Bull. Korean Math. Soc. **40** (2003), No. 1, pp. 149–157

AFFINENESS OF DEFINABLE C^r MANIFOLDS AND ITS APPLICATIONS

TOMOHIRO KAWAKAMI

ABSTRACT. Let $\mathcal{M}$ be an exponentially bounded o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. We prove that if r is a non-negative integer, then every definable C^r manifold is affine. Let $f : X \to Y$ be a definable C^1 map between definable C^1 manifolds. We show that the set S of critical points of f and $f(S)$ are definable and $\dim f(S) < \dim Y$. Moreover we prove that if $1 \le s < r \le \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^s diffeomorphism.

1. Introduction

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$", and any manifold in this paper does not have boundary, unless otherwise stated. Several properties of definable C^r manifolds and definable C^r maps are studied in [9], [10], [8]. The Nash category coincides with the definable C^∞ category based on $\mathcal{R}$ [15], and definable C^r categories based on $\mathcal{M}$ are generalizations of the C^r Nash category. General references on o-minimal structures are [3], [5], see also [14]. Further properties and constructions of them are studied in [4], [6], [12].

We say that $\mathcal{M}$ is *polynomially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number k and a real number x_0 such that $|f(x)| \le x^k$ for any $x > x_0$. Otherwise, $\mathcal{M}$ is called *exponential*. One of typical examples of polynomially bounded structures is $\mathcal{R}$. By a result of C. Miller [11], if $\mathcal{M}$ is exponential,

Received January 25, 2002.

2000 Mathematics Subject Classification: 14P20, 14P10, 57R35, 57R55, 58A05, 03C64.

Key words and phrases: definable C^r manifolds, definable C^r maps, o-minimal, Sard's theorem, exponentially bounded.

then the exponential function $\mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is definable. We call $\mathcal{M}$ *exponentially bounded* if for every function $h : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number l and a real number x_1 such that $|h(x)| \leq exp_l(x)$ for any $x > x_1$, where $exp_l(x)$ denotes the lth iterate of the exponential function, e.g. $exp_2(x) = e^{e^x}$. Note that the problem that every o-minimal expansion $\mathcal{M}$ of $\mathcal{R}$ is exponentially bounded is still open (e.g. [2]).

THEOREM 1.1. *If $\mathcal{M}$ is exponentially bounded and $0 \leq r < \infty$, then every definable C^r manifold is affine.*

Theorem 1.1 is a generalization of 1.1 [10] and an equivariant C^∞ version of Theorem 1.1 is true if $\mathcal{M}$ is exponential and the manifold is compact (see 1.2 [10]). If $\mathcal{M} = \mathcal{R}$ and $r = \infty$, then Theorem 1.1 is not true [13].

As applications of Theorem 1.1, we have the following two results.

Let A be a subset of an n-dimensional definable C^r manifold X with a definable C^r atlas $\{(U_i, \phi_i : U_i \to \mathbb{R}^n)\}_i$ and $r > 0$. We say that A *has measure* 0 *in* X if each $\phi_i(U_i \cap A) \subset \mathbb{R}^n$ has measure 0 (e.g. see P.68 [7]).

THEOREM 1.2. *Let X and Y be definable C^1 manifolds and $f : X \to Y$ a definable C^1 map. If $\mathcal{M}$ is exponentially bounded, then the set S of critical points of f and $f(S)$ are definable and $\dim f(S) < \dim Y$. In particular, the measure of $f(S)$ in Y is* 0.

Without assuming that f is definable, there exists a C^1 map from $\mathbb{R}^2$ to $\mathbb{R}^1$ whose critical point set has positive measure [17]. Note that if $\dim X < \dim Y$ and f is a definable C^1 imbedding, then $S = X$, in particular, $\dim f(S) = \dim X$. Thus in Theorem 1.2, one cannot replace $\dim f(S) < \dim Y$ by $\dim f(S) < \min(\dim X, \dim Y)$.

THEOREM 1.3. *If $\mathcal{M}$ is exponentially bounded and $1 < s < r < \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^r diffeomorphism.*

By [13], there exists an uncountable family $\{X\}_{\lambda \in \Lambda}$ of Nash manifolds such that they are C^2 Nash diffeomorphic and that X_λ is not Nash diffeomorphic to X_μ for $\lambda \neq \mu$. Thus if $\mathcal{M} = \mathcal{R}$ and $r = \infty$, then Theorem 1.3 does not hold.

2. Proof of results

To prove Theorem 1.1, we need the following three results.

PROPOSITION 2.1 (3.2 [10]). *Let X be an affine definable C^r manifold and $0 \leq r < \infty$. Then X can be definably C^r imbeddable into some $\mathbb{R}^n$ such that X is closed in $\mathbb{R}^n$. Moreover it is possible to definably C^r imbeddable into some $\mathbb{R}^k$ such that X is bounded and $\overline{X} - X$ consists of at most one point, where $\overline{X}$ denotes the closure of X in $\mathbb{R}^k$.*

Let $e_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$ be the function defined by

$$e_n(x) = \begin{cases} e^{-exp_{n-1}(1/x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

where $exp_0(x) = x$. Then elementary computations show the following proposition.

PROPOSITION 2.2. (1) *For any polynomial function $P(x_1, \cdots, x_n)$ in n variables,*

$$\lim_{x \to 0} P\left(\frac{1}{x}, exp_1(\frac{1}{x^2}), \cdots, exp_{n-1}(\frac{1}{x^2})\right) e_n(x) = 0.$$

(2) *Every e_n is a C^∞ function.*

Since $\mathcal{M}$ is exponentially bounded, in the proof of C.5 [5], we can take $\phi(t) = te_n(t)$ for some $n \in N$. Hence a similar proof of C.14 [5] proves the following proposition.

PROPOSITION 2.3 ([5]). *Let A be a non-empty compact definable subset of $\mathbb{R}^n$ and f, g two continuous definable functions on A such that $f^{-1}(0) \subset g^{-1}(0)$. If $\mathcal{M}$ is exponentially bounded, then there exist a natural number k and a positive constant c such that $e_k(g) \leq c|f|$ on A.*

Proof of Theorem 1.1. Let X be a definable C^r manifold. If dim $X = 0$, then X consists of finitely many points. Thus the result holds.

Assume that $m :=$ dim $X \geq 1$. Let $\{\phi_i : U_i \to \mathbb{R}^m\}_{i=1}^l$ be a definable C^r atlas of X. Then each $\phi_i(U_i)$ is a noncompact definable C^r submanifold of $\mathbb{R}^m$. Hence by Proposition 2.1, we have a definable C^r imbedding $\phi'_i : \phi_i(U_i) \to \mathbb{R}^{m'}$ such that the image is bounded in $\mathbb{R}^{m'}$ and

$$\overline{\phi'_i \circ \phi_i(U_i)} - \phi'_i \circ \phi_i(U_i)$$

consists of one point, say 0. For a sufficiently large positive integer n, set

$$\eta : \mathbb{R}^{m'} \to \mathbb{R}^{m'}, \eta(x_1, \cdots, x_{m'}) = \Big(\sum_{j=1}^{m'} e_n(x_j)x_1, \cdots, \sum_{j=1}^{m'} e_n(x_j)x_{m'}\Big),$$

$$g_i : U_i \to \mathbb{R}^{m'}, \eta \circ \phi'_i \circ \phi_i.$$

Then g_i is a definable C^r imbedding of U_i into $\mathbb{R}^{m'}$.

We now prove that the extension $\tilde{g}_i : X \to \mathbb{R}$ of g_i is defined by $\tilde{g}_i = 0$ on $X - U_i$ is of class definable C^r. It is sufficient to see this on each definable C^r coordinate neighborhood of X. Hence we may assume that X is open and bounded in $\mathbb{R}^m$. We only have to prove that for any sequence $\{a_t\}_{t=1}^{\infty}$ in U_i convergent to a point of $X - U_i$ and for any $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| \le r$, $\{D^\alpha g_i(a_t)\}_{t=1}^{\infty}$ converges to 0. On the other hand, $g_i = (\sum_{j=1}^{m'} e_n(\phi_{ij})\phi_{i1}, \cdots \sum_{j=1}^{m'} e_n(\phi_{ij})\phi_{im'})$, where $\phi'_i \circ \phi_i = (\phi_{i1}, \cdots, \phi_{im'})$. By the construction of ϕ_{ij}, $\{\phi_{ij}(a_t)\}_{t=1}^{\infty}$ converges to 0. Hence for any natural number k, $\{e_k(\phi_{ij}(a_t))\phi_{is}(a_t)\}_{t=1}^{\infty}$ converges to 0. Assume that if $|\alpha| \le r-1$, then there exists some $K \in \mathbb{N}$ such that if $k \ge K$, then $D^\alpha(e_k(\phi_{ij}(a_t))\phi_{is}(a_t)) \to 0$ as $t \to \infty$. Let $D^\alpha(e_k(\phi_{ij}(x)))\phi_{is}(x) = F(x)e_k(\phi_{ij}(x))$. Then F is a definable $C^{r-|\alpha|}$ map on U.

Let

$$\psi = \max\left\{1, |\frac{\partial F}{\partial x_1}|, |\frac{\partial \phi_{ij}}{\partial x_1}|\right\}.$$

Define

$$\theta_{ij} = \begin{cases} \min\{|\phi_{ij}|, 1/\psi\} & \text{on } U_i \\ 0 & \text{on } X - U_i, \end{cases} \quad \tilde{\phi_{ij}} = \begin{cases} \phi_{ij} & \text{on } U_i \\ 0 & \text{on } X - U_i. \end{cases}$$

Then θ_{ij} and $\tilde{\phi_{ij}}$ are continuous definable maps on X such that

$$X - U_i \subset (\theta_{ij})^{-1}(0) = (\tilde{\phi_{ij}})^{-1}(0).$$

Moreover by the construction of ϕ_{ij}, θ_{ij} and $\tilde{\phi_{ij}}$, θ_{ij} and $\tilde{\phi_{ij}}$ are extendable to continuous definable maps on $\mathbb{R}^m$. Hence by Proposition 2.3, there exist a positive integer a, a positive number b and a definable open neighborhood V of $X - U_i$ in X such that $e_a(\tilde{\phi_{ij}}) \le b|\theta_{ij}|$ on V.

On the other hand, by the definition of θ_{ij}, $|\psi\theta_{ij}| \le 1$ on U_i. Thus $|\psi| e_a(\tilde{\phi_{ij}}) \le b$. Hence if $n \ge N := K + a + 1$, then

$$\begin{aligned} \frac{\partial}{\partial x_1}(D^\alpha(e_n(\phi_{ij})\phi_{is})) &= \frac{\partial}{\partial x_1}(Fe_n(\phi_{ij})) \\ &= \frac{\partial F}{\partial x_1} e_n(\phi_{ij}) + FR_1 e_n(\phi_{ij}) \\ &= \frac{\partial F}{\partial x_1} e_n(\phi_{ij}) + (Fe_K(\phi_{ij}))(R_1 \frac{e_n(\phi_{ij})}{e_K(\phi_{ij})}), \end{aligned}$$

where $R_1 = 2(\frac{\partial \phi_{ij}}{\partial x_1}/\phi_{ij}^3)exp_1(\frac{1}{\phi_{ij}^2})\cdots exp_{n-1}(\frac{1}{\phi_{ij}^2})$. Thus using the inductive hypothesis and Proposition 2.2,

$$\begin{aligned}
&|\frac{\partial}{\partial x_1}(D^\alpha(e_n(\phi_{ij})\phi_{is}))| \\
&\leq |\frac{\partial F}{\partial x_1}|e_n(\phi_{ij}) + |Fe_K(\phi_{ij})||R_1|\frac{e_n(\phi_{ij})}{e_K(\phi_{ij})} \\
&\leq b\frac{e_n(\phi_{ij})}{e_a(\phi_{ij})} + |Fe_K(\phi_{ij})|\frac{2be_n(\phi_{ij})}{e_a(\phi_{ij})e_K(\phi_{ij})}\frac{exp_1(\frac{1}{\phi_{ij}^2})\cdots exp_{n-1}(\frac{1}{\phi_{ij}^2})}{|\phi_{ij}^3|} \to 0.
\end{aligned}$$

By the above argument, replacing some larger N, if $|\alpha| \leq r$ and $n \geq N$, then $|D^\alpha(e_n(\phi_{ij})\phi_{is})| \to 0$. Therefore if $n \geq N$, then each $\tilde{g}_i$ is a definable C^r map and the function $h_i : X \to \mathbb{R}$ defined by $h_i = \sqrt{(\tilde{g}_{i1})^2 + \cdots + (\tilde{g}_{im'})^2 + 1}$ is a definable C^r function with $h_i(X - U_i) = 1$, $(1 \leq i \leq l)$, where $\tilde{g}_i = (\tilde{g}_{i1}, \cdots, \tilde{g}_{im'})$, $(1 \leq i \leq l)$. It is easy to see that

$$(\tilde{g_1}, \cdots, \tilde{g_l}, h_1, \cdots, h_l) : X \to \mathbb{R}^{lm'} \times \mathbb{R}^l$$

is a definable C^r imbedding. □

Proof of Theorem 1.2. Since $\mathcal{M}$ is exponentially bounded and by Theorem 1.1, we may assume that X and Y are affine.

The first half of the theorem is obvious. We have only to prove the latter half. If $\dim X < \dim Y$, then $\dim f(S) \leq \dim f(X) \leq \dim X < \dim Y$. Thus we assume that $\dim Y \leq \dim X$.

By Sard's theorem (e.g. 3.1.3 [7]), if $r > \max(0, \dim X - \dim Y)$, then the set of critical values of every C^r map from X to Y has measure 0 in Y. Fix such an r.

By the definable C^r cell decomposition theorem (e.g. 7.3.3 [3]), there exists a finite partition $\{C_i\}_i$ of X into definable C^r cells such that each $f|C_i : C_i \to Y$ is a definable C^r map. Note that every C_i is a definable C^r submanifold of X and that C_i is open in X if $\dim C_i = \dim X$.

Let K_i denote the set of critical values of $f|C_i : C_i \to Y$ and let $K = f(S)$. Then by Sard's theorem, each K_i has measure 0 in Y. Thus $\dim K_i < \dim Y$. Hence $\dim \cup_i K_i < \dim Y$.

We now prove $K \subset \cup_i K_i \cup_{\dim C_i < \dim Y} f(C_i)$. Let $y \in K$. Then there exists an $x \in X = \cup_i C_i$ such that $y = f(x)$ and the rank of the Jacobian of f at x is smaller than $\dim Y$. Assume that $x \in C_i$. If $\dim C_i < \dim Y$, then $y = f(x) \in \cup_{\dim C_i < \dim Y} f(C_i)$. If $\dim C_i = \dim X$, then $y = f(x) \in K_i$ because C_i is open in X. Assume that $\dim Y \leq \dim C_i < \dim X$. Since C_i is a definable C^r submanifold of X.

there exists a definable C^r chart $\phi : U \to V \subset \mathbb{R}^k$ of X around x such that $\phi(x) = 0$ and $\phi(C_i \cap U) = V \cap \mathbb{R}^l$, where $k = \dim X, l = \dim C_i$ and $\mathbb{R}^l = \mathbb{R}^l \times 0 \subset \mathbb{R}^k$. The Jacobian A of $(f|C_i) \circ \phi^{-1}$ at $\phi(x)$ is a submatrix of the Jacobian B of $f \circ \phi^{-1}$ at $\phi(x)$. Then the determinant of every minor of B of degree $\dim Y$ at x is 0 because $\dim Y \leq \dim C_i < \dim X$. Hence the rank of A at $\phi(x)$ is smaller than $\dim Y$. Thus $y \in K_i$. Therefore $K \subset \cup_i K_i \cup_{\dim C_i < \dim Y} f(C_i)$.

Since $\dim \cup_i K_i < \dim Y$ and $\dim f(C_i) \leq \dim C_i$, $\dim K = \dim f(S) < \dim Y$. □

To prove Theorem 1.3, we need the following several results.

PROPOSITION 2.4 (1.3 [8]). *Let $1 \leq r < \infty$. Then every definable C^r submanifold X of $\mathbb{R}^n$ has a definable C^r tubular neighborhood (U, p) of X in $\mathbb{R}^n$, namely U is a definable open neighborhood of X in $\mathbb{R}^n$ and $p : U \to X$ is a definable C^r map with $p|X = id_X$.*

THEOREM 2.5 (1.2 [9]). *If $0 < r < \infty$, then every noncompact affine definable C^r manifold is definably C^r diffeomorphic to the interior of some compact affine definable C^r manifold with boundary.*

THEOREM 2.6 (5.8 [8]). *If $2 \leq r < \infty$, then every compact affine definable C^r manifold with boundary admits a definable C^r collar, namely there exists a definable C^r imbedding $\phi : \partial X \times [0,1] \to X$ such that $\phi|(\partial X \times \{0\})$ is the inclusion $\partial X \to X$, where the action on $[0,1]$ is trivial.*

Note that Proposition 2.4, Theorem 2.5 and 2.6 are true in more general settings (see 1.3 [8], 1.2 [9] and 5.8 [8]).

The following two results are algebraic realizations of compact C^∞ manifolds.

THEOREM 2.7 ([16]). *Every compact C^∞ manifold is C^∞ diffeomorphic to a nonsingular algebraic set.*

THEOREM 2.8 ([1]). *Let X' be a compact C^∞ submanifold of a compact C^∞ manifold X. Then there exist a nonsingular algebraic set Y and its nonsingular algebraic subset Y' such that $(X; X')$ is C^∞ diffeomorphic to $(Y; Y')$.*

The following is a result for raising differentiability of manifolds

THEOREM 2.9 (2.2.9 [7]). *If $1 \leq s < \infty$, then every C^s manifold admits a compatible C^∞ manifold structure. In other words, for any C^s manifold (X, θ), there exists a C^∞ structure θ' on X such that $id_X : (X, \theta) \to (X, \theta')$ is a C^s diffeomorphism.*

Some refinement of the proof of 2.2.9 [7] proves the following relative version of it.

THEOREM 2.10. *Let X' be a compact C^s submanifold of a compact C^s manifold X and $1 \leq s < \infty$. Then there exist a compact C^∞ manifold Y and its compact C^∞ submanifold Y' such that $(X; X')$ is C^s diffeomorphic to $(Y; Y')$.*

The following is useful to approximate a relative C^1 diffeomorphism by relative definable C^r diffeomorphisms.

THEOREM 2.11. *Let X and Y compact definable C^r manifolds and $1 \leq r < \infty$. Suppose that X' and Y' are compact definable C^r submanifolds of X and Y, respectively, and that $f : (X; X') \to (Y; Y')$ is a C^1 diffeomorphism. Then there exists a definable C^r diffeomorphism $h : (X; X') \to (Y; Y')$ as an approximation of f in the C^1 Whitney topology.*

Proof. Since X, Y are compact and by 1.1 [10] and 1.2 [10], we may assume that X and Y are definable C^r submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Since $f|X' : X' \to Y'$ is a C^1 diffeomorphism and by the polynomial approximation theorem and Proposition 2.4, there exists a definable C^r diffeomorphism $f_1 : X' \to Y'$ as an approximation of $f|X' : X' \to Y'$ in the C^1 Whitney topology. Similarly, one can find a definable C^r diffeomorphism $f_2 : X \to Y$ as an approximation of $f : X \to Y$ in the C^1 Whitney topology.

By Proposition 2.4, there exists a definable C^r tubular neighborhood (U, p) of X' in $\mathbb{R}^n$ (resp. (V, q) of Y in $\mathbb{R}^m$). Then $U' := U \cap X$ is a definable open neighborhood of X' in X. Thus we have a definable C^r map $f_3 : U' \to Y'$ with $f_3|X' = f_1$. Take a definable open neighborhood U_1 of X' in U' such that the closure of U_1 in X is properly contained in U' and take a definable C^r function $\lambda : X \to \mathbb{R}$ such that $\lambda = 1$ on U_1 and its support lies in U'. Then we have a definable C^r map $h : (X; X') \to (Y; Y'), h(x) = q(\lambda(x) f_3(x) + (1 - \lambda(x)) f_2(x))$ as an approximation of $f : (X; X') \to (Y; Y')$ in the C^1 Whitney topology. If our approximation is sufficiently close, then h is the required definable C^r diffeomorphism. □

One can define the definable C^s topology on the set of definable C^s maps between affine definable C^s manifolds (see [9]). This definable C^s topology is different from the C^s Whitney topology in general, but they coincide if the domain manifold is compact.

THEOREM 2.12 ([14], 4.11 [9]). *Let $0 \leq s < r < \infty$. Then every definable C^s map between affine definable C^r manifolds is approximated in the definable C^s topology by definable C^r maps.*

Note that Theorem 2.12 are true in a more general setting (see 1.1 [8]).

PROPOSITION 2.13 ([14], 4.10 [9]). *Let X and Y be definable C^s submanifolds of $\mathbb{R}^n$ and $0 < s < \infty$. If $f : X \to Y$ is a definable C^s diffeomorphism, then an approximation of f in the definable C^s topology is a definable C^s diffeomorphism.*

Proof of Theorem 1.3. Let X be a definable C^s manifold. Then by Theorem 1.1 and since $\mathcal{M}$ is exponentially bounded, X is affine.

Assume that X is compact. By Theorem 2.9, X is C^s diffeomorphic to a compact C^∞ manifold X'. Thus by Theorem 2.7, X' is C^∞ diffeomorphic to a nonsingular algebraic set X''. In particular, X is C^s diffeomorphic to an affine definable C^∞ manifold X''. By Theorem 2.11, X is definably C^s diffeomorphic to X''. Thus in this case, X admits a definable C^r manifold structure.

Assume that X is not compact. By Theorem 2.5, X is definably C^s diffeomorphic to the interior of some compact affine definable C^s manifold Y with boundary ∂Y. Thus by Theorem 2.6, Y admits a definable C^s collar. Hence we have the double D of Y. By Theorem 1.1, D is affine and compact. Using Theorem 2.10, there exist a compact C^∞ manifold D' and a compact C^∞ submanifold Z of D' such that $(D, \partial Y)$ is C^s diffeomorphic to (D', Z). By Theorem 2.8, one can find a nonsingular algebraic set D'' and a nonsingular algebraic subset Z' of D'' such that (D', Z) is C^∞ diffeomorphic to (D'', Z'). In particular, D'' is an affine definable C^∞ manifold, Z' is a definable C^∞ submanifold of D'' and $(D, \partial Y)$ is C^s diffeomorphic to (D'', Z'). Using Theorem 2.11, $(D, \partial Y)$ is definably C^s diffeomorphic to (D'', Z'). Thus X is definably C^s diffeomorphic to some union of connected components of $D'' - Z'$. Therefore X admits a definable C^r manifold structure.

Uniqueness follows from Theorem 1.1, Theorem 2.12 and Proposition 2.13. □

Remark that the above proof shows that every definable C^s manifold is definably C^s diffeomorphic to an affine definable C^∞ manifold.

References

[1] S. Akbulut and H. King, *A relative Nash theorem*, Trans. Amer. Math. Soc. **267** (1981), 465–481.

[2] L. van den Dries, *O-minimal structures and real analytic geometry*, Current developments in Math. (1998), 105–152.

[3] ———, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

[4] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. Math. **140** (1994), 183–205.

[5] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

[6] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.

[7] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York-Heidelberg, 1976.

[8] T. Kawakami, *Equivariant definable C^r approximation theorem, definable C^rG triviality of G invariant definable C^r functions and compactifications*, preprint.

[9] ———, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), no. 2, 323–349.

[10] ———, *Imbeddings of manifolds defined on an o-minimal structure on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), no. 1, 183–201.

[11] C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257–259.

[12] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), 4397–4419.

[13] M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155–162.

[14] ———, *Geometry of subanalyitc and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.

[15] A. Tarski, *A decision method for elementary algebra and geometry*, 2nd edition. revised, Berkeley and Los Angeles, 1951.

[16] A. Tognoli, *Su una congettura di Nash*, Annali Sc. Norm. Sup. Pisa **27** (1973), 167–185.

[17] H. Whitney, *A functions not constant on a connected set of critical points*, Duke Math. J. **1** (1935), 514–517.

Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan

E-mail: kawa@center.wakayama-u.ac.jp