

## CYCLIC PRESENTATIONS OF GROUPS AND CYCLIC BRANCHED COVERINGS OF $(1, 1)$ -KNOTS

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ABSTRACT. In this paper we study the connections between cyclic presentations of groups and cyclic branched coverings of  $(1, 1)$ -knots. In particular, we prove that every  $n$ -fold strongly-cyclic branched covering of a  $(1, 1)$ -knot admits a cyclic presentation for the fundamental group encoded by a Heegaard diagram of genus  $n$ .

### 1. Introduction and preliminaries

The problem of determining whether a balanced presentation of a group is geometric (i.e., induced by a Heegaard diagram of a closed orientable 3-manifold) is of considerable interest in geometric topology and has already been examined by many authors (see [11], [22], [26]–[29], [31]). Furthermore, the connections between cyclic coverings of  $\mathbf{S}^3$  branched over knots and cyclic presentations of groups induced by suitable Heegaard diagrams have recently been discussed in several papers (see [1], [5], [7], [15], [16], [18]–[21], [32]).

Note that a finite balanced presentation of a group  $\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$  is said to be a *cyclic presentation* if there exists a word  $w$  in the free group  $F_n$  generated by  $x_1, \dots, x_n$  such that the relators of the presentation are  $r_k = \theta_n^{k-1}(w)$ ,  $k = 1, \dots, n$ , where  $\theta_n : F_n \rightarrow F_n$  denotes the automorphism defined by  $\theta_n(x_i) = x_{i+1} \pmod{n}$ ,  $i = 1, \dots, n$ . This cyclic presentation (and the related group) will be denoted by  $G_n(w)$ . For further details see [17].

We list the most interesting examples:

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- the Fibonacci group  $F(2n) = G_{2n}(x_1 x_2 x_3^{-1}) = G_n(x_1^{-1} x_2^2 x_3^{-1} x_2)$  is the fundamental group of the  $n$ -fold cyclic covering of  $\mathbf{S}^3$  branched over the figure-eight knot, for all  $n > 1$  (see [16]);
- the Sieradsky group  $S(n) = G_n(x_1 x_3 x_2^{-1})$  is the fundamental group of the  $n$ -fold cyclic covering of  $\mathbf{S}^3$  branched over the trefoil knot, for all  $n > 1$  (see [5]);
- the fractional Fibonacci group  $\tilde{F}_{l,k}(n) = G_n((x_1^{-l} x_2^l)^k x_2 (x_3^{-l} x_2^l)^k)$  is the fundamental group of the  $n$ -fold cyclic covering of  $\mathbf{S}^3$  branched over the genus one two-bridge knot with Conway coefficients  $[2l, -2k]$ , for all  $n > 1$  and  $l, k > 0$  (see [32]).

Moreover, all the above cyclic presentations are geometric (i.e., they arise from suitable Heegaard diagrams).

In order to investigate these relations, Dunwoody introduced in [7] a class of Heegaard diagrams depending on six integers, having cyclic symmetry and defining cyclic presentations for the fundamental group of the represented manifolds. In [12] it has been shown that the 3-manifolds represented by these diagrams are cyclic coverings of lens spaces, branched over  $(1, 1)$ -knots (also called genus one 1-bridge knots). As a corollary, it has been proved that for some determined cases the manifolds turn out to be cyclic coverings of  $\mathbf{S}^3$ , branched over suitable knots. This gives a positive answer to a conjecture formulated by Dunwoody in [7]. Section 2 resumes the main statements of [12] concerning this topic.

The above results suggest that cyclic presentations of groups are actually related to cyclic branched coverings of  $(1, 1)$ -knots. As a basic result in this direction, we prove in Section 3 that every  $n$ -fold strongly-cyclic branched covering of a  $(1, 1)$ -knot admits a Heegaard diagram of genus  $n$  which encodes a cyclic presentation for the fundamental group. The definition of strongly-cyclic branched coverings of  $(1, 1)$ -knots will be introduced in Section 3. It is interesting to note that the construction used to prove our main result is strictly related to the concept of  $p$ -symmetric Heegaard splittings introduced by Birman and Hilden in [3].

In what follows, we shall deal with  $(1, 1)$ -knots, i.e., knots in lens spaces (possibly in  $\mathbf{S}^3$ ), which admit a certain decomposition. A knot  $K$  in a lens space  $L(p, q)$  is called a  $(1, 1)$ -knot (or also a *genus one 1-bridge knot*) if there exists a Heegaard splitting of genus one  $(L(p, q), K) = (T, A) \cup_\phi (T', A')$ , where  $T$  and  $T'$  are solid tori,  $A \subset T$  and  $A' \subset T'$  are properly embedded trivial arcs, and  $\phi : (\partial T, \partial A) \rightarrow (\partial T', \partial A')$  is the attaching homeomorphism. This means that there exists a disk  $D \subset T$

(resp.  $D' \subset T'$ ) with  $A \cap D = A \cap \partial D = A$  and  $\partial D - A \subset \partial T$  (resp.  $A' \cap D' = A' \cap \partial D' = A'$  and  $\partial D' - A' \subset \partial T'$ ).

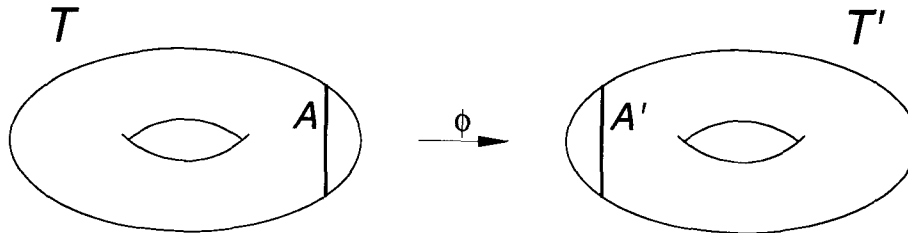


FIGURE 1. A (1, 1)-knot decomposition.

Notice that (1, 1)-knots are only a particular case of the notion of  $(g, b)$ -links in closed orientable 3-manifolds (see [6] and [13]), which generalize the classical concept of bridge decomposition of links in  $S^3$ .

The class of (1, 1)-knots has recently been investigated in several papers (see [2, 8, 9, 10, 12, 13, 14, 23, 24, 25, 33, 34, 35]) and appears very important in the light of some results and conjectures involving Dehn surgery on knots. It is well known that the subclass of (1, 1)-knots in  $S^3$  contains all torus knots (trivially) and all 2-bridge knots (i.e., (0, 2)-knots) [23].

## 2. Dunwoody manifolds

M. J. Dunwoody introduced in [7] a class of Heegaard diagrams having a cyclic symmetry, depending on six integers  $a, b, c, n, r, s$  such that  $n > 0, a, b, c \geq 0$  and  $a + b + c > 0$  (see Figure 2). This construction gives rise to a wide class of closed orientable 3-manifolds  $D(a, b, c, n, r, s)$ , called *Dunwoody manifolds*, admitting geometric cyclic presentations for their fundamental groups.

The diagram is an Heegaard diagram of genus  $n$ . It contains  $n$  upper cycles  $C'_1, \dots, C'_n$  and  $n$  lower cycles  $C''_1, \dots, C''_n$ , each having  $d = 2a + b + c$  vertices. For each  $i = 1, \dots, n$ , the cycle  $C'_i$  (resp.  $C''_i$ ) is connected to the cycle  $C'_{i+1}$  (resp.  $C''_{i+1}$ ) by  $a$  parallel arcs, to the cycle  $C''_i$  by  $c$  parallel arcs and to the cycle  $C''_{i+1}$  by  $b$  parallel arcs. The cycle  $C'_i$  is glued to the cycle  $C''_{i-s}$  (mod  $n$ ) so that equally labelled vertices are identified together (the labelling of the cycles is pointed out in Figure 3).

It is evident that the diagram (as well as the identification rule) is invariant with respect to a cyclic action of order  $n$ .

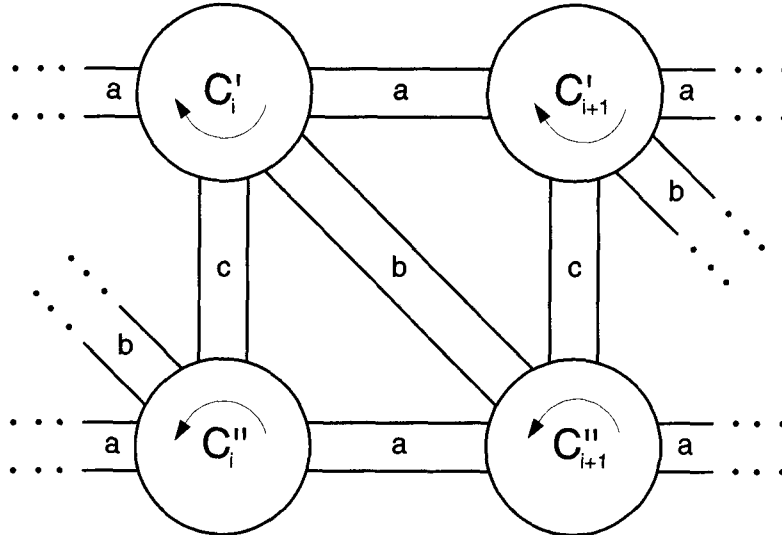


FIGURE 2. Heegaard diagram of Dunwoody type.

In [12] it has been shown that each Dunwoody manifold is a cyclic covering of a lens space (possibly  $\mathbf{S}^3$ ), branched over a  $(1, 1)$ -knot.

**THEOREM 2.1.** [12] *The Dunwoody manifold  $D(a, b, c, n, r, s)$  is the  $n$ -fold cyclic covering of a lens space  $D'$  (possibly  $\mathbf{S}^3$ ) branched over a  $(1, 1)$ -knot  $K \subset D'$ , both only depending on the integers  $a, b, c, r$ .*

As a consequence, for certain particular values of the parameters a Dunwoody manifold turn out to be a cyclic covering of  $\mathbf{S}^3$  branched over a knot. This gives a positive answer to a conjecture formulated by Dunwoody in [7], which has also been independently proved in [30].

**COROLLARY 2.2.** [12] *Each Dunwoody manifold  $D(a, b, c, n, r, s)$  of Section 3 of [7] is an  $n$ -fold cyclic covering of  $\mathbf{S}^3$ , branched over a  $(1, 1)$ -knot  $K \subset \mathbf{S}^3$ , which only depends on  $a, b, c, r$ .*

### 3. Strongly-cyclic branched coverings of $(1, 1)$ -knots

As well known, an  $n$ -fold cyclic branched covering between two orientable closed manifolds  $f : M \rightarrow N$ , with branching set  $L$ , is completely defined by an epimorphism  $\omega_f : H_1(N - L) \rightarrow \mathbf{Z}_n$ , where  $\mathbf{Z}_n$  is

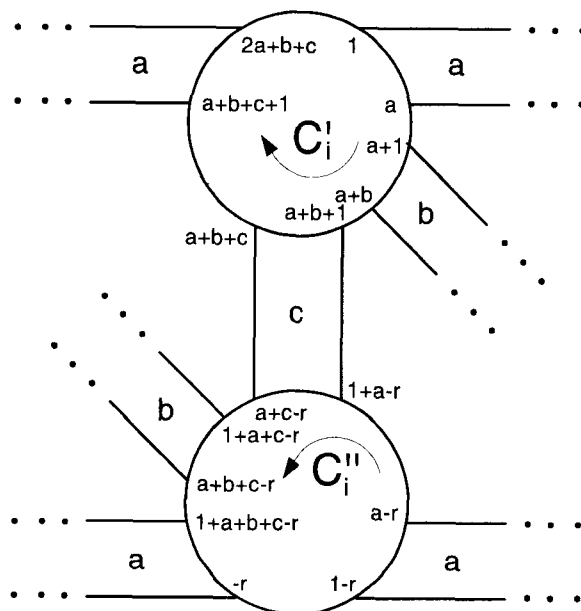


FIGURE 3

the cyclic group of order  $n$ . If  $N = \mathbf{S}^3$  and the branching set is a knot  $K$ , the covering is uniquely determined, up to covering equivalence, since  $H_1(\mathbf{S}^3 - K) \cong \mathbf{Z}$  and the homology class  $[m]$  of a meridian loop around the knot have to be mapped by  $\omega_f$  in a generator of  $\mathbf{Z}_n$ . Therefore the index of the branching set is exactly  $n$ .

Obviously, this property does not hold for a  $(1, 1)$ -knot in a lens space. Moreover, we would like to obtain cyclic branched coverings producing a cyclic presentation for the fundamental group of the manifold. In order to achieve this, we will select cyclic branched coverings of "special type", and this will be a very natural generalization of the case of knots in  $\mathbf{S}^3$ .

An  $n$ -fold cyclic covering of  $L(p, q)$  branched over the  $(1, 1)$ -knot  $K$  will be called *strongly-cyclic* if the branching index of  $K$  is  $n$ . This means that the homology class of a meridian loop  $m$  around  $K$  is mapped by  $\omega_f$  in a generator of  $\mathbf{Z}_n$  (up to covering equivalence we can always suppose  $\omega_f[m] = 1$ ).

Strongly-cyclic branched coverings of  $(1, 1)$ -knots appear to be a suitable tool for producing 3-manifolds with fundamental group admitting cyclic presentation. For example, it is easy to see that all Dunwoody manifolds are coverings of this type.

**THEOREM 3.1.** *Every  $n$ -fold strongly-cyclic branched covering of a  $(1, 1)$ -knot admits a Heegaard diagram of genus  $n$ , which induces a cyclic presentation of the fundamental group of the manifold.*

*Proof.* Let  $f : (M, f^{-1}(K)) \rightarrow (L(p, q), K) = (T, A) \cup_{\phi} (T', A')$  be an  $n$ -fold strongly-cyclic branched covering of the  $(1, 1)$ -knot  $K$ . Then  $Y_n = f^{-1}(T)$  and  $Y'_n = f^{-1}(T')$  are both handlebodies of genus  $n$ . Moreover,  $f^{-1}(A)$  and  $f^{-1}(A')$  are both properly embedded arcs in  $Y_n$  and  $Y'_n$  respectively. We get a genus  $n$  Heegaard splitting  $(M, f^{-1}(K)) = (Y_n, f^{-1}(A)) \cup_{\Phi} (Y'_n, f^{-1}(A'))$ , where  $\Phi : \partial Y_n \rightarrow \partial Y'_n$  is the lifting of  $\phi$  with respect to  $f$ . Let  $m \subset T - A$  be a meridian loop around  $A$  and  $\alpha \subset T - A$  be a generator of  $\pi_1(T, P)$  such that  $\omega_f[\alpha] = 0$ , where the base point  $P$  is any point of  $m$ . The loop  $\alpha$  exists: take a generator  $\alpha' \subset T - A$  of  $\pi_1(T, P)$ ; if  $\omega_f[\alpha'] = k$  then choose any  $\alpha$  homotopic to  $\alpha' m^{-k}$ . Moreover, take a point  $Q \in A$  and let  $\gamma$  be an arc from  $P$  to  $Q$  such that  $\gamma \cap A = Q$ . Then  $f^{-1}(Q)$  is a single point  $* \in f^{-1}(A)$  and  $f^{-1}(P)$  consists of  $n$  points  $P_1, \dots, P_n$ . For  $i = 1, \dots, n$ , let  $\tilde{\alpha}_i$  and  $\tilde{\gamma}_i$  be the lifting (with respect to  $f$ ) of  $\alpha$  and  $\gamma$  respectively, both containing  $P_i$ . Then the  $n$  loops  $\bar{\alpha}_1 = \tilde{\gamma}_1^{-1} \cdot \tilde{\alpha}_1 \cdot \tilde{\gamma}_1, \dots, \bar{\alpha}_n = \tilde{\gamma}_n^{-1} \cdot \tilde{\alpha}_n \cdot \tilde{\gamma}_n$  generate  $\pi_1(Y_n, *)$  and they are cyclically permuted by a generator  $\Psi$  of the group of covering transformations. Let  $E'$  be a meridian disk for the torus  $T'$  such that  $E' \cap A' = \emptyset$ , then  $f^{-1}(E')$  is a system of meridian disks  $\{\tilde{E}'_1, \dots, \tilde{E}'_n\}$  for the handlebody  $Y'_n$ , and they are cyclically permuted by  $\Psi$ . The curves  $\Phi^{-1}(\partial \tilde{E}'_1), \dots, \Phi^{-1}(\partial \tilde{E}'_n)$  give the relators for the presentation of  $\pi_1(M, *)$  induced by the Heegaard splitting. Since both generator and relator curves are cyclically permuted by  $\Psi$ , we get the statement.  $\square$

Obviously several problems arise:

- study the relations between the attaching homeomorphism  $\phi$  producing  $K$  and the monodromy  $\omega_f$  of the strongly-cyclic branched covering;
- find the word  $w$  associated to the cyclic presentation, starting from the  $(1, 1)$ -knot description;
- find some strongly-cyclic branched covering of a  $(1, 1)$ -knot (possibly in  $\mathbf{S}^3$ ) which is not a Dunwoody manifold.

A discussion of the first two problems can be found in [4].

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