

A CONSTRUCTION OF MAXIMAL COMMUTATIVE SUBALGEBRA OF MATRIX ALGEBRAS

YOUNGKWON SONG

ABSTRACT. Let (B, m_B, k) be a maximal commutative k -subalgebra of $M_m(k)$. Then, for some element $z \in \text{Soc}(B)$, a k -algebra $R = B[X, Y]/I$, where $I = (m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$ will create an interesting maximal commutative k -subalgebra of a matrix algebra which is neither a C_1 -construction nor a C_2 -construction. This construction will also be useful to embed a maximal commutative k -subalgebra of matrix algebra to a maximal commutative k -subalgebra of a larger size matrix algebra.

1. Introduction

Let (B, m_B, k) be a maximal commutative k -subalgebra of $M_m(k)$. In this paper, we are interested in the following problem:

“How can we construct a maximal commutative k -subalgebra (R, m, k) of $M_n(k)$ for some n with $m < n$?”

In [2], Brown introduced a construction to produce a maximal commutative k -subalgebra (R, m, k) of $M_{m+1}(k)$ from (B, m_B, k) . In this paper, we will present a construction to produce a maximal commutative k -subalgebra (R, m, k) of $M_{m+2}(k)$ from (B, m_B, k) . In fact, the k -subalgebra R has dimension two more than the dimension of B . We will call this construction a C_2^2 -construction.

Moreover, we will show the C_1 -construction (C_2 -construction) does not imply the C_2^2 -construction in the next section and we can conclude that C_2^2 -construction is another construction to produce maximal commutative k -subalgebras of matrix algebra.

Recall the C_1 -construction and C_2 -construction given by Brown and Call in [1] and [2].

Received May 23, 2002.

2000 Mathematics Subject Classification: 15A27, 15A33.

Key words and phrases: C_2^2 -construction.

The present research has been conducted by the Research Grant of Kwangwoon University in 2001.

DEFINITION 1.1. Let (B, m_B, k) be a finite dimensional commutative local k -algebra with identity and N a finitely generated faithful B -module. Then $R = B \oplus N^\ell$ is a commutative k -algebra and $M = B^\ell \oplus N$ is a faithful R -module via the following multiplications:

$$\begin{aligned}\alpha(b, n_1, \dots, n_\ell) &= (\alpha b, \alpha n_1, \dots, \alpha n_\ell) \\ (b, n_1, \dots, n_\ell)(b', n'_1, \dots, n'_\ell) &= (bb', n_1b' + n'_1b, \dots, n_\ell b' + n'_\ell b) \\ (b_1, \dots, b_\ell, n)(b, n_1, \dots, n_\ell) &= (b_1b, \dots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i).\end{aligned}$$

Moreover, it is known that $R \cong \text{Hom}_R(M, M)$ via the regular representation. Thus, R is isomorphic to a maximal commutative subalgebra of $M_n(k)$, where $n = \dim_k(M)$. The k -algebra R of this form is called a C_1 -construction.

The next theorem presents an equivalent condition to be a C_1 -construction and the proof can be found in [1].

THEOREM 1.2. Let (R, m, k) be a commutative local k -algebra. Then, R is a C_1 -construction if and only if there is an ideal I satisfying the following conditions:

- (1) $\text{Ann}_R(I) = I$
- (2) $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ splits as k -algebras.

Throughout this paper, the socle of an algebra R will be denoted by $\text{Soc}(R)$ and the following theorem can be found in [2].

THEOREM 1.3. Let (B, m_B, k) be a finite dimensional commutative local k -algebra with identity and N a finitely generated faithful B -module. Suppose $B \cong \text{Hom}_B(N, N)$ via the regular representation. Then there exists an element $z \in \text{Soc}(B)$ with $\dim_k(Nz) = 1$.

The following definition can be found in [3] and is a kind of generalization of the definition of C_2 -construction in [2].

DEFINITION 1.4. Let (B, m_B, k) be a finite dimensional commutative local k -algebra with identity. If $R \cong B[X]/(m_B X, X^p - z)$ for some $z \in \text{Soc}(B) - \{0\}$ and a positive integer $p > 1$, then the k -algebra R of this form is called a C_2 -construction.

Here is an equivalent condition to be a C_2 -construction and can be found in [3].

THEOREM 1.5. Let (R, m, k) be a commutative local k -algebra. Then, R is a C_2 -construction if and only if R contains a commutative

k -subalgebra (B, m_B, k) and an element $x \in m$ satisfying the following conditions:

- (1) $0 \neq x^p \in Soc(B)$ for some positive integer $p > 1$,
- (2) $m_B x = (0)$,
- (3) $dim_k(R) = dim_k(B) + p - 1$.

2. C_2^2 -construction

In this section, we will introduce a method to produce a maximal commutative k -subalgebra (R, m, k) of $M_{m+2}(k)$ from a maximal commutative k -subalgebra (B, m_B, k) of $M_m(k)$.

THEOREM 2.1. *Let (B, m_B, k) be a finite dimensional commutative local k -algebra with identity and N a finitely generated faithful B -module. Suppose $B \cong Hom_B(N, N)$ via the regular representation. Let $R = B[X, Y]/(m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$ and let $z \in Soc(B) - \{0\}$ with $dim_k(Nz) = 1$. If we let $M = N \oplus Nz \oplus Nz$, then the k -algebra R is isomorphic to $Hom_R(M, M)$ via the regular representation. In other words, R is isomorphic to a maximal commutative k -subalgebra of $M_n(k)$, where $n = dim_k(M)$.*

Proof. Obviously, $M = N \oplus Nz \oplus Nz$ is a $B[X, Y]$ -module via the following operations:

$$\begin{aligned} (n, n_1z, n_2z)b &= (nb, n_1zb, n_2zb) \\ (n, n_1z, n_2z)X &= (n_1z, nz, n_2z^2) = (n_1z, nz, 0) \\ (n, n_1z, n_2z)Y &= (n_2z, n_1z^2, nz) = (n_2z, 0, nz) \end{aligned}$$

for all $n, n_1, n_2 \in N$ and $b \in B$. If we let x and y be the images of X and Y in R , then M is an R -module via the following operations:

$$\begin{aligned} (n, n_1z, n_2z)x &= (n_1z, nz, n_2z^2) = (n_1z, nz, 0) \\ (n, n_1z, n_2z)y &= (n_2z, n_1z^2, nz) = (n_2z, 0, nz). \end{aligned}$$

Now let

$$(n, 0, 0)(b + \alpha x + \beta y) = (0, 0, 0)$$

for $n \in N$ and $\alpha, \beta \in k$. Then

$$(nb, n\alpha z, n\beta z) = (0, 0, 0)$$

and hence by the faithfulness of N , we obtain

$$b = 0, \quad \alpha = \beta = 0,$$

which implies M is a finitely generated faithful R -module.

Let $f \in \text{Hom}_R(M, M)$ and define $g : N \rightarrow M$ and $h : M \rightarrow N$ by

$$g(n) = (n, 0, 0), \quad h(n, n_1z, n_2z) = n$$

for $n, n_1, n_2 \in N$. Then, obviously g and h are B -module homomorphisms and hence the composition map $\pi = hfg$ is a B -module homomorphism.

Since $B \cong \text{Hom}_B(N, N)$ via the regular representation, $\pi = \mu_a$ for some $a \in B$. Thus,

$$h(f(n, 0, 0)) = \pi(n) = \mu_a(n) = na.$$

This implies that there are two functions

$$\phi_1 : N \rightarrow Nz, \quad \phi_2 : N \rightarrow Nz$$

such that

$$f(n, 0, 0) = (na, \phi_1(n), \phi_2(n)).$$

Then, it is easy to show that ϕ_1 and ϕ_2 are B -module homomorphisms.

Since $\dim_k(Nz) = 1$, there exists an element $n' \in N$ such that $\{n'z\}$ is a basis of k -vector space Nz . Thus, there exist $\gamma_1, \gamma_2 \in k$ such that

$$\phi_1(n') = \gamma_1 n'z, \quad \phi_2(n') = \gamma_2 n'z.$$

Then, $a + \gamma_1x + \gamma_2y \in R$ and we want to show

$$f = \mu_{a+\gamma_1x+\gamma_2y}.$$

To prove this, it is enough to show the following two identities:

$$(1) f(0, n_1z, n_2z) = (\gamma_1n_1z + \gamma_2n_2z, n_1az, n_2az),$$

$$(2) \phi_1(n) = \gamma_1nz, \quad \phi_2(n) = \gamma_2nz.$$

It suffices to show the identity (1) for $n_1 = n', n_2 = n'$. In fact,

$$\begin{aligned} f(0, n_1z, n_2z) &= f((n_1, 0, 0)x + (n_2, 0, 0)y) \\ &= f((n_1, 0, 0))x + f((n_2, 0, 0))y \\ &= (n_1a, \phi_1(n_1), \phi_2(n_1))x + (n_2a, \phi_1(n_2), \phi_2(n_2))y \\ &= (\phi_1(n_1), n_1az, \phi_2(n_1)z) + (\phi_2(n_2), \phi_1(n_2)z, n_2az) \\ &= (\gamma_1n_1z, n_1az, 0) + (\gamma_2n_2z, 0, n_2az) \\ &= (\gamma_1n_1z + \gamma_2n_2z, n_1az, n_2az). \end{aligned}$$

Thus, identity (1) is satisfied.

For identity (2), note that

$$\begin{aligned} (\gamma_1nz, naz, 0) &= f(0, nz, 0) = f((n, 0, 0)x) = f(n, 0, 0)x \\ &= (na, \phi_1(n), \phi_2(n))x = (\phi_1(n), naz, \phi_2(n)z) \end{aligned}$$

and from these identities, we obtain

$$\phi_1(n) = \gamma_1nz.$$

Similarly, we have the following identities:

$$\begin{aligned} (\gamma_2nz, 0, naz) &= f(0, 0, nz) = f((n, 0, 0)y) = f(n, 0, 0)y \\ &= (na, \phi_1(n), \phi_2(n))y = (\phi_2(n), \phi_1(n)z, naz). \end{aligned}$$

Thus we have

$$\phi_2(n) = \gamma_2nz.$$

The identity (2) is thus satisfied and finally we obtain

$$\begin{aligned} &f(n, n_1z, n_2z) \\ &= (na + n_1\gamma_1z + n_2\gamma_2z, n_1az + n\gamma_1z, n_2az + n\gamma_2z) \\ &= (na, n_1az, n_2az) + (n_1\gamma_1z, n\gamma_1z, 0) + (n_2\gamma_2z, 0, n\gamma_2z) \\ &= (na, n_1az, n_2az) + (n\gamma_1, n_1\gamma_1z, n_2\gamma_1z)x \\ &\quad + (n\gamma_2, n_1\gamma_2z, n_2\gamma_2z)y \\ &= (n, n_1z, n_2z)(a + \gamma_1x + \gamma_2y) \\ &= \mu_{a+\gamma_1x+\gamma_2y}(n, n_1z, n_2z). \end{aligned}$$

Therefore, we have the following result:

$$f = \mu_{a+\gamma_1x+\gamma_2y}.$$

Since M is a faithful R -module, R is isomorphic to $Hom_R(M, M)$ via the regular representation and hence R is isomorphic to a maximal commutative k -subalgebra of $M_n(k)$, where $n = dim_k(M)$.

We will call the k -algebra R of the form in Theorem 2.1 a C_2^2 -construction.

If (B_1, m_{B_1}, k) is a commutative k -algebra which is isomorphic to k -algebra (B, m_B, k) . Then, B -module N is a B_1 -module via $nb_1 = n\phi(b)$, where $\phi : B \rightarrow B_1$ is an isomorphism from B to B_1 and $\phi(b) = b_1$. Thus, the following corollary can be proved. \square

COROLLARY 2.2. *Let (B, m_B, k) be a finite dimensional commutative local k -algebra with identity and N a finitely generated faithful B -module. Suppose B is isomorphic to $Hom_B(N, N)$ via the regular representation. Let (B_1, m_{B_1}, k) be a commutative k -algebra which is isomorphic to B . If we let $R = B_1[X, Y]/(m_{B_1}X, m_{B_1}Y, X^2 - z, Y^2 - z, XY)$, $z \in Soc(B_1) - \{0\}$ with $dim_k(Nz) = 1$, and $M = N \oplus Nz \oplus Nz$, then the k -algebra R is isomorphic to $Hom_R(M, M)$ via the regular representation. Thus, R is isomorphic to a maximal commutative k -subalgebra of $M_n(k)$, where $n = dim_k(M)$.*

REMARK 2.3. With the C_2^2 -construction, a maximal commutative k -subalgebra B of $M_m(k)$ with $dim_k(B) = t$ can be embedded in a maximal commutative k -subalgebra R of $M_{m+2}(k)$ with $dim_k(R) = t+2$.

Moreover, if $t < m$, then we can construct a maximal commutative k -subalgebra of matrix algebra whose dimension is not greater than the size of the matrix by applying C_2^2 -construction.

For example, the k -algebra R in [4] by Courter is a maximal commutative k -subalgebra of $M_{14}(k)$ whose dimension is 13. Now, by applying C_2^2 -construction successively, we can construct maximal commutative k -subalgebras R_1, R_2, \dots, R_t of $M_{16}(k), M_{18}(k), \dots, M_{14+2t}(k)$ for $t \in \mathbb{N}$, respectively. Here, the dimension of R_t is obviously $13 + 2t$ for each $t \in \mathbb{N}$.

Here is an example of a C_2^2 -construction. We will let E_{ij} be the (i, j) -th matrix unit.

EXAMPLE 2.4. Let B be a local k -subalgebra of $M_4(k)$ having maximal ideal consisting of the following matrices:

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix}$$

for $a, b, c \in k$. Then, B is a local maximal commutative k -subalgebra of $M_4(k)$.

Note that we can embed B into $M_6(k)$ via the following k -algebra homomorphism:

$$f \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix} + \alpha I_4 \right] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ c & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \alpha I_6.$$

If $B_1 = f(B)$, then the map $f : B \rightarrow B_1$ is a k -algebra isomorphism. Now let

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, the following identities hold:

$$m_{B_1} X = (0) = m_{B_1} Y, \quad X^2 = E_{41} = Y^2, \quad XY = 0$$

and hence

$$(m_{B_1}X, m_{B_1}Y, X^2 - E_{41}, Y^2 - E_{41}, XY) = (0).$$

Thus, the k -algebra R is given as follows:

$$R = B_1[X, Y] = B_1[X, Y]/(m_{B_1}X, m_{B_1}Y, X^2 - E_{41}, Y^2 - E_{41}, XY).$$

In fact,

$$R = \{r + \alpha I_6 \mid \alpha \in k\},$$

where r is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ c & b & a & 0 & d & e \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $a, b, c, d, e \in k$. Thus, the k -algebra R is a C_2^2 -construction. Therefore, we construct a maximal commutative k -subalgebra R which is a C_2^2 -construction and of dimension 6 from a maximal commutative k -subalgebra B of dimension 4.

REMARK 2.5. In the above Example, a maximal commutative k -subalgebra B of $M_4(k)$ is embedded to a maximal commutative k -subalgebra R of $M_6(k)$.

From the definition of C_2^2 -construction, the following property can be obtained.

THEOREM 2.6. *Let (R, m, k) be a finite dimensional local commutative k -algebra. Then, R is a C_2^2 -construction if and only if there exist a commutative k -subalgebra (B, m_B, k) and elements $x, y \in m$ satisfying the following properties:*

- (1) $x^2 = y^2 \in Soc(B) - \{0\}$,
- (2) $xy = 0$,
- (3) $m_Bx = (0) = m_By$,
- (4) $dim_k(R) = dim_k(B) + 2$.

Proof. Suppose R is a C_2^2 -construction. Then, by the definition of C_2^2 -construction, there exist a finite dimensional local commutative k -algebra (B, m_B, k) and a finitely generated faithful B -module N such that

$$R = B[X, Y]/(m_BX, m_BY, X^2 - z, Y^2 - z, XY)$$

for some $z \in \text{Soc}(B) - \{0\}$ with $\dim_k(Nz) = 1$. Let x and y be the image of X and Y , respectively. Then, the conditions (1),(2),(3) and (4) can be shown by straightforward calculations. Conversely, suppose there exist a k -subalgebra B and elements x and y in m such that the given conditions are satisfied. Let $x^2 = y^2 = z \in \text{Soc}(B)$ and define a map

$$\psi : B[X, Y]/(m_B X, m_B Y, X^2 - z, Y^2 - z, XY) \longrightarrow R$$

by

$$\psi(b + I) = b, \quad \psi(X + I) = x, \quad \psi(Y + I) = y,$$

where $b \in B$ and $I = (m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$. Then, obviously ψ is a k -algebra homomorphism. Suppose $b + cX + dY + I \in \ker \psi$. Then, we have

$$\psi(b + cX + dY + I) = b + cx + dy = 0.$$

Here, we may assume $c, d \in k$ since $m_B x = m_B y = (0)$. If $b \neq 0$, then $b \notin m_B$. For, if $b \in m_B$, then

$$cz = bx + cx^2 + dxy = 0, \quad dz = by + cxy + dy^2 = 0$$

and $c = 0 = d$. This implies $b = 0$ which is impossible. Thus, $b \notin m_B$ and the element $b + cx + dy$ is a unit which is impossible. Thus, $b = 0$ and $cx + dy = 0$. If $c \neq 0$, then c^{-1} exists and hence $x + (c^{-1}d)y = 0$. By multiplying by x each side, we get

$$0 = x^2 + (c^{-1}d)xy = z.$$

This is also impossible and we should have $c = 0$. Finally we have $dy = 0$ and we can show $d = 0$ from the identity $dz = dy^2 = 0$. Thus,

$$b = c = d = 0$$

and this implies $b + cX + dY + I = I$ and ψ is a monomorphism. Since $\dim_k(\text{im}(\psi)) = \dim_k(B[x, y])$ and $\dim_k(R) = \dim_k(B) + 2$, the map ψ should be an isomorphism and we can conclude the k -algebra R is a C_2^2 -construction. \square

Now, in the rest of this paper, we want to prove C_i -construction does not imply C_2^2 -construction for each $i = 1, 2$.

COROLLARY 2.7. C_1 -construction does not imply C_2^2 -construction.

Proof. Let (R, m, k) be a Schur algebra of size 4. That is, the element $r \in R$ is of the following form:

$$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha I_4$$

for some $a, b, c, d \in k$. It is known in [1] that the Schur algebra of size 4 is a C_1 -construction. Note that the index of m is 2 and hence by Theorem 2.6, the k -algebra R can't be a C_2^2 -construction. For, if R is a C_2^2 -construction, then there exist elements x and y in m whose squares are not zero. But, this is impossible since the index of m is 2. \square

COROLLARY 2.8. C_2 -construction does not imply C_2^2 -construction.

Proof. Let $R = m \oplus kI_4$ be a maximal commutative k -subalgebra of $M_4(k)$ such that each element $r \in m$ is of the following form:

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix}$$

for some $a, b, c \in k$. If we let $B = k[E_{41}]$ and $s = E_{21} + E_{32} + E_{43}$, then

- (1) $s^3 \in Soc(B) - \{0\}$
- (2) $m_B s = (0)$
- (3) $dim_k(R) = dim_k(B) + 2$.

This implies R is a C_2 -construction.

Now, suppose R is a C_2^2 -construction. Then R contains a k -subalgebra B and an element $x \in m$ satisfying the following conditions:

$$x^2 \in Soc(B) - \{0\}, \quad m_B x = (0).$$

If we let

$$u = E_{21} + E_{32} + E_{43}, \quad v = E_{31} + E_{42}, \quad w = E_{41},$$

then $x = au + bv + cw$ for some $a, b, c \in k$. Note that

$$x^2 = a^2v + abw \in Soc(B) \subseteq m_B.$$

Since $m_B x = (0)$, we obtain $a = 0$ from the identities

$$a^3 E_{41} = x^3 = 0.$$

Therefore, $x = bv + cw$ and $x^2 = 0$ which is impossible since $x^2 \in Soc(B) - \{0\}$. Now, we can conclude that the k -algebra R is not a C_2^2 -construction. \square

REMARK 2.9. Example 2.4 and the proof of Corollary 2.8 show that a C_2^2 -construction can be constructed from a maximal commutative k -subalgebra that is not a C_2^2 -construction.

References

- [1] W. C. Brown and F. W. Call, *Maximal Commutative Subalgebras of $n \times n$ Matrices*, Comm. Algebra **59** (1993), no. 12, 4439–4460.
- [2] W. C. Brown, *Two Constructions of Maximal Commutative Subalgebras of $n \times n$ Matrices*, Comm. Algebra **22** (1994), no. 10, 4051–4066.
- [3] ———, *Constructing Maximal Commutative Subalgebras of Matrix Rings in Small Dimensions*, Comm. Algebra, **25** (1997), no. 12, 3923–3946.
- [4] R. C. Courter, *The Dimension of Maximal Commutative Subalgebras of K_n* , Duke Math. J. **32** (1965), 225–232.
- [5] M. Gerstenhaber, *On Dominance and Varieties of Commuting Matrices*, Ann. of Math. **73** (1961), no. 2, 324–348.
- [6] N. Jacobson, *Schur's Theorem on Commutative Matrices*, Bull. of the Amer. Math. Soc. **50** (1944), 431–436.
- [7] Y. Song, *On the Maximal Commutative Subalgebras of 14 by 14 Matrices*, Comm. Algebra **25** (1997), no. 12, 3823–3840.
- [8] ———, *Maximal Commutative Subalgebras of Matrix Algebras*, Comm. Algebra **27** (1999), no. 4, 1649–1663.
- [9] ———, *Notes on the Constructions of Maximal Commutative Subalgebra of $M_n(k)$* , Comm. Algebra **29** (2001), no. 10, 4333–4339.

Department of Mathematics
Research Institute of Basic Science
Kwangwoon University
Seoul 139–701, Korea
E-mail: yksong@kw.ac.kr