Bull. Korean Math. Soc. **40** (2003), No. 1, pp. 123–137

(WEAK) IMPLICATIVE HYPER K-IDEALS

A. Borumand Saeid, R. A. Borzooei, and M. M. Zahedi

Abstract. In this note first we define the notions of weak implicative and implicative hyper K-ideals of a hyper K-algebra H. Then we state and prove some theorems which determine the relationship between these notions and (weak) hyper K-ideals. Also we give some relations between these notions and all types of positive implicative hyper K-ideals. Finally we classify the implicative hyper K-ideals of a hyper K-algebra of order 3.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [9] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a BCK-algebra. Recently [2, 3, 12] Borzooei, Jun and Zahedi et.al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Now, in this note we define the notions of (weak) implicative hyper K-ideals, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. [2] Let H be a nonempty set and "$\circ$" be a hyperoperation on H, that is "$\circ$" is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y,\ y < x \Rightarrow x = y$

Received March 8, 2002.
2000 Mathematics Subject Classification: 06F35, 03G25.
Key words and phrases: hyper K-algebra, hyper K-ideal, (weak) implicative hyper K-ideal.

(HK5) $0 < x$
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{\substack{a \in A \\ b \in B}} a \circ b$ of H.

EXAMPLE 2.2. [2] Define the hyperoperation " $\circ$ " on $H = [0, +\infty)$ as follows:

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \neq 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper K-algebra.

THEOREM 2.3. [2] *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following hold:*

(i) $x \circ y < z \Leftrightarrow x \circ z < y$,
(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \circ y < x$,
(v) $A \subseteq B$ *implies* $A < B$,
(vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$.

DEFINITION 2.4. [2] Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then,

(i) I is called a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

(ii) I is called a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

THEOREM 2.5. [2] *Any hyper K-ideal of a hyper K-algebra H, is a weak hyper K-ideal.*

DEFINITION 2.6. [1] Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ such that $0 \in I$. Then I is called a *positive implicative hyper K-ideal of*

(i) *type* 1, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that

$x \circ z \subseteq I$,

(ii) *type* 2, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$,

(iii) *type* 3, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z \subseteq I$,

(iv) *type* 4, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply that $x \circ z \subseteq I$,

(v) *type* 5, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z < I$,

(vi) *type* 6, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z < I$,

(vii) *type* 7, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply that $x \circ z < I$,

(viii) *type* 8, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply that $x \circ z < I$.

DEFINITION 2.7. [3] Let I be a nonempty subset of H. Then we say that I satisfies the *additive condition* if for all $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

DEFINITION 2.8. [1] Let H be a hyper K-algebra. An element $a \in H$ is called a *left* (*resp. right*) *scalar* if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is an *scalar element.*

DEFINITION 2.9. [1] We say that the hyper K-algebra H satisfies the *transitive condition* if for all $x, y, z \in H$, $x < y$ and $y < z$ imply that $x < z$.

3. Some results on hyper K-ideals

From now on H is a hyper K-algebra, unless otherwise is stated.

PROPOSITION 3.1. *Let I be a hyper K-ideal of H, and $A, B \subseteq H$. If $A \circ B < I$ and $B \subset I$, then $A < I$.*

Proof. We have $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$ and $A \circ B < I$. Thus there exist $t \in a \circ b$ for some $a \in A$, $b \in B$ and $s \in I$ such that $t < s$. Hence $a \circ b < I$. Since I is a hyper K-ideal and $b \in I$ we conclude that $a \in I$, thus $A < I$. □

REMARK 3.2. (i) In the above proposition it is not necessary that $A \subseteq I$. To show this, let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Now, $I = \{0, 1\}$ is a hyper K-ideal of H, $\{1, 2\} \circ \{0, 1\} = \{0, 1, 2\} < I$ and $\{0, 1\} \subseteq I$, but $\{1, 2\} \not\subseteq I$.

(ii) If in Proposition 3.1, we use $B < I$ instead of $B \subseteq I$, then the result does not hold. Because consider $H = \{0, 1, 2\}$, then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

Let $I = \{0\}$, clearly I is a hyper K-ideal. We have $\{1\} \circ \{0, 1, 2\} < I$ and $\{0, 1, 2\} < I$, but $\{1\} \not< I$.

LEMMA 3.3. *Let I be a weak hyper K-ideal of H. If for all $A, B \subseteq H$, $A \circ B \subseteq I$ and $B \subseteq I$, then $A \subseteq I$.*

Proof. For all $a \in A$, $b \in B$ we have $a \circ b \subseteq A \circ B \subseteq I$ and $b \in I$. Since I is a weak hyper K-ideal, we get that $a \in I$, thus $A \subseteq I$. □

REMARK 3.4. In the above lemma the condition $B \subseteq I$ can not be replaced by $B < I$. Because let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

Now, $I = \{0,1\}$ is a weak hyper K-ideal of H, $2\circ(1\circ 2) \subseteq I$ and $1\circ 2 < I$, while $\{2\} \nsubseteq I$.

DEFINITION 3.5. We say that H satisfies the *strong transitive condition* if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

COROLLARY 3.6. *Let H satisfies the strong transitive condition. Then it satisfies the transitive condition.*

Proof. It is easy. □

The following example shows that the converse of the above corollary is not true in general. To show this let $H = \{0,1,2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

It is easy to check that H satisfies the transitive condition, while it does not satisfy the strong transitive condition. Because $\{2\} < \{1,2\}$ and $\{1,2\} < \{1\}$, but $\{2\} \nless \{1\}$.

PROPOSITION 3.7. *Let H satisfies the strong transitive condition. If I is a hyper K-ideal of H and $A, B \subseteq H$, $A \circ B < I$ and $B < I$, then $A < I$.*

Proof. Let $A \circ B < I$. Then by Theorem 2.3 (ix) we have $A \circ I < B$, and $B < I$. Since H satisfies the strong transitive condition we get that $A \circ I < I$. Now by Proposition 3.1 we have $A < I$. □

4. Implicative hyper K-ideal

DEFINITION 4.1. A nonempty subset I of H is called a *weak implicative hyper K-ideal* if it satisfies:
(i) $0 \in I$
(ii) $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

EXAMPLE 4.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$

Then $I = \{0, 2\}$ is a weak implicative hyper K-ideal of H.

DEFINITION 4.3. A nonempty subset I of H is called an *implicative hyper K-ideal* if it satisfies:
(i) $0 \in I$
(ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

EXAMPLE 4.4. Let $H = \{0, 1, 2\}$. The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 2\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 2\}$

Then $I = \{0, 2\}$ is an implicative hyper K-ideal, while $I = \{0, 1\}$ is not an implicative hyper K-ideal, because $(2 \circ 0) \circ (1 \circ 2) < I$, and $0 \in I$ but $2 \notin I$.

PROPOSITION 4.5. *Each implicative hyper K-ideal of H is a weak implicative.*

Proof. Let I be an implicative hyper K-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then by Theorem 2.3 (v) we have $(x \circ z) \circ (y \circ x) < I$, thus $x \in I$. So I is a weak implicative hyper K-ideal. □

The following example shows that the converse of the above proposition is not correct in general. Consider $H = \{0, 1, 2\}$. The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $I = \{0,1\}$ is a weak implicative hyper K-ideal, while it is not an implicative hyper K-ideal, because $(2 \circ 0) \circ (1 \circ 2) < I$, $0 \in I$ but $2 \notin I$.

THEOREM 4.6. *Every implicative hyper K-ideal of H is a hyper K-ideal.*

Proof. Let I be an implicative hyper K-ideal of H, $x \circ y < I$ and $y \in I$. Then there exist $t \in x \circ y$ and $z \in I$ such that $t < z$. We have $t \in t \circ 0 \subseteq (x \circ y) \circ (0 \circ x)$. Thus $(x \circ y) \circ (0 \circ x) < I$ and $y \in I$, therefore $x \in I$. □

The following example shows that the converse of the above theorem is not correct in general. Let $H = \{0,1,2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

Now, we can see that $I = \{0,2\}$ is a hyper K-ideal, while it is not an implicative hyper K-ideal, since $(1 \circ 0) \circ (2 \circ 1) = \{0,1,2\} < I$ and $0 \in I$, but $1 \notin I$.

REMARK 4.7. (i) In general, a weak implicative hyper K-ideal does not need to be a weak hyper K-ideal. To show this, consider $H = \{0,1,2\}$, then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

We can check that $I = \{0,1\}$ is a weak implicative hyper K-ideal, while it is not a weak hyper K-ideal, because $2 \circ 1 \subseteq I$ and $1 \in I$, but $2 \notin I$.

(ii) In general, a weak hyper K-ideal does not need to be a weak implicative hyper K-ideal. For this consider the hyper K-algebra H of Remark 3.4. Then $I = \{0,1\}$ is a weak hyper K-ideal, while it is not a weak implicative hyper K-ideal, since $(2\circ 0)\circ(1\circ 2)\subseteq I$, and $0\in I$, but $2\notin I$.

THEOREM 4.8. *Let I be a weak hyper K-ideal of H. Then the following statements hold:*
(i) If for all $x,y,z\in H$, $x\circ(y\circ x)\subseteq I$ implies $x\in I$, then I is a weak implicative hyper K-ideal.
(ii) Let $0\in H$ be a right scalar element and I be a weak implicative hyper K-ideal. Then for all $x,y\in H$, $x\circ(y\circ x)\subseteq I$, implies that $x\in I$.

Proof. (i) Let I be a weak hyper K-ideal, $(x\circ z)\circ(y\circ x)\subseteq I$ and $z\in I$. Then $(x\circ(y\circ x))\circ z\subseteq I$. By Lemma 3.3, we have $x\circ(y\circ x)\subseteq I$. Now by hypothesis $x\in I$. So I is a weak implicative hyper K-ideal.

(ii) Let I be a weak implicative hyper K-ideal, $x\circ(y\circ x)\subseteq I$ and $0\in H$ is a right scalar element. We have $(x\circ 0)\circ(y\circ x)=x\circ(y\circ x)\subseteq I$ and $0\in I$, thus $x\in I$. □

The following theorem shows that if we restrict to a hyper K-algebra of order 3, then we can omit the condition "$0\in H$ be a right scalar element", in the above theorem.

THEOREM 4.9. *Let $H=\{0,1,2\}$ be a hyper K-algebra of order 3, and I be a proper weak hyper K-ideal of H. Then I is a weak implicative hyper K-ideal if and only if for all $x,y\in H$, $x\circ(y\circ x)\subseteq I$ implies $x\in I$.*

Proof. Let $I=\{0,1\}$ be a proper weak hyper K-ideal and also a weak implicative hyper K-ideal of H. If $x\circ(y\circ x)\subseteq I$, for arbitrary elements $x,y\in H$, then we show that $x\in I$. If $x=0$ or 1, then it is done. So let $x=2$, therefore

$$2\circ(y\circ 2)\subseteq I. \tag{1}$$

We know that $0\notin 2\circ 0$ and $2\in 2\circ 0$. Thus $2\circ 0=\{2\}$ or $2\circ 0=\{1,2\}$. If $2\circ 0=\{2\}$, then $(2\circ 0)\circ(y\circ 2)=2\circ(y\circ 2)\subseteq I$, by (1). Since $0\in I$ and I is a weak implicative hyper K-ideal, we get that $2\in I$, which is a contradiction.

If $2\circ 0=\{1,2\}$, then we consider the following different cases.

(i) If $y=0$, then $2\in 2\circ 0\subseteq 2\circ(0\circ 2)\subseteq I$, by (1), which is a contradiction.
(ii) If $y=1$ and $1<2$, then $0\in 1\circ 2$. Thus $2\in 2\circ 0\subseteq 2\circ(1\circ 2)\subseteq I$, by (1). Which is a contradiction.

If $y = 1$ and $1 \not< 2$, then $1 \circ 2 = \{1\}$ or $\{1,2\}$ or $\{2\}$. So we must discuss on the above different cases:

(a) If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$, by (1). Since $1 \in I$ and I is a weak hyper K-ideal, we conclude that $2 \in I$, which is a contradiction.

(b) If $1 \circ 2 = \{1,2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1,2\} = 2 \circ (1 \circ 2) \subseteq I$, by (1). Hence $2 \circ 1 \subseteq I$. Therefore $2 \in I$, which is a contradiction.

(c) If $1 \circ 2 = \{2\}$, then we claim that $1 \circ 0 = \{1\}$. Suppose $1 \circ 0 \neq \{1\}$. Since $1 \in 1 \circ 0$ and $0 \notin 1 \circ 0$, we must have $1 \circ 0 = \{1,2\}$. Then $0 \in 2 \circ 2 \subseteq \{2\} \cup 2 \circ 2 = 1 \circ 2 \cup 2 \circ 2 = \{1,2\} \circ 2 = (1 \circ 0) \circ 2$, so

$$0 \in (1 \circ 0) \circ 2. \tag{2}$$

On the other hand $(1 \circ 0) \circ 2 = (1 \circ 2) \circ 0 = 2 \circ 0$. Since $0 \notin 2 \circ 0$, we get that $0 \notin (1 \circ 0) \circ 2$, which is a contradiction by (2). Thus we must have $1 \circ 0 = \{1\}$. Therefore

$$(1 \circ 2) \circ 0 = 2 \circ 0 = \{1,2\} \tag{3}$$

and

$$(1 \circ 0) \circ 2 = 1 \circ 2 = \{2\}. \tag{4}$$

Since $(1 \circ 2) \circ 0 = (1 \circ 0) \circ 2$. So (3), (4) given a contradiction. Thus $y = 1$ does not happen.

(iii) Let $y = 2$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$, by (1). Which is a contradiction. Therefore the above argument shows that $x \neq 2$, i.e., $x \in I$. Finally by considering Theorem 4.8, the proof of the converse is obvious. □

DEFINITION 4.10. [11] Let $H = \{0,1,2\}$ be a hyper K-algebra of order 3. We say that H *satisfies the simple condition* if $1 \not< 2$ and $2 \not< 1$.

THEOREM 4.11. *Let $H = \{0,1,2\}$ be a hyper K-algebra of order 3, that satisfies the simple condition, and let $\{0\} \neq I \subset H$. Then I is a weak hyper K-ideal of H if and only if I is a weak implicative K-ideal of H.*

Proof. Let I be a weak hyper K-ideal of H. By hypothesis we have $I = \{0,1\}$ or $\{0,2\}$. Let $I = \{0,1\}$. By Theorem 4.9 it is enough to show that if $x \circ (y \circ x) \subseteq I$, for any two arbitrary elements x, y of H, then $x \in I$. So let $x \circ (y \circ x) \subseteq I$. If $x = 0$ or 1, then it is done. Thus let $x = 2$. Consider the following different cases:

Case (1). If $y = 0$, then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 0) \subseteq I$ and hence $2 \in I$, which is a contradiction.

Case (2). If $y = 1$, since H satisfies the simple condition then $1 \not< 2$ and $0 \notin 1 \circ 2$. Hence $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$.

(i) If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$. Since I is a weak hyper K-ideal and $1 \in I$ then we get that $2 \in I$, which is a contradiction.

(ii) The case $1 \circ 2 = \{2\}$ does not happen, by Theorem 3.17 of [11].

(iii) If $1 \circ 2 = \{1, 2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$. Thus $2 \circ 1 \subseteq I$. Now $1 \in I$ implies that $2 \in I$, which is a contradiction.

Case (3). If $y = 2$, then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$. Hence $2 \in I$, which is a contradiction.

Thus $x \neq 2$. Hence x is in I. Note that the proof of the case $I = \{0, 2\}$ is similar as above.

Conversely, let I be a weak implicative hyper K-ideal of H. Without loss of generality we assume that $I = \{0, 1\}$. Let $x \circ y \subseteq I$ and $y \in I$. If $x = 0$ or 1, then $x \in I$. So let $x = 2$. We consider the following cases:

Case (1). The case $y = 0$ does not happen, because $2 = 2 \circ 0 \not\subseteq I$.

Case (2). If $y = 1$, since $2 \not< 1$, then $0 \notin 2 \circ 1$. Hence $2 \circ 1 = \{1\}$, $\{2\}$ or $\{1, 2\}$. Since H satisfies the simple condition, then by Theorem 3.17 of [11] $2 \circ 1 \neq \{1\}$. So the cases $2 \circ 1 = \{2\}$ or $\{1, 2\}$ do not happen, since $2 \circ 1 \not\subseteq I$.

Case (3). The case of $y = 2$ does not happen, because $2 \notin I$.

Consequently $x \neq 2$, hence $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$. This shows that I is a weak implicative hyper K-ideal. Note that the proof of the case $I = \{0, 2\}$ is similar as above. □

THEOREM 4.12. *Let I be a hyper K-ideal of H. Then I is an implicative hyper K-ideal if and only if*

(5) $\quad x \circ (y \circ x) < I$ *implies that* $x \in I$, *for any* $x, y \in H$.

Proof. Let I satisfies in (5) and $(x \circ z) \circ (y \circ x) < I$, $z \in I$. Then by Proposition 3.1 we have $x \circ (y \circ x) < I$. So by (5) we get that $x \in I$. Therefore I is an implicative hyper K-ideal.

Conversely, let I be an implicative hyper K-ideal, and $x \circ (y \circ x) < I$. Since $x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x)$, we conclude that $(x \circ 0) \circ (y \circ x) < I$. Thus $0 \in I$ implies that $x \in I$. □

THEOREM 4.13. *Let H satisfies the strong transitive condition. If I is an implicative hyper K-ideal of H, then I is a positive implicative hyper K-ideal of types 1-8.*

Proof. By considering Theorem 3.5 of [1], it is enough to show that I is a positive implicative hyper K-ideal of type 3. Let $(x \circ y) \circ z < I$, and $y \circ z < I$, we must show that $x \circ z \subseteq I$. Let $t \in x \circ z$. Then by (HK1) we have

$$(t \circ z) \circ (y \circ z) < t \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z < I.$$

Since H satisfies the strong transitive condition, then $(t \circ z) \circ (y \circ z) < I$. Since $y \circ z < I$ by Proposition 3.7, we conclude that $t \circ z < I$. Now, by Theorem 2.3 (ii) we have $(x \circ z) \circ (x \circ t) < t \circ z$, thus by hypothesis we get that $(x \circ z) \circ (x \circ t) < I$. Since $(x \circ z) \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z))$, we conclude that $(x \circ z) \circ (x \circ (x \circ z)) < I$. But for all $t \in x \circ z$ we have $t \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z))$, so by hypothesis $t \circ (x \circ t) < I$. Thus by Theorem 4.12, $t \in I$, and hence $x \circ z \subseteq I$. □

REMARK 4.14. In Theorem 4.13 the condition strong transitivity of H is essential. Because, let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

Now H does not satisfy the strong transitive condition, because $\{1\} < \{1,2\} < \{2\}$ and $\{1\} \not< \{2\}$. Clearly $I = \{0, 2\}$ is an implicative hyper K-ideal of H, but it is not a positive implicative hyper K-ideal of type 2 or 3. Because $(2 \circ 0) \circ 0 < I$ and $0 \circ 0 \subseteq I$, but $2 \circ 0 \not\subseteq I$.

THEOREM 4.15. *Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3, that satisfies the simple condition, and $\{0\} \neq I \subset H$. Then I is an implicative hyper K-ideal if and only if I is a positive implicative hyper K-ideal of type 3.*

Proof. Let I be a positive implicative hyper K-ideal of type 3. Without loss of generality assume that $I = \{0, 1\}$. Let $(x \circ z) \circ (y \circ x) < I$ and $z \in I$, we show that $x \in I$. By Theorems 17.3 and 19.3 of [11], we have $2 \circ 1 = \{2\}$, $2 \circ 0 = \{2\}$, $1 \circ 2 = \{1\}$, $1 \circ 0 = \{1\}$, $x \circ y \neq \{0, 2\}$ and $x \circ y \neq \{0, 1, 2\}$ for all $x, y \in H$. Thus

$$x \circ y \subseteq \{0, 1\}, \quad \text{for all } x, y \in H. \tag{6}$$

Now, let $x = 2$. In the following we show that, this case is impossible. To this end consider three different cases:

(i) Let $z = 0$. We consider the following subcases:

(a) If $y = 0$, then by (6) we have $0 \circ 2 \subseteq \{0, 1\}$. Hence $(2 \circ 0) \circ (0 \circ 2) = 2 \circ (0 \circ 2) \subseteq 2 \circ \{0, 1\} = (2 \circ 0) \cup \{2 \circ 1\} = \{2\} \cup \{2\} = \{2\}$. So by hypothesis $(2 \circ 0) \circ (0 \circ 2) < \{0, 1\}$, therefore $\{2\} < \{0, 1\}$, which implies that $2 < 1$. Thus we obtain a contradiction, because H satisfies the simple condition.

(b) If $y = 1$, then $(2 \circ 0) \circ (1 \circ 2) = \{2\} \circ \{1\} = \{2\}$. By hypothesis $\{2\} < \{0, 1\}$. Therefore $2 < 1$, which is a contradiction.

(c) If $y = 2$, then by (6), $2 \circ 2 \subseteq \{0, 1\}$. So $(2 \circ 0) \circ (2 \circ 2) = 2 \circ (2 \circ 2) \subseteq 2 \circ \{0, 1\} = (2 \circ 0) \cup (2 \circ 1) = \{2\} \cup \{2\} = \{2\}$. By hypothesis $\{2\} < \{0, 1\}$, hence $2 < 1$, which is a contradiction.

(ii) Let $z = 1$. Then a similar argument as the case of (i), gives a contradiction.

Note that by hypothesis $z \in I$ so $z \neq 2$. Hence $x = 2$ is impossible i.e., $x \neq 2$. Thus $x \in I$, which implies that I is an implicative hyper K-ideal. Conversely, let I be an implicative hyper K-ideal. Without loss of generality assume that $I = \{0, 1\}$. Let $(x \circ y) \circ z < I$ and $y \circ z < I$ for $x, y, z \in H$, we must show that $x \circ z \subseteq I$. By Theorem 3.17 [11], we know that $1 \circ 0 = \{1\}$, $2 \circ 0 = \{2\}$, $1 \circ 2 \neq \{2\}$ and $2 \circ 1 \neq \{1\}$. Now we show that

(I) $1 \circ 2 = \{1\}$

(II) $2 \circ 1 = \{2\}$

(III) $x \circ y \neq \{0, 2\}$, $x \circ y \neq \{0, 1, 2\}$; for all $x, y \in H$.

(I): Let $1 \circ 2 \neq \{1\}$. Then $1 \not< 2$, since H is simple. Thus $0 \notin 1 \circ 2$, therefore we must have $1 \circ 2 = \{1, 2\}$. But

$$0 \in 2 \circ 2 \subseteq (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) = (2 \circ 0) \circ (1 \circ 2).$$

So $(2 \circ 0) \circ (1 \circ 2) < I$. Since $0 \in I$, we conclude that $2 \in I$, which is a contradiction. Hence $1 \circ 2 = \{1\}$.

(II): Suppose $2 \circ 1 \neq \{2\}$. Since $2 \not< 1$, $0 \notin 2 \circ 1$ and since $2 \circ 1 \neq \{1\}$, thus we must have $2 \circ 1 = \{2, 1\}$. Now $\{1, 2\} = 2 \circ 1 = (2 \circ 0) \circ (1 \circ 2)$, by (I), that is $(2 \circ 0) \circ (1 \circ 2) < I$. Since $0 \in I$ and I is implicative we get that $2 \in I$ which is a contradiction. Hence $2 \circ 1 = \{2\}$.

(III): By considering (I) and (II), it remains to show that none of $0 \circ 0$, $0 \circ 1$, $1 \circ 1$ and $2 \circ 2$ are equal to $\{0, 2\}$ or $\{0, 1, 2\}$. Clearly all of them contain 0, so we show that none of them contain 2.

(a) $2 \notin 2 \circ 2$: Let $2 \in 2 \circ 2$. Then by (II) we have $0 \in 2 \circ 2 \subseteq 2 \circ (2 \circ 2) = (2 \circ 1) \circ (2 \circ 2)$, hence $(2 \circ 1) \circ (2 \circ 2) < I$. Since $1 \in I$, then $2 \in I$, which is a contradiction. Therefore $2 \notin 2 \circ 2$.

(b) The proof of $2 \notin 0 \circ 2$ is similar as (a).

(c) $2 \notin 0 \circ 1$: Let $2 \in 0 \circ 1$. Then by (HK3) and (HK2) we have $2 \in 0 \circ 1 \subseteq (2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. By (I), $(2 \circ 1) \circ 2 = 2 \circ 2$, so $2 \in 2 \circ 2$, which is in contradiction with (a).

(d) $2 \notin 1 \circ 1$: Let $2 \in 1 \circ 1$. Then by (HK2) and (I) we have

$$2 \in 1 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2. \tag{7}$$

Since $0 \in 1 \circ 1$ and $2 \in 1 \circ 1$, then $1 \circ 1$ contains $\{0, 2\}$. Thus $1 \circ 1 = \{0, 2\}$ or $\{0, 1, 2\}$. If $1 \circ 1 = \{0, 1, 2\}$, then by (7), (I) and (II) we have

$$2 \in (1 \circ 1) \circ 2 = \{0, 1, 2\} \circ 2 = (0 \circ 2) \cup (1 \circ 2) \cup (2 \circ 2) \subseteq \{0, 1\},$$

which is a contradiction. If $1 \circ 1 = \{0, 2\}$, then similarly we get a contradiction.

(e) $2 \notin 0 \circ 0$: Let $2 \in 0 \circ 0$. Then by (HK2), (HK3) and (d) we have $2 \in 0 \circ 0 \subseteq (1 \circ 1) \circ 0 = (1 \circ 0) \circ 1 = 1 \circ 1 \subseteq \{0, 1\}$, which is a contradiction. Thus (III) is proved.

Now, (III) imposes that $(H, \circ, 0)$ must have the following hyper structure table:

$\circ$	0	1	2
0	$\{0\}$or$\{0,1\}$	$\{0\}$or$\{0,1\}$	$\{0\}$or$\{0,1\}$
1	$\{1\}$	$\{0\}$or$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$or$\{0,1\}$

As we see, in the above table except the cases $2 \circ 0 = \{2\}$ and $2 \circ 1 = \{2\}$, the other possible cases of $x \circ z$ are subsets of I. That is $x \circ z \subseteq I$. Now we prove that if $x = 2$, $z = 0$ or $x = 2$, $z = 1$, then $(x \circ y) \circ z \nless I$, or $y \circ z \nless I$. Therefore the proof will be completed.

First let $x = 2$ and $z = 0$. If $y = 0$, then we have

$$2 = 2 \circ 0 = (2 \circ 0) \circ 0 < I = \{0, 1\},$$

which is a contradiction. Similarly for $y = 1$ or $y = 2$ we obtain a contradiction.

Now, if $x = 2$ and $z = 1$, then by a similar argument as above we give a contradiction. Hence we proved that if $(x \circ y) \circ z < I$, and $y \circ z < I$, then $x \circ z \subseteq I$, for all $x, y, z \in H$. Thus I is a positive implicative hyper K-ideal of type 3. □

COROLLARY 4.16. *Let $H = \{0,1,2\}$ be a hyper K-algebra of order 3, that satisfies the simple condition and I be an implicative hyper K-ideal of H. Then I is a positive hyper K-ideal of types 1-8.*

Proof. The proof follows from Theorem 4.15 and Theorem 3.5 of [1]. □

THEOREM 4.17. *There are 12 non-isomorphic hyper K-algebras of order 3, with simple condition such that they have at least one proper implicative hyper K-ideal.*

Proof. The proof follows from Theorems 3.20 and 3.21 of [11] and Theorem 4.15. □

THEOREM 4.18. *Let I be an implicative hyper K-ideal of H, that satisfies the strong transitive condition, A be a hyper K-ideal of H that contains I. Then A is an implicative hyper K-ideal of H.*

Proof. Let $x \circ (y \circ x) < A$, we prove that $x \in A$. By Theorem 2.3 (ix) we have $x \circ A < y \circ x$. Since $I \subseteq A$, we get that $x \circ I < x \circ A$, hence $x \circ I < y \circ x$. Thus $x \circ (y \circ x) < I$, by Theorem 2.3 (ix). Since I is an implicative hyper K-ideal we get that $x \in I$, so $x \in A$. Therefore by Theorem 4.12 A is an implicative hyper K-ideal of H. □

THEOREM 4.19. *If $\{I_i | i \in \Lambda\}$ is a family of (weak) implicative hyper K-ideals, then $\bigcap_{i\in\Lambda} I_i$ is also a (weak) implicative hyper K-ideal.*

Proof. The proof is straightforward. □

THEOREM 4.20. *Let $(H, *, 0)$ be a BCK-algebra and I be a nonempty subset of H which satisfies the additive condition. If we consider the hyperoperation $x \circ y = \{x * y\}$ on H, then I is a weak implicative hyper K-ideal of H if and only if I is an implicative hyper K-ideal of H.*

Proof. The proof is easy. □

OPEN PROBLEM. *Under what suitable condition each weak implicative hyper K-ideal is an implicative hyper K-ideal?*

References

[1] R. A. Borzooei, P. Corsini and M. M. Zahedi, *Some kinds of positive Implicative hyper K-ideals,* Journal of Discrete Mathematical Sciences and Cryptography, Delhi, (to appear).

[2] R. A. Borzooei, A. Hasankhani, M. M. Zahedi and Y. B. Jun, *On hyper K-algebras,* Math. Japon. **52** (2000), no. 1, 113–121.

[3] R. A. Borzooei and M. M. Zahedi, *Positive Implicative hyper K-ideals,* Sci. Math. Jpn. **53** (2001), no. 3, 525–533.
[4] P. Corsini, *Prolegomena of Hypergroup Theory,* Aviani Editore, 1993.
[5] Y. Imai and K. Iseki, *On axiom systems of propositional calculi,* XIV Proc. Japan Academy **42** (1966), 19–22.
[6] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras,* Math. Japon **23** (1978), 1–26.
[7] Y. B. Jun, X. L. Xin, E. H. Roh and M. M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebras,* Math. Japon **51** (2000), no. 3, 493–498.
[8] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei, *On hyper BCK-algebras,* Italian Journal of Pure and Applied Mathematics, (2000), no. 8, 127–136.
[9] F. Marty, *Sur une generalization de la notion de groups,* 8th congress Math. Scandinaves, Stockholm, (1934), 45–49.
[10] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moonsa, Seoul, Korea, 1994.
[11] M. M. Zahedi, R. A. Borzooei and H. Rezaei, *Some classification of hyper K-algebras of order 3,* Sci. Math. Jpn. **53** (2001), no. 1, 133–142.
[12] M. M. Zahedi, R. A. Borzooei, Y. B. Jun and A. Hasankhani, *Some results on hyper K-algebra,* Sci. Math. **3** (2000), no. 1, 53–59.

A. Borumand Saeid, Department of Mathematics, Islamic Azad University of Kerman, Kerman, Iran
E-mail: arsham@iauk.ac.ir

R. A. Borzooei, Department of Mathematics, Sistan and Baluchestan University, Zahedan, Iran
E-mail: borzooei@hamoon.usb.ac.ir

M. M. Zahedi, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
E-mail: zahedi_mm@mail.uk.ac.ir