and Proposition 2.3.

We have the following two properties on the symmetric differences of orthomodular lattices.

THEOREM 2.4. *Let L be an OML and $+$ be a symmetric difference on L. Then*
(1) $x+y = x+_1 y$ *if and only if* $x+y = (x\wedge y')+(x'\wedge y)$ $\forall x, y \in L$ *and*
(2) $x+y = x+_2 y$ *if and only if* $x\vee y = (x+y)+(x\wedge y)$ $\forall x, y, \in L$.

PROOF. (1) The conclusion follows since $(x \wedge y') + (x' \wedge y) = (x \wedge y') \vee (x' \wedge y)$ by (3) of Proposition 2.3.
(2) Assume that $x + y = x +_2 y = (x \vee y) \wedge (x' \vee y')$ $\forall x, y \in L$. Then

$$\begin{aligned}(x + y) + (x \wedge y) &= (x +_2 y) +_2 (x \wedge y)\\ &= (((x \vee y) \wedge (x' \vee y')) \vee (x \wedge y)))\\ &\quad \wedge(((x' \wedge y') \vee (x \wedge y)) \vee (x' \vee y'))\\ &= ((x \vee y) \wedge (x' \vee y')) \vee (x \wedge y)\\ &= ((x \vee y) \vee (x \wedge y)) \wedge ((x' \vee y') \vee (x \wedge y))\\ &= (x \vee y) \vee (x \wedge y)\\ &= x \vee y.\end{aligned}$$

Conversely, assume that $x \vee y = (x+y)+(x\wedge y)$ $\forall x, y \in L$. Then

$$\begin{aligned}&x +_2 y\\ &= (x \vee y) \wedge (x' \vee y')\\ &= ((x + y) + (x \wedge y)) \wedge ((x' + y') + (x' \wedge y'))\\ &= ((x + y) \vee (x \wedge y)) \wedge ((x' + y') \vee (x' \wedge y')) \quad \text{since} \quad (x + y) \perp (x \wedge y)\\ &= ((x + y) \vee (x \wedge y)) \wedge ((x + y) \vee (x' \wedge y')) \quad \text{since} \quad x + y = x' + y'\\ &= (((x + y) \vee (x \wedge y)) \wedge (x + y)) \vee (((x + y) \vee (x \wedge y)) \wedge (x' \wedge y'))\\ &= (x + y) \vee (((x + y) + (x \wedge y)) \wedge (x' \wedge y'))\\ &= (x + y) \vee ((x \vee y) \wedge (x' \wedge y'))\\ &= x + y.\end{aligned}$$

□

PROPOSITION 2.5. *Let L be an OML and x, y in L. Then $(x+_1 y)' = x' +_2 y$ and $x +_1 y \le x +_2 y$.*

PROOF. The conclusions follow by two definitions of $x +_1 y$ and $x +_2 y$. □

We know that $x * y = (x +_1 y)' \wedge (x' +_1 y)'$ and $x * y = (x +_2 y) \wedge (x' +_2 y)$. Now, we are ready to prove the following theorem.

THEOREM 2.6. *Let L be an OML and x, y in L. Then the following conditions are equivalent.*

(1) $x \mathbf{C} y$

(2) $x +_1 y = x +_2 y$

(3) $x +_2 y = (x \wedge y') +_2 (x' \wedge y)$

(4) $x +_1 y = (x \vee y) +_1 (x \wedge y)$

(5) $x = (x \wedge y) +_i (x \wedge y')$ *for each* $i = 1, 2$.

PROOF. (1) $\Longleftrightarrow$ (2)

Assume that $x \mathbf{C} y$. Then

$$\begin{aligned} x +_1 y &= (x' \wedge y) \vee (x \wedge y') \\ &= ((x' \wedge y) \vee x) \wedge ((x' \wedge y) \vee y') \\ &= ((x' \vee x) \wedge (y \vee x)) \wedge ((x' \vee y') \wedge (y \vee y')) \\ &= (y \vee x) \wedge (x' \vee y') \\ &= x +_2 y. \end{aligned}$$

Conversely, assume that $x +_1 y = x +_2 y$. Then $(x \wedge y') \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y')$. Thus

$$\begin{aligned} x * y &= ((x \vee y) \wedge (x' \vee y')) \wedge (x' \vee y) \wedge (x \vee y') \\ &= (((x \wedge y') \vee (x' \wedge y)) \wedge (x' \vee y)) \wedge (x \vee y')) \\ &= (((x \wedge y') \wedge (x' \vee y)) \vee ((x' \wedge y) \wedge (x' \vee y))) \wedge (x \vee y') \\ &= (x \wedge y') \wedge (x' \vee y) \wedge (x \vee y') \\ &= (x' \wedge y) \wedge (x \vee y') \\ &= 0. \end{aligned}$$

Therefore $x \mathbf{C} y$ since $x * y = 0$ if and only if $x \mathbf{C} y$ [3].

(2) $\Longleftrightarrow$ (3)

$x +_2 y = x +_1 y = (x \wedge y') \vee (x' \wedge y) = (x \wedge y') +_2 (x' \wedge y)$.

(2) $\Longleftrightarrow$ (4)

$$\begin{aligned} x +_1 y &= x +_2 y \\ &= (x \vee y) \wedge (x' \vee y') \\ &= ((x' \wedge y') \vee (x \wedge y))' \\ &= ((x' \wedge y') +_2 (x \wedge y))' && \text{by (3) of Proposition 2.3} \\ &= (x' \wedge y')' +_1 (x \wedge y) && \text{by Proposition 2.5} \\ &= (x \vee y) +_1 (x \wedge y) \end{aligned}$$

(1) $\Longleftrightarrow$ (5)

$x \mathbf{C} y \Longleftrightarrow x = (x \wedge y) \vee (x \wedge y') \Longleftrightarrow x = (x \wedge y) +_i (x \wedge y')$ $\forall i = 1, 2$. □

The following three corollaries follow.

COROLLARY 2.7. *Let L be an OML. Then L is a Boolean algebra if and only if* $x +_1 y = x +_2 y \quad \forall x, y \in L$.

PROOF. Suppose that L is a Boolean. Then $x +_1 y = x +_2 y \quad \forall x, y \in L$ by (13) of Proposition 1.1.

Conversely, if $x +_1 y = x +_2 y \quad \forall x, y \in L$, then $x\mathbf{C}y \quad \forall x, y \in L$ by (1) and (2) of Theorem 2.6. This completes the proof. □

COROLLARY 2.8. *Let L be an OML and $x, y \in L$. Then $x \leq y$ if and only if* $x +_1 y = x +_2 y = x' \wedge y$.

PROOF. Assume that $x \leq y$. Then $x +_1 y = x +_2 y = x' \wedge y$ by (2) of Proposition 2.3.

Conversely, assume that $x +_1 y = x +_2 y$. Then $x \, \mathbf{C} \, y$ by Theorem 2.6. Thus $y \, \mathbf{C} \, x$ since $y \, \mathbf{C} \, x$ if and only if $x \, \mathbf{C} \, y$ [5]. Therefore

$$
\begin{aligned}
y &= (x \wedge y) \vee (x' \wedge y) \\
&= (x \wedge y) \vee ((x \vee y) \wedge (x' \vee y')) \quad \text{for} \quad x +_1 y = x' \wedge y \\
&= ((x \wedge y) \vee (x \vee y)) \wedge ((x \wedge y) \vee (x' \vee y')) \\
&= x \vee y.
\end{aligned}
$$

Thus $y \geq x$. □

COROLLARY 2.9. *Let L be an OML and $x, y \in L$. Then $x \perp y$ if and only if* $x +_1 y = x +_2 y = x \vee y$.

PROOF.

$$
\begin{aligned}
x \perp y &\Longleftrightarrow x \leq y' \\
&\Longleftrightarrow x +_1 y' = x +_2 y' = x' \wedge y' && \text{by Corollary 2.8} \\
&\Longleftrightarrow (x +_1 y')' = (x +_2 y')' = x \vee y \\
&\Longleftrightarrow x' +_2 y' = x' +_1 y' = x \vee y && \text{by Proposition 2.5} \\
&\Longleftrightarrow x +_2 y = x +_1 y = x \vee y && \text{by (1) of Proposition 2.3.}
\end{aligned}
$$

□

We have the following equivalent condition for an ortholattice to be an OML.

THEOREM 2.10. *Let L be an ortholattice. Then L is an OML if and only if* $x \vee y = (x +_2 y) +_2 (x \wedge y) \quad \forall x, y \in L$.

PROOF. Assume that L is an OML. Then

$$\begin{aligned} x \vee y &= (x \wedge y) \vee ((x \wedge y)' \wedge (x \vee y)) \\ &= (x \wedge y) \vee ((x' \vee y') \wedge (x \vee y)) \\ &= (x \wedge y) \vee (x +_2 y) \\ &= (x \wedge y) +_2 (x +_2 y) \\ &= (x +_2 y) +_2 (x \wedge y). \end{aligned}$$

Conversely, assume $x \vee y = (x +_2 y) +_2 (x \wedge y)$ and $x \leq y$. Then

$$\begin{aligned} y &= x \vee y \\ &= (x +_2 y) +_2 (x \wedge y) \\ &= (x' \wedge y) +_2 x \quad \text{by (4) and (3) of Proposition 2.3 and the Remark} \\ &= (x' \wedge y) \vee x \quad \text{since } (x' \wedge y) \perp x. \end{aligned}$$

Thus $x \leq y$ implies $y = (x' \wedge y) \vee x$. □

The associativity of symmetric differences in an OML has the following property [1].

THEOREM 2.11. *An OML L is Boolean if and only if one of two symmetric differences of L is associative.*

PROOF. We know that if an OML L is Boolean then two symmetric differences in L are associative by (5) and (13) of Proposition 1.1.

Conversely, suppose that $+_1$ is associative in L. Then $x = x +_1 0 = x +_1 (y + y) = (x +_1 y) +_1 y = (((x \wedge y') \vee (x' \wedge y)) \wedge y') \vee (((x' \vee y) \wedge (x \vee y')) \wedge y) = (x \wedge y') \vee (y \wedge (x \vee y'))$ since $(x \wedge y') \, \mathbf{C} \, y$ and $(x' \wedge y) \, \mathbf{C} \, y$. Thus $x \wedge y = ((x \wedge y') \vee (y \wedge (x \vee y'))) \wedge y = (x \wedge y' \wedge y) \vee (y \wedge (x \vee y') \wedge y) = y \wedge (x \vee y')$. Hence $x = (x \wedge y') \vee (y \wedge (x \vee y')) = (x \wedge y') \vee (x \wedge y)$. This means that $x \, \mathbf{C} \, y$. Therefore L is Boolean. Similarly , we can show that if $+_2$ is associative, then $x \, \mathbf{C} \, y$. □

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