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THE HILBERT-KUNZ MULTIPLICITY OF TWO-DIMENSIONAL TORIC RINGS

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ABSTRACT. Recently, K. Watanabe showed that the Hilbert-Kunz multiplicity of a toric ring is a rational number. In this paper we give an explicit formula to compute the Hilbert-Kunz multiplicity of two-dimensional toric rings. This formula also shows that the Hilbert-Kunz multiplicity of a two-dimensional non-regular toric ring is at least 3/2.

1. Introduction

Every ring in this paper is assumed to be commutative and Noetherian.

Let $(A, \mathbf{m})$ be a d-dimensional local ring with maximal ideal $\mathbf{m}$, I an $\mathbf{m}$-primary ideal and M a finitely generated A-module. Then the length of M/I^nM can be expressed for $n >> 0$ as a polynomial in n with rational coefficients and degree equal to dim$M$, therefore at most d. So we can write

$$l(M/I^nM) = e_0\binom{n+d}{d} + e_1\binom{n+d-1}{d-1} + \cdots + e_d,\ e_i \in \mathbb{Z},\ n >> 0,$$

where l denotes the length. Then $e_0 = e(I, M)$ is called *the multiplicity of M with respect to I.* Hence

$$e(I, M) = d! \cdot \lim_{n\to\infty} \frac{l(M/I^nM)}{n^d}.$$

Note that $e(I, M) > 0$ if and only if $\dim M = \dim A$.

By definition *the multiplicity of I*, $e(I)$ is $e(I, A)$ and *the multiplicity of A*, $e(A)$ is $e(\mathbf{m}, A)$.

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The notion of Hilbert-Kunz multiplicity was defined implicitly by Kunz([3]) using the Frobenius morphism in characteristic $p > 0$ and it was formulated explicitly by Monsky([4]).

DEFINITION 1.1. (Monsky, [4]) Let $(A, \mathbf{m})$ be a d-dimensional local ring of characteristic $p > 0$, I an $\mathbf{m}$-primary ideal of A. Then *the Hilbert-Kunz multiplicity, $e_{HK}(I, A)$ of I* is

$$e_{HK}(I, A) := \lim_{e \to \infty} \frac{l_A(A/I^{[p^e]})}{p^{de}},$$

where $I^{[q]}$ $(q = p^e)$ is the ideal generated by the q-th powers of all elements of I .

By definition *the Hilbert-Kunz multiplicity of A, $e_{HK}(A)$* is $e_{HK}(\mathbf{m}, A)$.

LEMMA 1.2. (Huneke, [2]) *Let $(A, \mathbf{m})$ be a local ring of characteristic $p > 0$. Set $d = \dim A$, and let I an $\mathbf{m}$-primary ideal. Then*

$$\frac{e(I)}{d!} \leq e_{HK}(I) \leq e(I).$$

As an immediate consequence of this we have the following.

COROLLARY 1.3. *Let $(A, \mathbf{m})$ be a local ring of characteristic $p > 0$, and I an $\mathbf{m}$-primary ideal. If $\dim A = 1$, then $e(I) = e_{HK}(I)$. In particular, the Hilbert-Kunz multiplicity exists and is an integer.*

In general, the Hilbert-Kunz multiplicity exists and is a real number([2], [4]). However, it remains open whether it is a rational number or not. This multiplicity has many nice properties as usual multiplicity and is proved to be more sensitive than the usual one. For example, the Hilbert-Kunz multiplicity of 2-dimensional F-rational double point has been calculated explicitly, and their values give more information than the values of usual one.

THEOREM 1.4. ([7], Theorem 5.4) *Let A be a 2-dimensional Cohen-Macaulay local ring of characteristic $p > 0$. Then $1 < e_{HK}(A) < 2$ if and only if A is an F-rational double point. In this case, $e_{HK}(A) = 2 - 1/|G|$, where G is the finite subgroup of SL(2, k) attached to the corresponding singularity in characteristic 0.*

Usually, the Hilbert-Kunz multiplicity is very difficult to compute and has been calculated for few cases. However, the Hilbert-Kunz multiplicity is also known to be a rational number in the following cases.

REMARK 1.5. Let $(A, \mathbf{m})$ be a local ring of characteristic $p > 0$, and I an $\mathbf{m}$-primary ideal.

(1) If A has a regular overring B which is a finite A-module, then $r \cdot e_{HK}(I) \in \mathbb{Z}$ where $\mathrm{rank}_A B = r$ ([7]).
(2) If A is a Cohen-Macaulay ring and has finite Cohen-Macaulay type. That is, if the number of the isomorphism classes of indecomposable maximal Cohen-Macaulay module is finite. Then $e_{HK}(I)$ is a rational number ([5]).

In this paper, we develop a computational method suggested in [6, Theorem 2.1] and derive a formula for computing the Hilbert-Kunz multiplicity of two-dimensional toric rings. As a result of this, the smallest value of the Hilbert-Kunz multiplicity of non-regular 2-dimensional toric rings is sharply 3/2.

2. Two-dimensional toric rings

Let $H \subset \mathbb{Z}^n$ be a finitely generated additive subsemigroup of $\mathbb{Z}^n$. We always assume that $0 \in H$ and $H \cap -H = \{0\}$.

Let k be a fixed ground field of characteristic $p > 0$ and we put

$$k[H] = k[t^h | h \in H] \subset k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}],$$

where we denote $t^h = t_1^{h_1} \cdots t_n^{h_n}$ for $h = (h_1, \ldots, h_n) \in H$.

We denote M the subgroup of $\mathbb{Z}^n$ generated by H. We say that H is *normal* if $nh \in H$ for some positive integer n and $h \in M$ then $h \in H$. It is known that H is normal if and only if $k[H]$ is normal ([1]).

Recently, Watanabe has proved the Hilbert-Kunz multiplicity of a toric (normal semigroup) ring is a rational number.

THEOREM 2.1. (Watanabe, [6]) *Let $k[H]$ be a normal semigroup ring as above and A be the local ring of $k[H]$ at the maximal ideal $\mathbf{m} = \{t^h | h \in H, h \neq 0\}$ and I be a monomial $\mathbf{m}$-primary ideal of A. Then $e_{HK}(I) \in \mathbb{Q}$.*

In the proof of the above theorem, $e_{HK}(I)$ is expressed as a finite sum of the products, where each product is a multiplication of the number of generators of a module and the volume of a subregion of the unit cube $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_i \leq 1 \text{ for } i = 1, \ldots, d\}$. As the subregion is defined by linear inequalities with integer coefficients, its volume is a rational number. Consequently, $e_{HK}(I)$ is a rational number.

Now we focus on two dimensional toric rings and their Hilbert-Kunz multiplicity.

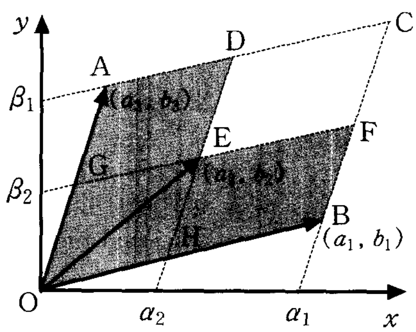
Let H be a subsemigroup of $\mathbb{Z}^2$ generated by $(a_1, b_1), \ldots, (a_n, b_n)$ with $a_1 > a_2 > \cdots > a_n$. Assume that H is normal and $H \cap -H = \{0\}$. If $A = k[s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, \ldots, s^{a_n}t^{b_n}]_{\mathbf{m}}$ with $\mathbf{m} = (s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, \ldots, s^{a_n}t^{b_n})$, then $e_{HK}(A)$ is the area of $2n$-gon Δ where Δ satisfies the following:

1. The $n+1$ points, O, $P_1(a_1, b_1)$, $\ldots$, $P_n(a_n, b_n)$ are vertices of Δ.
2. Each side of Δ is parallel with either $\overrightarrow{OP_1}$ or $\overrightarrow{OP_n}$.

THEOREM 2.2. *Let k be a field of characteristic $p > 0$ and let $H = \langle (a_1, b_1), (a_2, b_2), (a_3, b_3) \rangle$ in $\mathbb{Z}^2$ be normal with $a_1 > a_2 > a_3$. If $A = k[s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, s^{a_3}t^{b_3}]_{\mathbf{m}}$ with $\mathbf{m} = (s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, s^{a_3}t^{b_3})$, then $e_{HK}(A)$ is equal to*

$$\frac{(a_1b_3 - a_3b_1)^2 - \{(a_1b_3 - a_3b_1 - a_2b_3 + a_3b_2)(a_1b_3 - a_3b_1 - a_1b_2 + a_2b_1)\}}{a_1b_3 - a_3b_1}.$$

Proof. Let $\overrightarrow{OA} = (a_3, b_3)$, $\overrightarrow{OE} = (a_2, b_2)$, $\overrightarrow{OB} = (a_1, b_1)$. Then $e_{HK}(A)$ is the area of $OADEFB$. Consider the parallelograms $OACB$, $OGEH$ and compute the x-intercepts α_1, α_2 and the y-intercepts β_1, β_2 of the straight lines in the diagram.



Since the line segments $\overline{OA}$, $\overline{HD}$ and $\overline{BC}$ are parallel we have the following,

$$\alpha_1 = a_1 - \frac{a_3}{b_3}b_1 = \frac{a_1b_3 - a_3b_1}{b_3},$$
$$\alpha_2 = a_2 - \frac{a_3}{b_3}b_2 = \frac{a_2b_3 - a_3b_2}{b_3}.$$

Similarly, we calculate that

$$\beta_1 = \frac{a_1b_3 - a_3b_1}{a_1}, \quad \beta_2 = \frac{a_1b_2 - a_2b_1}{a_1}.$$

Let $B'(\alpha_1, 0)$, $H'(\alpha_2, 0)$ and consider the similar triangles OBB', OHH'. Then

$$\frac{|\overrightarrow{HB}|}{|\overrightarrow{OB}|} = \frac{\alpha_1 - \alpha_2}{\alpha_1}.$$

Also

$$\frac{|\overrightarrow{AG}|}{|\overrightarrow{OA}|} = \frac{\beta_1 - \beta_2}{\beta_1}.$$

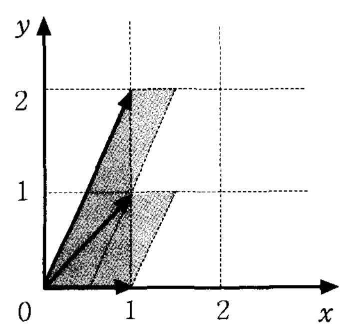
Now the area of parallelogram $DEFC$ is Suv, where S is the area of parallelogram $OACB$. Therefore

$$\begin{aligned} & e_{HK}(A) \\ = \; & S(1 - uv) \\ = \; & (a_1 b_3 - a_3 b_1)\left\{1 - \frac{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{\alpha_1 \beta_1}\right\} \\ = \; & \frac{(a_1 b_3 - a_3 b_1)^2}{a_1 b_3 - a_3 b_1} \\ & - \frac{(a_1 b_3 - a_3 b_1 - a_2 b_3 + a_3 b_2)(a_1 b_3 - a_3 b_1 - a_1 b_2 + a_2 b_1)}{a_1 b_3 - a_3 b_1}. \end{aligned}$$

□

The formula in the above theorem shows that the Hilbert-Kunz multiplicity of a toric ring is not an integer in general. Also there is a non-regular local ring whose Hilbert-Kunz multiplicity is less than 2.

EXAMPLE 2.3. Suppose that $A = k[s, st, st^2]_{\mathbf{m}}$ with $\mathbf{m} = (s, st, st^2)$. Then the area of the hexagon determined by the 3 vectors $(1, 0)$, $(1, 1)$ and $(1, 2)$ is $3/2$ as below. Hence $e_{HK}(A) = 3/2$.



THEOREM 2.4. *Let k be a field of characteristic $p > 0$ and $H = \langle(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\rangle \subset \mathbb{Z}^2$ be normal with $a_1 > a_2 > \cdots > a_n$. If $A = k[s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, \dots, s^{a_n}t^{b_n}]_{\mathbf{m}}$ with $\mathbf{m} = (s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, \dots, s^{a_n}t^{b_n})$ for $n \geq 3$, then $e_{HK}(A)$ is equal to*

$$(a_1b_n - a_nb_1) - \sum_{i=1}^{n-2} \frac{(a_ib_n - a_nb_1 - a_{i+1}b_n + a_nb_{i+1})(a_1b_n - a_nb_1 - a_1b_{i+1} + a_{i+1}b_1)}{(a_ib_n - a_nb_i)}.$$

Proof. Consider the points $P_1(a_1, b_1)$, $P_2(a_2, b_2), \dots, P_n(a_n, b_n)$ and draw the straight lines that are through these points and parallel with $\overrightarrow{OP_1}$ or $\overrightarrow{OP_n}$. Call the x-intercepts of the straight lines $\alpha_1, \dots, \alpha_{n-1}$ and the y-intercepts $\beta_1, \dots, \beta_{n-1}$ with $\alpha_1 > \cdots > \alpha_{n-1}$ and $\beta_1 > \cdots > \beta_{n-1}$. Then

$$\alpha_i = \frac{a_ib_n - a_nb_i}{b_n}, \quad \beta_i = \frac{a_1b_{n-i+1} - a_{n-i+1}b_1}{a_1}.$$

Since $e_{HK}(A)$ is the area of $2n$-gon having $P_1, \dots, P_n$ as vertices and each side parallel with either $\overrightarrow{OP_1}$ or $\overrightarrow{OP_n}$,

$$e_{HK}(A) = (a_1b_n - a_nb_1)\{1 - (u_1v_1 + \cdots + u_{n-2}v_{n-2})\},$$

where $u_i = \dfrac{\alpha_i - \alpha_{i+1}}{\alpha_i}$ and $v_i = \dfrac{\beta_1 - \beta_{n-i}}{\beta_1}$.

Now substitute for u_i, v_i and α_i, β_i. Then

$$e_{HK}(A) = (a_1b_n - a_nb_1) - \sum_{i=1}^{n-2} \frac{(a_ib_n - a_nb_1 - a_{i+1}b_n + a_nb_{i+1})(a_1b_n - a_nb_1 - a_1b_{i+1} + a_{i+1}b_1)}{(a_ib_n - a_nb_i)}.$$

□

In the following example we calculate the Hilbert-Kunz multiplicity directly from the area of Δ or by using the formula in the above theorem.

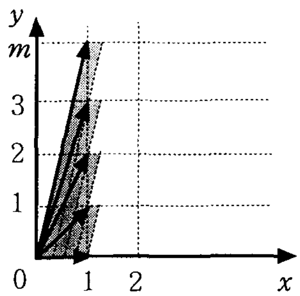
EXAMPLE 2.5.

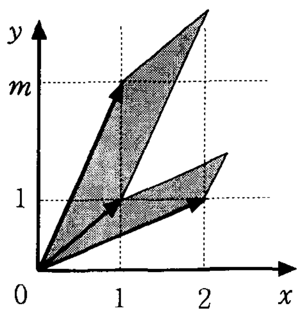
(1) Let $A = k[s, st, st^2, st^3, \dots, st^m]_{\mathbf{m}}$ with $\mathbf{m} = (s, st, st^2, \dots, st^m)$. Then the area in the diagram shows that

$$e_{HK}(A) = \frac{m}{2} + \frac{1}{2m}m = \frac{m+1}{2}.$$

(2) Let $A = k[s^2t, st, st^m]_{\mathbf{m}}$ with $\mathbf{m} = (s^2t, st, st^m)$. Then

$$e_{HK}(A) = \frac{(2m-1)^2 - 2m(m-1)}{2m-1} = \frac{2m^2 - 2m + 1}{2m-1}.$$





If A is unmixed, then $e_{HK}(A) = 1$ if and only if A is regular[8]. Also in [7, Question 1.2] Watanabe asked what is the minimal value of $e_{HK}(A) > 1$ in dimension d? If $d = 2$, then the smallest value after 1 is $3/2$ [7]. In the following theorem, it is shown that this holds for two-dimensional toric rings.

THEOREM 2.6. *Let A be a two-dimensional non-regular toric ring, then the smallest value of $e_{HK}(A)$ is $3/2$.*

Proof. Note that $e_{HK}(A) = 3/2$ for $A = k[s, st, st^2]_{\mathbf{m}}$ with $\mathbf{m} = (s, st, st^2)$ [Example 2.3].

Let $H = \langle (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \rangle \subset \mathbb{Z}^2$ be normal with $a_1 > a_2 > \cdots > a_n$ and $A = k[H]_{\mathbf{m}}$ with $\mathbf{m} = (s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, \ldots, s^{a_n}t^{b_n})$. Since A is not regular $n \geq 3$. Put $b_1 = 0$ (note that the rotation of axis does not change the area). To obtain the minimal area we may assume that $a_1 = 1$. That is, $(a_1, b_1) = (1, 0)$. Since H is normal we have $b_1 < b_2 < \cdots < b_n$.

If $b_2 \geq 2$, then

$$e_{HK}(A) > a_1 b_2 - a_2 b_1 \geq 2.$$

If $b_2 = 1$, then

$$e_{HK}(A) \geq (a_1 b_2 - a_2 b_1) + \sum_{i=1}^{b_n - 1} \frac{b_n - i}{b_n} \geq 1 + \frac{b_n - 1}{2} \geq \frac{3}{2}.$$

This finishes the proof of the theorem. □

The proof of Theorem 2.6 suggests more than the smallest value 3/2. That is, if a normal subsemigroup H is minimally generated by $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ and $0 \leq b_1 < b_2 < \cdots < b_n$, then

$$e_{HK}(A) \geq 1 + \frac{b_n - 1}{2} \geq \frac{n}{2}.$$

COROLLARY 2.7. *Let H be a normal subsemigroup of $\mathbb{Z}^2$ minimally generated by n vectors and $A = k[H]_m$. Then the smallest value of $e_{HK}(A)$ is $\frac{n}{2}$.*

Proof. Note that $e_{HK}(A) \geq \frac{n}{2}$ as above.

Let $A = k[s, st, st^2, st^3, \ldots, st^{n-1}]_{\mathbf{m}}$ as in Example 2.5 (1), then

$$e_{HK}(A) = \frac{n}{2}.$$

□

It has been suggested that the minimum value of the Hilbert-Kunz multiplicity is a rational function of the characteristic p. However, Watanabe's proof shows that the value of the Hilbert-Kunz multiplicity of a Toric ring does *not* depend on the characteristic. Also the Hilbert-Kunz multiplicity of a semigroup ring(whether it is normal or not) is always a rational number. Finally, we ask the following questions.

QUESTION 2.8. (1) Find a rational number that is Hilbert-Kunz multiplicity of 3-dimen-sional toric ring but is not Hilbert-Kunz multiplicity of 2-dimensional toric ring.

(2) Is it true that for any rational number $n/m \geq 3/2$, there is a toric ring A such that $e_{HK}(A) = n/m$?

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