THE BOUNDEDNESS OF SOLUTIONS FOR STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. We consider the stochastic differential inclusion of the form $dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt$, where σ, b are set-valued maps, B is a standard Brownian motion. We prove the boundedness of solutions under the assumption that σ and b satisfy the local Lipschitz property and linear growth.

1. Introduction

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathfrak{F}_t)_{t \geq 0}$ of sub σ–fields of $\mathfrak{F}$ each containing all P-null sets. Let $B = (B_t)_{t \geq 0}$ be an r-dimensional $(\mathfrak{F}_t)$-Brownian motion. We consider the following stochastic differential inclusion.

$$(1.1) \qquad dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt,$$

where $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d \otimes \mathbb{R}^r)$, $b : [0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ are set-valued maps. In recent years the study of the existence and properties of solution for these stochastic differential inclusions have been developed by many authors ([1], [5]). Furthermore the results for the viable solutions have been made ([3], [7]). For the stochastic differential equation associated with (1.1), many results for the existence, uniqueness and properties of solutions have been done under various conditions that σ and b are continuous and bounded or Lipschitzean or Hölder continuous ([4]). We proved the existence of solution for stochastic differential inclusion (1.1) under the condition that σ and b satisfy the local Lipschitz property and linear growth ([6]).

In this paper, we prove any solution for stochastic differential inclusion (1.1) is bounded.

2. Preliminaries

We prepare the definition of solution for stochastic differential inclusion and some results for the stochastic differential equation and selection theorems.

DEFINITION 2.1. An r-dimensional continuous process $B = (B_t)_{t \in [0,\infty)}$ is called an r-dimensional $(\mathfrak{F}_t)$-Brownian motion if it is $(\mathfrak{F}_t)$-adapted and satisfies

$$E[\exp[i < \xi, B_t - B_s >] \mid \mathfrak{F}_s] = \exp[-(t - s)|\xi|^2/2], \quad \text{a.s.}$$
$$\text{for every } \xi \in \mathbb{R}^r \text{ and } 0 \le s < t.$$

Let us consider the stochastic differential inclusion

$$dX_t \in \sigma(t, X_t)dB_t + b(t, X_t)dt$$

with the initial value $X_0 = x$, where $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d \otimes \mathbb{R}^r)$, $b : [0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ are set-valued maps and x is a $\mathbb{R}^d$-valued $\mathfrak{F}_0$-measurable function.

DEFINITION 2.2. A predictable continuous stochastic process $X = \{X_t, \ t \in [0, T]\}$ is said to be a solution of (1.1) on $[0, T]$ with the initial condition x_0 if there are predictable random processes $f : \Omega \times [0, T] \to \mathbb{R}^d \otimes \mathbb{R}^r$, $g : \Omega \times [0, T] \to \mathbb{R}^d$ such that $f(t) \in \sigma(t, X_t)$, $g(t) \in b(t, X_t)$ for every $t \in [0, T]$ almost surely and

$$X_t = x + \int_0^t f(s) \, dB_s + \int_0^t g(s) \, ds.$$

For the stochastic differential equation

$$(2.1) \qquad X_t = x + \int_0^t \sigma_1(s, X_s)dB_s + \int_0^t b_1(s, X_s)ds,$$

where $\sigma_1 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$, $b_1 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ are $\mathfrak{B}([0, T]) \otimes \mathfrak{B}(\mathbb{R}^d)$-measurable and x is $\mathfrak{F}_0$-measurable, the following theorems are well known.

THEOREM 2.3. ([4]) *If σ_1 and b_1 are continuous and satisfy the linear growth condition*

$$||\sigma_1(t,x)||^2 + |b_1(t,x)|^2 \le K(1+|x|^2),$$

for some positive constant K, then (2.1) has a solution on $[0,T]$.

THEOREM 2.4. ([4]) *Assume that for each $N > 0$, there exists a constant $C_N > 0$ such that*

$$||\sigma_1(t,x) - \sigma_1(t,y)||^2 + |b_1(t,x) - b_1(t,y)|^2 \le C_N \cdot |x-y|, \quad x,y \in B_N,$$

where $B_N = \{x \in \mathbb{R}^d, |x| \le N\}$ and $||\sigma_1||^2 = \sum_{j=1}^{r} \sum_{i=1}^{d} |(\sigma_1)_j^i|^2 \equiv tr(\sigma_1\sigma_1^*)$. Then (2.1) has a unique solution X_t up to explosion time.

3. Main results

For a Banach space X with the norm $||\cdot||$ and for non-empty sets A, A' in X, we denote $||A|| = \sup\{||a|| \mid a \in A\}$, $d(a,A') = \inf\{d(a,a') \mid a' \in A'\}$, $d(A,A') = \sup\{d(a,A') \mid a \in A\}$ and $d_H(A,A') = \max\{d(A,A'), d(A',A)\}$, a Hausdorff metric. Given a family of sets $\{F_\alpha \mid \alpha \in A\}$, a selection is a map $\alpha \to f_\alpha$ in F_α. The most famous continuous selection theorem is the following result by Michael.

THEOREM 3.1. ([2]) *Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semi-continuous. Then there exists $f : X \to Y$, a continuous selection from F.*

If σ and b satisfy the linear growth condition, then clearly any selections from σ and b satisfy the linear growth condition. Thus by Theorem 2.3 and Theorem 3.1, (1.1) has a solution if σ and b are closed convex valued lower semi-continuous satisfying the linear growth property.

THEOREM 3.2. *Assume that σ and b are lower semi-continuous with closed convex values satisfying the linear growth condition*

$$||\sigma(t,x)||^2 + |b(t,x)|^2 \le K(1+|x|^2),$$

for some constant K. Then (1.1) has a solution on $[0,T]$.

The existence of solution to (1.1) can be proved also under Lipschitz condition using the fixed point theorem.

THEOREM 3.3. ([6]) *Assume that* $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d \otimes \mathbb{R}^r)$, $b :$
$[0, T] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ *are closed convex set-valued functions which are*
Lipschitz, i.e., there exists a constant $L > 0$ *such that*

$$\begin{cases} d_H(\sigma(t, x), \sigma(t, y)) \le L \cdot |x - y|, \\ d_H(b(t, x), b(t, y)) \le L \cdot |x - y|. \end{cases}$$

Then there exists a solution for the stochastic differential inclusion
(1.1).

For the proof of the existence of solution for stochastic differential
inclusion (1.1) under the local Lipschitz condition, we prepare the local
Lipschitz barycentric selection based on that proposed by Aubin[2]. Let
$A \subset \mathbb{R}^n$ be a compact convex body, i.e., a compact set with nonempty in-
terior, and let m_n be an n-dimensional Lebesgue measure. Since $m_n(A)$
is positive, we can define the barycenter of A as

$$b(A) = \frac{1}{m_n(A)} \int_A x \, dm_n.$$

Then the barycenters of A and $A^1 = A + B$ belong to A, where B is the
closed unit ball in $\mathbb{R}^n$ if A be a compact and convex set ([2]). Using this,
we have the following local Lipschitz barycentric selection theorem.

PROPOSITION 3.4. ([6]) *Let* $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ *be a local Lipschitz*
set-valued map with compact convex images, i.e., there exists a constant
$K_N > 0$ *such that*

$$d_H(F(x), F(y)) \le K_N \cdot |x - y|, \quad \forall x, y \in B_N = \{x \in \mathbb{R}^n, |x| \le N\}.$$

Assume moreover that there exists a constant $C > 0$ *such that* $||F(x)|| \le$
$C \cdot (1 + |x|)$, *for every* $x \in \mathbb{R}^n$. *Then there exist a constant* $\hat{C}_N > 0$ *and*
a single valued map $f : \mathbb{R}^n \to \mathbb{R}^n$, *local Lipschitzean with constant* $\hat{C}_N$,
a selection from F.

Proof. Since the single valued map $b^1 : x \to b(F(x) + B)$ is a selection
of F, we have to prove that it is a local Lipschitzean selection.

Fix $x, y \in B_N$. Call $\Phi(x) = F(x) + B$ and $\Phi(y) = F(y) + B$. Since
$||\Phi(x)|| \le ||F(x) + B|| \le ||F(x)|| + 1 \le C \cdot (1 + |x|) + 1 \le C \cdot (1 + N) + 1 =$

$C_{N'}$ and $m_n(\Phi(x)) \leq C_{N''}$, we have

$$\frac{1}{m_n(\Phi(x))} \int_{\Phi(x)} x \, dm_n - \frac{1}{m_n(\Phi(y))} \int_{\Phi(y)} x \, dm_n$$

$$\leq \left| \left(\frac{1}{m_n(\Phi(x))} - \frac{1}{m_n(\Phi(y))} \right) \int_{\Phi(x) \cap \Phi(y)} x \, dm_n \right|$$

$$+ \left| \frac{1}{m_n(\Phi(x))} \int_{\Phi(x) \backslash \Phi(y)} x \, dm_n \right.$$

$$\left. - \frac{1}{m_n(\Phi(y))} \int_{\Phi(y) \backslash \Phi(x)} x \, dm_n \right|$$

$$\leq |m_n(\Phi(x)) - m_n(\Phi(y))| \cdot C_{N'} \cdot C_{N''} / (m_n(B))^2$$

$$+ \{ m_n(\Phi(x) \backslash \Phi(y)) + m_n(\Phi(y) \backslash \Phi(x)) \}$$

$$\cdot C_{N'} \cdot C_{N''} / m_n(B).$$

The above can be estimated in terms of $d_H(\Phi(x), \Phi(y))$(see [2]). Thus we have

$$|b^1(x) - b^1(y)| \leq C \cdot d_H(F(x), F(y))$$

$$\leq C \cdot K_N \cdot |x - y| = \hat{C}_N \cdot |x - y|,$$

for some constants $C, \hat{C}_N > 0$, i.e., $f = b^1$ is the required local Lipschitzean selection. $\square$

By Theorem 2.4 and Proposition 3.4, we have the following theorem.

THEOREM 3.5. ([6]) *Assume that for each $N > 0$, there exist constants $C > 0$ and $C_N > 0$ such that*

$$\begin{cases} d_H(\sigma(t,x), \sigma(t,y)) \leq C_N \cdot |x - y|, & x, y \in B_N, \\ d_H(b(t,x), b(t,y)) \leq C_N \cdot |x - y|, & x, y \in B_N, \\ \|\sigma(t,x)\| + |b(t,x)| \leq C \cdot (1 + |x|), & x \in \mathbb{R}^n, \end{cases}$$

where $B_N = \{ x \in \mathbb{R}^d, |x| \leq N \}$. Then (1.1) has a solution X_t.

Furthermore, we have the following main theorem for boundedness of solutions.

THEOREM 3.6. *Let X_t be any solution of (1.1). Then X_t is bounded, i.e., $E[\sup_{0 < s < t} |X_s|^p] < \infty$ for $p \geq 2$.*

Proof. Let X_t be a solution. Then there exist $f_s \in \sigma(X_s)$ and $g_s \in b(X_s)$ such that

$$X_t = x + \int_0^t f_s dB_s + \int_0^t g_s ds.$$

Since

$$E[\sup_{0 \leq s \leq t} |X_s|^p] \leq 3^{p-1}|x|^p + 3^{p-1}C_1 E \left[\{ \int_0^t |f_s|^2 ds \}^{p/2} \right]$$

$$+ 3^{p-1} E \left[\{ \int_0^t |g_s|^2 ds \}^p \right]$$

$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1 E \left[\{ \int_0^t |f_s|^p ds \} \{ \int_0^t 1 ds \}^{\frac{p-2}{2}} \right]$$

$$+ 3^{p-1} E \left[\int_0^t |g_s|^p ds \{ \int_0^t 1 ds \}^{p-1} \right]$$

$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1 T^{\frac{p-2}{2}} \int_0^t E[|f_s|^p] ds$$

$$+ 3^{p-1} T^{p-1} \int_0^t E[|g_s|^p] ds$$

$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1 T^{\frac{p-2}{2}} \int_0^t E[|\sigma(X_s)|^p] ds$$

$$+ 3^{p-1} T^{p-1} \int_0^t E[|b(X_s)|^p] ds$$

$$\leq 3^{p-1}|x|^p + 3^{p-1}C_1 T^{\frac{p-2}{2}} \int_0^t K^p(1 + E[|X_s|^p]) 2^{p-1} ds$$

$$+ 3^{p-1} T^{p-1} \int_0^t K^p(1 + E[|X_s|^p]) 2^{p-1} ds,$$

if we put $\varphi(t) = E[\sup_{0 \leq s \leq t} |X_s|^p]$,

$$\varphi(t) \leq 3^{p-1}|x|^p + 6^{p-1}K^p T^{\frac{p}{2}} C_1 + 6^{p-1}K^p T^{\frac{p-2}{2}} C_1 \int_0^t \varphi(s) ds$$

$$+ 6^{p-1}K^p T^p + 6^{p-1}K^p T^{p-1} \int_0^t \varphi(s) ds$$

$$= 3^{p-1}|x|^p + 6^{p-1}K^p T^{\frac{p}{2}}(C_1 + 1)$$

$$+ 6^{p-1}K^p (T^{\frac{p-2}{2}} C_1 + T^{p-1}) \int_0^t \varphi(s) ds.$$

By Gronwall's inequality,

$$\varphi(t) \le (3^{p-1}|x|^p + 6^{p-1}K^pT^{\frac{p}{2}}(C_1 + 1)) \cdot \exp(6^{p-1}K^p(T^{\frac{p-2}{2}}C_1 + T^{p-1})t).$$

Hence X_t is bounded.   □

References

[1] N. U. Ahmed, *Nonlinear stochastic differential inclusions on Banach space*, Stochastic Anal. Appl. **12** (1994), no. 1, 1–10.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[3] J. P. Aubin and G. D. Prato, *The viability theorem for stochastic differential inclusions*, Stochastic Anal. Appl. **16** (1998), no. 1, 1–15.

[4] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion prosesses* (1981); North Holland-Kodansha, Tokyo.

[5] A. A. Levakov, *Asymptotic behavior of solutions of stochastic differential inclusions*, Differ. Uravn. **34** (1998), no. 2, 204–210.

[6] Y. S. Yun and I. Shigekawa, *The existence of solutions for stochastic differential inclusion,*, to appear in Far East J. Math. Sci.

[7] B. Truong-Van and X. D. H. Truong, *Existence results for viability problem associated to nonconvex stochastic differential inclusions*, Stochastic Anal. Appl. **17** (1999), no. 4, 667–685.

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