Bull. Korean Math. Soc. **40** (2003), No. 1, pp. 139–147

ON SET-VALUED CHOQUET INTEGRALS AND CONVERGENCE THEOREMS (II)

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ABSTRACT. In this paper, we consider Choquet integrals of interval number-valued functions(simply, interval number-valued Choquet integrals). Then, we prove a convergence theorem for interval number-valued Choquet integrals with respect to an autocontinuous fuzzy measure.

1. Introduction

In this paper, we consider autocontinuity fuzzy measures [12, 15] and interval number-valued functions [16]. It is well-known that closed set-valued functions had been used repeatedly in many papers [1, 2, 5, 6, 7, 8, 9, 13, 15, 16]. Jang et al. [7, 9] studied closed set-valued Choquet integrals and convergence theorems under some sufficient conditions, for examples; (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions(see [7]), (ii) convergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see [9]).

The aim of this paper is to prove a convergence theorem for convergent sequences of Choquet integrably bounded interval number-valued functions in the metric $\triangle_S$ (see Definition 3.4). In Section 2, we list various definitions and notations which are used in the proof of the convergence theorem and discuss some properties of measurable interval number-valued functions. In Section 3, using these definitions and properties, we prove the convexity of interval number-valued Choquet

Received March 21, 2002.
2000 Mathematics Subject Classification: 28A45, 28A48, 28E10.
Key words and phrases: fuzzy measures, autocontinuous, Choquet integrals, Hausdorff metric, convergence theorem.
This work was supported by grant No. 2001-1-10100-003-1 from the Basic Research Program of the Korea Science and Engineering Foundation.

integrals and discuss the concepts of convergence sequences of measurable interval number-valued functions in the metric $\triangle_S$.

2. Definitions and preliminaries

DEFINITION 2.1. [8, 12] (1) A fuzzy measure on a measurable space $(X, \mathcal{A})$ is an extended real-valued function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \mathcal{A},\ A \subset B$.

(2) A fuzzy measure μ is said to be autocontinuous from above [resp., below] if $\mu(A \cup B_n) \to \mu(A)$ [resp., $\mu(A \sim B_n) \to \mu(A)$] whenever $A \in \mathcal{A}$, $\{B_n\} \subset \mathcal{A}$ and $\mu(B_n) \to 0$.

(3) If μ is autocontinuous both from above and from below, it is said to be autocontinuous.

Recall that a function $f : X \to [0, \infty]$ is said to be measurable if $\{x | f(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$.

DEFINITION 2.2. [12] (1) A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure, in symbols $f_n \to_M f$ if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \mu(\{x | (f_n(x) - f(x)| > \epsilon\}) = 0.$$

(2) A sequence $\{f_n\}$ of measurable functions is said to converge to f in distribution, in symbols $f_n \to_D f$ if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \mu_{f_n}(r) = \mu_f(r)\ e.c.,$$

where $\mu_f(r) = \mu(\{x | f(x) > r\})$ and "e.c." stands for "except at most countably many values of r".

DEFINITION 2.3. [10, 11, 12] (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet integral of f can be defined and its value is finite.

Throughout this paper, R^+ will denote the interval $[0, \infty)$, $I(R^+) = \{[a, b] | a, b \in R^+ \text{ and } a \leq b\}$. Then an element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$$[a, b] + [c, d] = [a + c, b + d],$$

$[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$,
$k[a, b] = [ka, kb]$,
$[a, b] \leq [c, d]$ if and only if $a \leq c$ and $b \leq d$.

Then $(I(R^+), d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

PROPOSITION 2.4. *For each pair* $[a, b], [c, d] \in I(R^+)$, $d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}$.

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F : X \to C(R^+)\backslash\{\emptyset\}$ and an interval number-valued function $F : X \to I(R^+)\backslash\{\emptyset\}$. We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$.

DEFINITION 2.5. [1, 6, 7] A closed set-valued function F is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x \in X | F(x) \cap O \neq \emptyset\} \in \mathcal{A}.$$

DEFINITION 2.6. [1] Let F be a closed set-valued function. A measurable function $f : X \to R^+$ satisfying

$$f(x) \in F(x) \text{ for all } x \in X$$

is called a measurable selection of F.

We say $f : X \to R^+$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C) \int f d\mu < \infty$. We note that "$x \in X$ $\mu - a.e.$" stands for "$x \in X$ μ-almost everywhere". The property $p(x)$ holds for $x \in X$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $p(x)$ holds for all $x \in A^c$, where A^c is the complement of A.

DEFINITION 2.7. [6, 7](1) Let F be a closed set-valued function and $A \in \mathcal{A}$. The Choquet integral of F on A is defined by

$$(C) \int_A F d\mu = \{(C) \int_A f d\mu |\ f \in S_c(F)\},$$

where $S_c(F)$ is the family of $\mu - a.e.$ Choquet integrable selections of F, that is,

$$S_c(F) = \{f \in L_c^1(\mu) |\ f(x) \in F(x)\ x \in X\ \mu - a.e.\}.$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \neq \emptyset$.

(3) A closed set-valued function F is said to be Choquet integrably bounded if there is a function $g \in L_c^1(\mu)$ such that

$$\|F(x)\| = \sup_{r \in F(x)} |r| \leq g(x) \text{ for all x} \in \text{X}.$$

Instead of $(C)\int_X F d\mu$, we will write $(C)\int F d\mu$. Let us discuss some basic properties of measurable closed set-valued functions. Since $R^+ = [0, \infty)$ is a complete separable metric space in the usual topology, using Theorem 8.1.3 ([1]) and Theorem 1.0(2^0) ([5]), we have the following two theorems.

THEOREM 2.8. [1, 5] *A closed set-valued function F is measurable if and only if there exists a sequence of measurable selections $\{f_n\}$ of F such that*

$$F(x) = cl\{f_n(x)\} \text{ for all x} \in \text{X}.$$

THEOREM 2.9. [1, 5] *If F is a measurable closed set-valued function and Choquet integrably bounded, then it is Choquet integrable.*

3. Main results

In this section, we prove the convexity of interval number-valued Choquet integrals and discuss the concepts of convergent sequences of measurable interval number-valued functions in the metric $\triangle_S$. Since $(X, \mathcal{A})$ is a measurable space and R^+ is a separable metric space, Theorem 1.0(2^0) ([5]) implies the following theorem. Recall that a measurable closed set-valued function is said to be convex-valued if $F(x)$ is convex for all $x \in X$ and that a set A is an interval number if and only if it is closed and convex.

THEOREM 3.1. *If F is a measurable closed set-valued function and Choquet integrably bounded, then there exists a sequence $\{f_n\}$ of Choquet integrable functions $f_n : X \to R^+$ such that $F(x) = cl\{f_n(x)\}$ for all $x \in X$.*

Proof. By Theorem 1.0 (2^0) ([5]), there exists a sequence $\{f_n\}$ of measurable functions $f_n : X \to R^+$ such that $F(x) = cl\{f_n(x)\}$ for all

$x \in X$. Since F is Choquet integrably bounded, there is a measurable function $g \in L_c^1(\mu)$ such that

$$\|F(x)\| = \sup\{r | r \in F(x)\} \leq g(x), \text{ for all } x \in X.$$

Since $f_n(x) \in F(x)$ for all $x \in X$ and all $n = 1, 2, \cdots$, $f_n(x) \leq g(x)$ for all $x \in X$. By Proposition 3.2 ([11]),

$$(C) \int f_n d\mu \leq (C) \int g d\mu < \infty, \text{ for all } n = 1, 2, \cdots.$$

So, f_n is Choquet integrable for all $n = 1, 2, \cdots$. The proof is complete. □

THEOREM 3.2. *If F is a measurable closed set-valued function and Choquet integrably bounded and if we define $f^*(x) = \sup\{r | r \in F(x)\}$ and $f_*(x) = \inf\{r | r \in F(x)\}$ for all $x \in X$, then f^* and f_* are Choquet integrable selections of F.*

Proof. Since F is Choquet integrably bounded, there exists a function $g \in L_c^1(\mu)$ such that $\|F(x)\| \leq g(x)$ for all $x \in X$. Theorem 3.1 implies that there is a sequence $\{f_n\}$ of Choquet integrable selections of F such that $F(x) = cl\{f_n(x)\}$ for all $x \in X$. Then

$$f^*(x) = \sup\{r | r \in F(x)\} = \sup_n f_n(x)$$

and

$$f_*(x) = \inf\{r | r \in F(x)\} = \inf_n f_n(x).$$

Since the supremum and the infimum of a sequence $\{f_n\}$ of measurable functions are measurable, f^* and f_* are measurable. And also, we have

$$0 \leq f_*(x) \leq f^*(x) = \|F(x)\| \leq g(x) \text{ for all } x \in X.$$

Since $g \in L_c^1(\mu)$, f^* and f_* belong to $L_c^1(\mu)$. By the closedness of $F(x)$ for all $x \in X$, $f_*(x) \in F(x)$ and $f^*(x) \in F(x)$ for all $x \in X$. Therefore, f^* and f_* are Choquet integrable selections of F. □

ASSUMPTION (A). For each pair $f, g \in S_c(F)$, there exists $h \in S_c(F)$ such that $f \sim h$ and $(C) \int g d\mu = (C) \int h d\mu$.

We consider the following classes of interval number-valued functions;

$\mathcal{F} = \{F | F : X \to I(R^+)$ is measurable and Choquet integrably bounded$\}$ and

$\mathcal{F}_1 = \{F \in \mathcal{F} | F$ is convex $-$ valued and satisfies the assuption(A)$\}$.

THEOREM 3.3. *If $F \in \mathcal{F}_1$, then we have*

(1) *$cF \in \mathcal{F}_1$ for all $c \in R^+$,*
(2) *$(C) \int F d\mu$ is convex,*

(3) $(C)\int Fd\mu = [(C)\int f_* d\mu, (C)\int f^* d\mu]$.

Proof. (1) The proof of (1) is trivial.

(2) If $(C)\int F\mu$ is a single point set, then it is convex. Otherwise, let $y_1, y_2 \in (C)\int Fd\mu$ and $y_1 < y_2$. Then, there exist $f_1, f_2 \in S_c(F)$ such that

$$y_1 = (C)\int f_1 d\mu \text{ and } y_2 = (C)\int f_2 d\mu.$$

Further, let $y \in (y_1, y_2)$ we need to a selection $f \in S_c(F)$ with $y = (C)\int f d\mu$. Since $y \in (y_1, y_2)$, there exists $\lambda_0 \in (0,1)$ such that $y = \lambda_0 y_1 + (1-\lambda_0)y_2$. For above two selections $f_1, f_2 \in S_c(F)$, the assumption (A) implies that there exists $g \in S_c(F)$ such that $f_1 \sim g$ and $(C)\int g d\mu = (C)\int f_2 d\mu$. We define a function $f = \lambda_0 f_1 + (1-\lambda_0)g$ and note that $\lambda_0 f_1 \sim (1-\lambda_0)g$. Since F is convex, $f(x) = \lambda_0 f_1(x) + (1-\lambda_0)g(x) \in F(x)$ for $x \in X$ μ-a.e. By Theorem 5.6 [11] and Proposition 3.2 (2) [11],

$$\begin{aligned} y &= \lambda_0 y_1 + (1-\lambda_0)y_2 \\ &= (C)\int \lambda_0 f_1 d\mu + (C)\int (1-\lambda_0) f_2 d\mu \\ &= \lambda_0 (C)\int f_1 d\mu + (1-\lambda_0)(C)\int f_2 d\mu \\ &= \lambda_0 (C)\int f_1 d\mu + (1-\lambda_0)(C)\int g d\mu \\ &= (C)\int \lambda_0 f_1 d\mu + (C)\int (1-\lambda_0) g d\mu \\ &= (C)\int (\lambda_0 f_1 + (1-\lambda_0)g) d\mu \\ &= (C)\int f d\mu. \end{aligned}$$

Thus, we have $f \in S_c(F)$ and $y = (C)\int f d\mu \in (C)\int Fd\mu$. The proof of (2) is complete.

(3) We note that $f_* \leq f \leq f^*$ for all $f \in S_c(F)$. Thus, by Proposition 3.2(2) [11],

$$(C)\int f_* d\mu \leq (C)\int f d\mu \leq (C)\int f^* d\mu$$

for all $f \in S_c(F)$. Theorem 3.2 implies $(C)\int f_* d\mu, (C)\int f^* d\mu \in (C)\int Fd\mu$. By (2), $(C)\int Fd\mu$ is convex in R^+ and hence $(C)\int Fd\mu = [(C)\int f_* d\mu, (C)\int f^* d\mu]$. □

We consider a function $\triangle_S$ on $\mathcal{F}_1$ defined by

$$\triangle_S(F,G) = \sup_{x\in X} d_H(F(x),G(x))$$

for all $F, G \in \mathcal{F}_1$. Then, it is easily to show that $\triangle_S$ is a metric on $\mathcal{F}_1$.

DEFINITION 3.4. Let $F \in \mathcal{F}_1$. A sequence $\{F_n\} \subset \mathcal{F}_1$ converges to F in the metric $\triangle_S$, in symbols, $F_n \to_{\triangle_S} F$ if

$$\lim_{n\to\infty} \triangle_S(F_n, F) = 0.$$

THEOREM 3.5 (CONVERGENCE THEOREM). *Let $F, G, H \in \mathcal{F}_1$ and $\{F_n\}$ be a sequence in $\mathcal{F}_1$. If a fuzzy measure μ is autocontinuous and if $F_n \to_{\triangle_S} F$ and $G \le F_n \le H$, then we have*

$$d_H - \lim_{n\to\infty} (C)\int F_n d\mu = (C)\int F d\mu.$$

Proof. By Proposition 2.4, $d_H(F_n(x), F(x)) = \max\{|f_{n*}(x) - f_*(x)|, |f_n^*(x) - f^*(x)|\}$ for all $x \in X$, where $f_{n*}(x) = \inf\{r|r \in F_n(x)\}$, $f_n^*(x) = \sup\{r|r \in F_n(x)\}$ for $n = 1, 2, \cdots$, $f_*(x) = \inf\{r|r \in F(x)\}$, and $f^*(x) = \sup\{r|r \in F(x)\}$. Since $\triangle_S(F_n, F) \to 0$ as $n \to \infty$, $\sup_{x\in X} |f_{n*}(x) - f_*(x)| \to 0$ and $\sup_{x\in X} |f_n^*(x) - f^*(x)| \to 0$. Given any $\varepsilon > 0$, there exist two natural numbers N_1, N_2 such that $|f_{n*}(x) - f_*(x)| < \varepsilon$ for all $n \ge N_1$ and all $x \in X$, and $|f_n^*(x) - f^*(x)| < \varepsilon$ for all $n \ge N_2$ and all $x \in X$. We put $N = \max\{N_1, N_2\}$. Thus for each $n \ge N$,

$$\mu\{x|\ |f_{n*}(x) - f_*(x)| > \varepsilon\} = \mu(\emptyset) = 0$$

and

$$\mu\{x|\ |f_n^*(x) - f^*(x)| > \varepsilon\} = \mu(\emptyset) = 0.$$

Then, clearly we have that for arbitrary $\varepsilon > 0$, $\mu\{x|\ |f_{n*}(x) - f_*(x)| \ge \varepsilon\} \to 0$ as $n \to \infty$ and $\mu\{x|\ |f_n^*(x) - f^*(x)| \ge \varepsilon\} \to 0$ as $n \to \infty$. That is, $f_{n*} \to_M f_*$ and $f_n^* \to_M f^*$ as $n \to \infty$. It is clearly to show that if $G \le F_n \le H$ then $\mu_{g_*}(r) \le \mu_{f_{n*}}(r) \le \mu_{h_*}(r)$ and $\mu_{g^*}(r) \le \mu_{f_n^*}(r) \le \mu_{h^*}(r)$ for all $r \in R^+$, where $g_*(x) = \inf\{r|r \in G(x)\}$, $g^*(x) = \sup\{r|r \in G(x)\}$, $h_*(x) = \inf\{r|r \in H(x)\}$, and $h^*(x) = \sup\{r|r \in H(x)\}$. Since μ is autocontinuous, by Theorem 3.2 [12], we have

$$\lim_{n\to\infty} (C)\int f_{n*} d\mu = (C)\int f_* d\mu \text{ and } \lim_{n\to\infty} (C)\int f_n^* d\mu = (C)\int f^* d\mu.$$

Therefore,

$$d_H[(C)\int F_n d\mu, (C)\int F d\mu] = \max\{|(C)\int f_{n*} d\mu - (C)\int f_* d\mu|,$$

$$|(C)\int f_n^* d\mu - (C)\int f^* d\mu|\} \to 0$$

as $n \to \infty$.

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