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YANG-MILLS OR YANG-MILLS-HIGGS FIELDS OVER KAEHLER AND CONTACT MANIFOLDS

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ABSTRACT. In this paper we give a characterization of an irreducible connection with harmonic curvature over a connected Kaehler manifold to be *self-dual*. Also we introduce new notions of c_i-self-dual or Kaehler Yang-Mills connections on compact Kaehler manifolds and investigate some fundamental properties of this kind of new connections. Moreover, on a compact odd dimensional Riemannian manifold we give a property of generalized monopole.

0. Introduction

Let M be a compact oriented Riemannian manifold, and $\mathbb{P}$ be a principal fiber bundle with compact structure group G . Now we denote by A a connection on a principal fiber bundle $\mathbb{P}$ and by F_A the curvature form of A which is the adjoint bundle $\mathfrak{g}_P = \mathbb{P}\times_{Ad}\mathfrak{g}$ valued 2-form defined on M, where $\mathfrak{g}$ denotes the Lie algebra of the Lie group G. Then Yang-Mills functional is defined by

$$\mathfrak{Y}M(A) = \frac{1}{2}\int_M \|F_A\|^2 dvol_M. \tag{0.1}$$

It is known that the curvature two form F_A of A satisfies Euler-Lagrange equation such that $d_A F_A = 0$ and $d_A * F_A = 0$. The first of this equation is called the second Bianchi identity and the second corresponds to the critical points of the Yang-Mills Functional (0.1), that is, *Yang-Mills* connection.

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When M is a Kaehler manifold of complex dimension 2, that is, a Kaehler surface, the Hodge * operator determines a decomposition

$$\Lambda^2 T^* M = \Lambda_+^2 \oplus \Lambda_-^2$$

of the space of 2-forms, where $\Lambda_\pm^2$ denotes the eigenspace subbundle of the Hodge $*$ operator corresponding to eigenvalues ± 1. So from $*^2 = id$ it follows that the adjoint bundle $\mathfrak{g}_P = \mathbb{P} \times_{Ad} \mathfrak{g}$ valued 2-form $F_A = dA + \frac{1}{2}[A \wedge A]$ can be splitted into $F^+ = \frac{1}{2}(F_A + *F_A)$ and $F^- = \frac{1}{2}(F_A - *F_A)$, which are said to be the *self-dual* part and the *anti-self-dual* part of F_A respectively. Thus a connection A on a principal fibre bundle $\mathbb{P}$ over a Kaehler surface M being *Yang-Mills* is equivalent to $d_A F^+ = 0$ or $d_A F^- = 0$.

When M is a compact oriented Riemannian manifold of odd dimension 3, we consider a 3-dimensional Yang-Mills-Higgs field and magnetic monopole (Φ, A), which satisfies Bogomolny equation such that $F_A = \pm * \nabla_A \Phi$. Then they correspond to respectively Yang-Mills field and instanton of the curvature $F_A = dA + [A \wedge A]$, which satisfies $*F_A = \pm F_A$ of the connection A defined on a Kaehler surface.

Now let us apply the above situation to higher dimensional manifolds. So in this paper as a base manifold we consider a higher dimensional Kaehler manifold of complex dimension n or a higher dimensional contact manifold $2n + 1$. Firstly we want to give a characterization of *self-duality* of the connection in a higher dimensional Kaehler manifold in terms of the second Chern class of the complex vector bundle $E = \mathbb{P} \times_{SU(r)} \mathbb{C}^r$. Namely we assert the following:

THEOREM 1. *Let M be a connected Kaehler manifold. Let A be an irreducible connection with harmonic curvature. Then*

$$\mathfrak{YM}(A) \geq -\frac{1}{2} \int_M C(\mathbb{P}) \wedge \frac{\Phi^{n-2}}{(n-2)!},$$

where $C(\mathbb{P}) = \mathrm{Tr} F_A \wedge F_A = 8\pi^2 c_2(E)$, $E = \mathbb{P} \times_{SU(r)} C^r$, where the equality holds if and only if A is a self-dual.

Secondly, we want to assert a property of the generalized monopole (A, ϕ) over a compact contact odd-dimensional Riemannian manifold.

THEOREM 2. *Let M be a compact oriented contact manifold and let (A, ϕ) be a generalized monopole. Then $F_A = 0$ and $\nabla_A \Phi = 0$.*

Moreover, in section 3 we introduce the notion of c_i-self-dual connection and find some *topological charge* of the principal fiber bundle $\mathbb{P}$ over a compact connected Kaehler manifold. Finally, we introduce the notion of Kaehler Yang-Mills connection and also assert that this connection could be a kind of Yang-Mills connection as the following:

THEOREM 3. *If a connection ∇_A is a Kaehler Yang-Mills connection, then ∇_A is a Yang-Mills connection.*

1. A characterization of self-dual connections over Kaehler manifolds

Let M be an n-dimensional compact complex manifold with a Kaehler metric g. Let us denote by Φ its Kaehler form. When M is a compact Kaehler surface, the Hodge $*$ operator is involutive. Thus it is natural that we consider a self-dual (or anti-self-dual) 2 form of the curvature form F_A. But in order to make a sense in a higher dimensional manifold we have introduced an operator # as follows (See S. Kobayashi [5], pages 60-63).

Let us denote by $A' = \sum A^p$ the exterior algebra of all smooth real valued forms on M. Then we can define the Lipschitz operator L by $L\phi = \phi \wedge \Phi$, $\phi \in A'$ and its adjoint $\Lambda : A' \to A'$. Then it is known that $*$, L and Λ satisfy the following relations:

$$\Lambda = L^* = *^{-1} \circ L \circ *, \; (\Lambda L - L\Lambda)|_{A^k} = n - k, \; \Lambda(\Phi) = n, \tag{1.1}$$

$$*^2|A^k = (-1)^{k(n-k)}, \tag{1.2}$$

$$*(\frac{\Phi^k}{k!}) = \frac{\Phi^{n-k}}{(n-k)!}, \quad k = 0, 1, \cdots, n. \tag{1.3}$$

Now let us denote by $A^{p,q}$ the space of $C^\infty - (p,q)$ forms on M and by $A_0^{p,q}$ the space of primitive (p,q) forms, that is,

$$A_0^{p,q} = \{\alpha \in A^{p,q} | \Lambda\alpha = 0\}.$$

Then the space of all of 2-forms A^2 can be decomposed in such a way that

$$A^2 = A^{2,0} + A^{0,2} + A_0^{1,1} + A_\Phi^{1,1},$$

where $A_\Phi^{1,1}$ denotes the space of (1,1)-type proportional to the Kaehler form Φ. Now we introduce an operator $\#$ which is defined in such a way that

$$\# : A^2 \overset{\frac{L^{n-2}}{(n-2)!}}{\longrightarrow} A^{2(n-1)} \overset{*^{-1}=*}{\longrightarrow} A^2, \quad \text{i.e.,} \quad \# = *^{-1} \circ \frac{L^{(n-2)}}{(n-2)!}.$$

Then by the above definition of the operator $\#$ and a lemma given by R.O. Well [7] we also assert the following

LEMMA 1.1.
(i) $A_0^{1,1} = \{\alpha \in A^2 | \#\alpha = -\alpha\}$,
(ii) $A^{2,0} + A^{0,2} = \{\alpha \in A^2 | \#\alpha = \alpha\}$,
(iii) $A_\Phi^{1,1} = \{\alpha \in A^2 | \#\alpha = (n-1)\alpha\}$.

Then by Lemma 1.1 we can define a new operator $\tilde{\#}$ in such a way that

$$\tilde{\#} = \begin{cases} \# & \text{on} \quad A^{2,0} + A^{0,2} + A_0^{1,1}, \\ \frac{\#}{n-1} & \text{on} \quad A_\Phi^{1,1}. \end{cases}$$

Then the fact $\tilde{\#}^2 = id$ implies that A^2 can be decomposed into the *self-dual* part $A_+^2 = A^{2,0} + A^{0,2} + A_\Phi^{1,1}$ and the anti-self-dual part $A_0^{1,1}$. Hence the curvature form F_A also can be splitted into the *self-dual* part $F^+ = F^{2,0} + F^{0,2} + F^0 \otimes \Phi$ and *anti-self-dual* part $F^- = F_0^{1,1}$. That is, we have

$$\tilde{\#} F^+ = F^+, \quad \text{and} \quad \tilde{\#} F^- = -F^-.$$

When the anti-self-dual part F^- (or self-dual part F^+) vanishes, the connection A is said to be *self-dual* (or *anti-self-dual*) respectively. Let $\mathbb{P}$ be a principal fibre bundle over a compact Kaehler manifold M with a compact semi-simple Lie group G. Let A be a connection on $\mathbb{P}$. Then in the paper [6] the second author proved that

PROPOSITION A. *The following conditions are equivalent.*
(i) *A is Yang-Mills, i.e.,* $d_A * F_A = 0$,
(ii) $d_A \# F_A = 0$,
(iii) $2\bar{\partial}_A^* F^{2,0} + n\partial_A(F^0 \otimes \Phi) = 0$,

(iv) $\partial_A^* F^{2,0} = -ni\partial_A F^0/(2n-1)$.

Let $\mathbb{P}$ be a principal fibre bundle over a compact Kaehler manifold M with structure group $G = SU(r)$. And let A be a connection in $\mathbb{P}$. Then it is well known that Yang-Mills functional $\mathfrak{YM}(A)$ is given by

$$\mathfrak{YM}(A) = \frac{1}{2}\int_M (-Tr)(F_A \wedge * F_A) = \frac{1}{2}\int_M \|F_A\|^2 \frac{\Phi^n}{n!},$$

where $\frac{\Phi^n}{n!}$ is the volume of the compact Kaehler manifold M. Now we have the following

LEMMA 1.2.

$$-TrF_A \wedge * F_A = -\ TrF_A \wedge F_A \wedge \frac{\Phi^n}{(n-2)!} + 2\|F_\Phi^{1,1}\|^2 vol_\Phi - (n-2)\|F^0 \otimes \Phi\|^2 vol_\Phi,$$

where $vol_\Phi = \frac{\Phi^n}{n!}$.

Proof. The curvature F_A can be decomposed into the self-dual part and the anti-self-dual part in such a way that

$$F_A = F^{2,0} + F^{0,2} + F^0 \otimes \Phi + F_0^{1,1}.$$

Then the definition of # yields

$$\# F_A = *(F_A \wedge \frac{\Phi^{n-2}}{(n-2)!}) = F^{2,0} + F^{0,2} + (n-1)F^0 \otimes \Phi - F_0^{1,1}.$$

By applying the Hodge * operator to the second equality we have

$$\begin{aligned} & *(F^{2,0} + F^{0,2}) + (n-1) * F^0 \otimes \Phi - *F_0^{1,1} \\ &= F_A \wedge \frac{\Phi^{n-2}}{(n-2)!} \\ &= (F^{2,0} + F^{0,2} + F^0 \otimes \Phi + F_0^{1,1}) \wedge \frac{\Phi^{n-2}}{(n-2)!}. \end{aligned}$$

Then it follows

$$(1.4) \qquad *F_A = (F^{2,0} + F^{0,2}) \wedge \frac{\Phi^{n-2}}{(n-2)!} + F^0 \otimes \frac{\Phi^{n-1}}{(n-1)!} - F_0^{1,1} \wedge \frac{\Phi^{n-2}}{(n-2)!}.$$

Combining the above equations, we have

(1.5)
$$\begin{aligned}\mathrm{Tr}F_A\wedge *F_A &= \mathrm{Tr}(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^{n-2}}{(n-2)!}\\ &\quad + \mathrm{Tr}F^0\otimes F^0\otimes\frac{\Phi^n}{(n-1)!} - \mathrm{Tr}F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^n}{(n-2)!},\end{aligned}$$

(1.6)
$$\begin{aligned}\mathrm{Tr}F_A\wedge F_A\wedge\frac{\Phi^n}{(n-2)!} &= \mathrm{Tr}(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\frac{\Phi^n}{(n-2)!}\\ &\quad + \mathrm{Tr}F^0\otimes F^0\cdot\frac{\Phi^n}{(n-2)!} + \mathrm{Tr}F_0^{1,1}\wedge F_0^{1,1}\wedge\frac{\Phi^{n-2}}{(n-2)!}.\end{aligned}$$

Combining (1.5) and (1.6), we get the Lemma 1.2. □

Now let us assume that a connection A on a Kaehler manifold M is said to be with harmonic curvature if $F^{2,0}$ is harmonic. Then by Proposition A we have

$$0 = \partial_A^* F^{2,0} = -ni\partial_A F^0/2(n-1).$$

Then from the irreducibility of the connection of M we have $F^0 = 0$. Thus Lemma 1.2 becomes

$$-\mathrm{Tr}F\wedge *F = -\mathrm{Tr}F\wedge F\wedge\frac{\Phi^n}{(n-2)!} + 2|F_0^{1,1}|^2 vol_\Phi.$$

From these formulas we complete the proof of our Theorem 1.

2. Generalized monopole over contact manifolds

In this section we will prove Theorem 2. Now let $\mathbb{P}\to M$ be a G-principal bundle over a complete open oriented Riemannian manifold of dimension $2n+1$. We call M a contact manifold if M has a 1-form η such that (A $(2n+1)$-form $\eta\wedge(d\eta)^n$ is non-zero over M. In this case such a 1-form η is called a contact form).

Set $\omega = d\eta$. Then the form ω is a closed 2-form. Let (A,Φ) be a smooth connection on $\mathbb{P}$ and a smooth section of the adjoint bundle

$\mathfrak{g}_P = \mathbb{P}\times_{Ad}\mathfrak{g}$, called a Higgs field. In what follows, we call a pair (A, Φ) a configuration. The Yang-Mills-Higgs functional $\mathcal{A}(A, \Phi)$ is defined as

$$\mathcal{A}(A, \Phi) = \frac{1}{2}\int_M \{|F_A|^2 + |\nabla_A \Phi|^2\} dv_g. \tag{2.1}$$

We call such a configuration Yang-Mills-Higgs field when the above functional $\mathcal{A}$ is stationary at this configuration (See M. Itoh [3]).

The Euler-Lagrange equations for the first variation of $\mathcal{A}$ are

$$d_A(*F_A) + [\Phi, *\nabla_A \Phi] = 0, \quad d_A(*\nabla_A \Phi) = 0. \tag{2.2}$$

Here $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature form of A and ∇_A, d_A are the covariant derivative and the covariant exterior derivative in the adjoint bundle $\mathfrak{g}_P$, respectively. Furthermore $*$ denotes the Hodge star operator.

A configuration (A, Φ) which satisfies the Bogomolny equation

$$*F_A = \pm \nabla_A \Phi \tag{2.3}$$

is said to be a (magnetic) monopole. It can be easily verified by using the Bianchi identity and the Ricci identity that a monopole satisfies the Euler-Lagrange equations and hence is Yang-Mills-Higgs.

DEFINITION. Let $\mathbb{P} \to M$ be a G-principal bundle over a complete open contact manifold M. A configuration (A, Φ) on P is called a generalized monopole if (A, Φ) satisfies the generalized Bogomolny equations

$$*F_A = c\nabla_A \Phi \wedge \omega^{n-1}, \quad *\nabla_A \Phi = cF_A \wedge \omega^{n-1}, \tag{2.4}$$

where c is a constant.

It is clear that when $dim\ M = 3$, the formula (2.4) reduces to the simple equation (2.3) which is free from any contact form on M. In this section we want to prove the following

THEOREM 2.1. *Let M be a compact oriented contact manifold and let (A, ϕ) be a generalized monopole. Then $F_A = 0$, and $\nabla_A \Phi = 0$.*

Proof. As is known to us, the generalized monopole satisfies

$$*\nabla_A\Phi = cF_A \wedge \omega^{n-1},$$

where $\omega = d\eta$ is a closed 2-form. From this it follows

$$\nabla_A\Phi = c * (F_A \wedge \omega^{n-1}).$$

Then by virtue of Bianchi identity and $d\omega^{n-1} = 0$ we have

$$\begin{aligned} \nabla_A^*\nabla_A\Phi &= c\nabla_A^*(*(F_A \wedge \omega^{n-1})) \\ &= - c * d_A * (*(F_A \wedge \omega^{n-1})) \\ &= - c * d_A(F_A \wedge \omega^{n-1}) \\ &= - c * \{d_A F_A \wedge \omega^{n-1} + F_A \wedge d\omega^{n-1}\} \\ &= 0. \end{aligned}$$

From this, integrating over M, we have

$$0 = \int_M (\nabla_A^*\nabla_A\Phi, \Phi) dv_g = \int_M \|\nabla_A\Phi\|^2 dv_g.$$

From this and the Bogomolny equation it follows

$$*F_A = \pm\nabla_A\Phi = 0.$$

Thus we conclude the proof of Theorem 2.1. □

3. c-self-dual connections

In this section we introduce a new notion of c-self-dual connection over compact Kaehler manifold and will investigate some fundamental properties of this kind of connections. For this we define the following notion.

DEFINITION 3.1. When $*F_A = cF_A \wedge \Phi^{n-2}$, we say the connection A is said to be c-self-dual. More explicitly, the connection A is said to be c_i*-self-dual* (resp. *anti-self-dual*) if $*F_A = c_i F_A \wedge \Phi^{n-2}$.

Then the definition in above gives the following

THEOREM 3.1. *Any c-self-dual connection is an extremum of the Yang-Mills energy functional. That is, it is a Yang-Mills connection.*

Proof. From the definition of the c-self-dual connection we know that

$$*F_A = cF_A \wedge \Phi^{n-2}$$

Then by the exterior derivative and Bianchi identity, we have

$$d_A(*F_A) = c d_A(F_A \wedge \Phi^{n-2}) = 0.$$

So by Proposition A in section 1 we assert that A is a Yang-Mills connection. □

In particular, when $dim_C M = 2$, that is M is a Kaehler surface, $c^2 = 1$. So we can divide two cases in this situation. Then the connection A is said to be self-dual if $c = 1$ and the connection A is said to be anti-self-dual if $c = -1$.

On the other hand, from the proof of Lemma 1.1 we know that

$$(3.1) \qquad \begin{cases} *(F^{2,0} + F^{0,2}) & = (F^{2,0} + F^{0,2}) \wedge \frac{\Phi^{n-2}}{(n-2)!}, \quad c_1 = \frac{1}{(n-2)!} \\ *(F^0 \otimes \Phi) & = (F^0 \otimes \Phi) \frac{\Phi^{n-2}}{(n-1)!}, \quad c_2 = \frac{1}{(n-1)!} \\ *F_0^{1,1} & = -F_0^{1,1} \wedge \frac{\Phi^{n-2}}{(n-2)!}, \quad c_3 = -\frac{1}{(n-2)!}. \end{cases}$$

From the definition of c_i-self-dual connections we have known that they are Yang-Mills connections. Now let us introduce a generalized Yang-Mills functional which is defined by

$$\mathfrak{YM}_C(A) = \frac{1}{2} \int_M [\|F\|^2 + c^2 \|F \wedge \Phi^{n-2}\|^2].$$

Note that

$$\begin{aligned} 0 &\leq \| *F - cF \wedge \Phi^{n-2}\|^2 = \| *F - cF \wedge \Phi^{n-2}\|^2 \\ &= \| *F\|^2 - 2 < *F, cF \wedge \Phi^{n-2} > + c^2 \|F \wedge \Phi^{n-2}\|^2 \\ &= \| *F\|^2 - 2c(trF \wedge F) \wedge \Phi^{n-2} + c^2 \|F \wedge \Phi^{n-2}\|^2 \\ &= \|F\|^2 - 16c\pi^2 c_2(E) \wedge \Phi^{n-2} + c^2 \|F \wedge \Phi^{n-2}\|^2, \end{aligned}$$

where $c_2(E)$ denotes the second Chern class of the vector bundle $E = \mathbb{P}\times_{SU(r)}C^r$.

Integrating over M, we get

$$8\pi^2 c\int_M c_2(E)\wedge\Phi^{n-2}vol(M)\leq\mathfrak{YM}_C(A),$$

where the above equality holds if and only if $*F_A = c_i F_A\wedge\Phi^{n-2}$. Moreover in this case it is equivalent to the fact that the connection A is a c_i-self-dual connection.

Now summing up all of situations in above, we summarize as follows:

THEOREM 3.2. *Any c-self-dual connection is minimum of the generalized Yang-Mills energy functional* $\mathfrak{YM}_C(A)$.

Now we define its lower bound in Theorem 3.1 by a topological charge of the bundle $\mathbb{P}$. Then it can be represented by

$$\begin{aligned}Q(\mathbb{P}) &= 8\pi^2\int_M c_2(E)\wedge\Phi^{n-2}vol(M)\\ &=\int_M Tr(F_A\wedge F_A)\wedge\Phi^{n-2}vol(M).\end{aligned}$$

In the above formula let us calculate more explicitly

$$\begin{aligned}Tr(F_A\wedge F_A)\wedge\Phi^{n-2} &= Tr(F^{2,0}+F^{0,2})\wedge(F^{2,0}+F^{0,2})\wedge\Phi^{n-2}\\ &\quad + TrF_2\wedge F_2\wedge\Phi^{n-2} + TrF_3\wedge F_3\wedge\Phi^{n-2}\\ &= \frac{1}{c_1}TrF_1\wedge *F_1 + \frac{1}{c_2}TrF_2\wedge *F_2 + \frac{1}{c_3}TrF_3\wedge *F_3\\ &= \frac{1}{c_1}\|F_1\|^2 + \frac{1}{c_2}\|F_2\|^2 + \frac{1}{c_3}\|F_3\|^2.\end{aligned}$$

Thus the topological charge $Q(\mathbb{P})$ of the bundle $\mathbb{P}$ becomes

$$\begin{aligned}Q(\mathbb{P}) &=\int_M Tr(F\wedge F)\wedge\Phi^{n-2}vol(M)\\ &=\int_M (\frac{1}{c_1}\|F_1\|^2 + \frac{1}{c_2}\|F_2\|^2 + \frac{1}{c_3}\|F_3\|^2)vol(M).\end{aligned}$$

On the other hand, we know that the Yang-Mills functional is given by

$$\begin{aligned}\mathfrak{YM}(A) &= \frac{1}{2}\int_M -Tr(F_A \wedge * F_A)\\ &= \frac{1}{2}\int_M (\|F^{2,0}\|^2 + \|F^{0,2}\|^2 + \|F^{1,1}\|^2) vol(M)\\ &= \frac{1}{2}\int_M (\|F_1\|^2 + \|F_2\|^2 + \|F_3\|^2) vol(M).\end{aligned}$$

From this we are able to write respectively the following formulas:

(3.2)

$$\begin{aligned}2\mathfrak{YM}(A) &= c_1 Q(\mathbb{P}) + \int_M ((1-\frac{c_1}{c_2})\|F_2\|^2 + (1-\frac{c_1}{c_3})\|F_3\|^2) vol(M)\\ &= c_1 Q(\mathbb{P}) + \int_M ((2-n)\|F_2\|^2 + 2\|F_3\|^2) vol(M)\end{aligned}$$

(3.3)

$$\begin{aligned}2\mathfrak{YM}(A) &= c_2 Q(\mathbb{P}) + \int_M \{(1-\frac{c_2}{c_1})\|F_1\|^2 + (1-\frac{c_2}{c_3})\|F_3\|^2\} vol(M)\\ &= c_2 Q(\mathbb{P}) + \int_M \{\frac{n}{n-1}\|F_1\|^2 + 2\|F_3\|^2\} vol(M).\end{aligned}$$

(3.4)

$$\begin{aligned}2\mathfrak{YM}(A) &= c_3 Q(\mathbb{P}) + \int_M \{(1-\frac{c_3}{c_1})\|F_1\|^2 + (1-\frac{c_3}{c_2})\|F_2\|^2\} vol(M)\\ &= c_3 Q(\mathbb{P}) + \int_M \{2\|F_1\|^2 + n\|F_2\|^2\} vol(M).\end{aligned}$$

Now let us denote by F_1, F_2 and F_3 in such a way that

$$F_1 = F^{2,0} + F^{0,2}, \ F_2 = F^0 \otimes \Phi, \text{ and } F_3 = F_0^{1,1}.$$

Then from (3.2), (3.3) and (3.4) together with the definition 3.1 we assert the following respectively.

THEOREM 3.3. *Let M be a compact Kaehler manifold and A be a connection on a principal fiber bundle $\mathbb{P}$ over M. Then we have the followings:*

(i) *When $F_2 = 0$, $\mathfrak{YM}(A) = \frac{1}{2}c_1 Q(\mathbb{P})$ holds on M if and only if the connection A is c_3-self-dual.*

(ii) *$\mathfrak{YM}(A) = \frac{1}{2}c_2 Q(\mathbb{P})$ holds on M if and only if $F = F^0 \otimes \Phi$. That is, the connection A is c_2-self-dual.*

(iii) *$\mathfrak{YM}(A) = \frac{1}{2}c_3 Q(\mathbb{P})$ holds on M if and only if the connection A is c_3-self-dual.*

REMARK 3.1. In paper [4] K. Galicki and Y.S. Poon have considered the notions of c-self-dual connection on quaternionic Kaehler manifold M. In such a case the topological charge $Q(\mathbb{P})$ was defined by the first Pontrjagin class of the bundle $\mathbb{P}$ of M and have obtained some fundamental properties different from ours.

4. A Kaehler Yang-Mills connection

In this section we give a complete proof of Theorem 3. In order to prove this let us introduce a new notion of Kaehler Yang-Mills connection.

DEFINITION 4.1. A connection A on a Riemannian vector bundle over a compact Kaehler manifold is called a Kaehler Yang-Mills connection if $\triangle_A(F_A \wedge \Phi^{n-2}) = 0$, where Φ is the Kaehler form.

THEOREM 4.1. *If a connection A is a Kaehler Yang-Mills connection, then A is a Yang-Mills connection.*

Proof. From $\triangle_A(F_A \wedge \Phi^{n-2}) = 0$ we should verify that

$$\delta_A F_A = 0.$$

Since M is compact, the assumption gives $\delta_A(F_A \wedge \Phi^{n-2}) = 0$. Then from this it sufficies to show that $\delta_A F_A = 0$. By the formulas

$$*(\frac{\Phi^k}{k!}) = \frac{\Phi^{n-k}}{(n-k)!},$$

and

$$\frac{1}{(n-2)!}\delta_A(F_A \wedge \Phi^{n-2}) = \delta_A\{F_A \wedge * \frac{\Phi^2}{2!}\} = 0$$

we should verify that the formula $\delta_A\{F_A \wedge * \Phi^2\} = 0$ is equivalent to $\nabla_i F_{ij} = 0$. Now let us calculate the following:

$$\begin{aligned}
F_A \wedge * (\Phi^2) =&(-1)^{\#}\sum_{i,j} F_{ij}\theta^i \wedge \theta^j \wedge \underbrace{\{\theta^1 \wedge \cdots \wedge \theta^n \wedge \bar\theta^1 \wedge \cdots \wedge \bar\theta^n\}}_{\text{(n-2,n-2)}} \\
=&(-1)^{\#}\Big[\sum_{ab} F_{ab}\theta^a \wedge \theta^b \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)} \\
&+ \sum_{\alpha\beta} F_{\alpha\beta}\bar\theta^\alpha \wedge \bar\theta^\beta \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)} \\
&+ \sum_{a\alpha} F_{a\alpha}\theta^a \wedge \bar\theta^\alpha \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)}\Big],
\end{aligned}$$

where $\#$ is given by $\# = \frac{n(n-1)}{2} + \frac{n-2}{2} = \frac{n^2-2}{2}$ and $i, j, \cdots : 1, \cdots, n, \bar 1, \cdots, \bar n$. Thus $\delta_A(F_A \wedge * \Phi^2) = 0$ holds if and only if

$$\begin{aligned}
\text{(4.1)} \qquad 0 =&\sum_{c,b} \nabla_c F_{cb}\theta^b \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)} \\
&+ \sum_{\gamma,\beta} \nabla_\gamma F_{\gamma\beta}\bar\theta^\beta \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)} \\
&+ \sum_{c,\alpha} \nabla_c F_{c\alpha}\bar\theta^\alpha \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)} \\
&- \sum_{\gamma,a} \nabla_\gamma F_{a\gamma}\theta^a \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{(a,b)} \wedge \overbrace{\bar\theta^1 \wedge \cdots \wedge \bar\theta^n}^{(\alpha,\beta)},
\end{aligned}$$

where $\overbrace{\cdots}^{(a,b)}$ means "delete θ^a and θ^b from $\theta^1 \wedge \cdots \wedge \theta^n$". Thus (4.1) implies

$$\begin{aligned}
&\sum_c \nabla_c F_{ca}\theta^a \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{b} - \sum_\gamma \nabla_\gamma F_{a\gamma}\theta^a \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{b} = 0, \\
&\sum_c \nabla_c F_{cb}\theta^b \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{a} - \sum_\gamma \nabla_\gamma F_{b\gamma}\theta^b \wedge \overbrace{\theta^1 \wedge \cdots \wedge \theta^n}^{a} = 0,
\end{aligned}$$

where $\overbrace{\cdots}^{b}$ also denotes "delete θ^b among $\theta^1\wedge\cdots\wedge\theta^n$". This gives

$$\sum_c \nabla_c F_{cb} + \sum_\gamma \nabla_\gamma F_{\gamma b} = 0. \tag{4.2}$$

Also (4.1) gives the following

$$\sum_\gamma \nabla_\gamma F_{\gamma\beta}\bar{\theta}^\beta \wedge \overbrace{\bar{\theta}^1\wedge\cdots\wedge\bar{\theta}^n}^{\alpha} + \sum_c \nabla_c F_{c\beta}\bar{\theta}^\beta \wedge \overbrace{\bar{\theta}^1\wedge\cdots\wedge\bar{\theta}^n}^{\alpha} = 0,$$

$$\sum_\gamma \nabla_\gamma F_{\gamma\alpha}\bar{\theta}^\alpha \wedge \overbrace{\bar{\theta}^1\wedge\cdots\wedge\bar{\theta}^n}^{\beta} + \sum_c \nabla_c F_{c\alpha}\bar{\theta}^\alpha \wedge \overbrace{\bar{\theta}^1\wedge\cdots\wedge\bar{\theta}^n}^{\beta} = 0.$$

Then from this it follows

$$\sum_\gamma \nabla_\gamma F_{\gamma\alpha} + \sum_c \nabla_c F_{c\alpha} = 0. \tag{4.3}$$

Thus summing up the above formulas, we have the followings

$$\sum_i \nabla_i F_{ia} = 0, \quad \sum_i \nabla_i F_{i\gamma} = 0$$

for any index $i, j, \cdots : 1, \cdots, n, \bar{1}, \cdots, \bar{n}$. This implies $\delta_A F_{Ai} = 0$. That is, the curvature F_A satisfies $\delta_A F_A = 0$. The connection A is a Yang-Mills connection. Thus it completes the proof of Theorem 4.1. □

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