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STRONGLY Π-REGULAR MORITA CONTEXTS

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ABSTRACT. In this paper, we show that if the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings is a strongly π-regular ring of bounded index if and only if so are A and B. Furthermore, we extend this result to the ring of a Morita context over quasi-duo strongly π-regular rings.

Let R be an associative ring with identity. We say that R is strongly π-regular if for each $x \in R$ there exists a positive integer $m = m(a)$, depending on a, such that $a^m R = a^{m+1} R$. This concept is left-right symmetric and is equivalent to the condition that every cyclic left or right R-module is co-hopfian. It is well known that every strongly π-regular ring has stable range one and every element in a strongly π-regular ring is either a two-sided zero divisor or a unit. Many authors have studied such rings such as [1], [3]-[6] and [9]-[12].

Recall that a Morita context denoted by (A, B, M, N, ψ, ϕ) consists of two rings A, B, two bimodules ${}_A N_B, {}_B M_A$ and a pair of bimodule homomorphisms (called pairings) $\psi : N \bigotimes_B M \to A$ and $\phi : M \bigotimes_A N \to B$ which satisfy the following associativity:

$$\psi(n \bigotimes m)n' = n\phi(m \bigotimes n'), \quad \phi(m \bigotimes n)m' = m\psi(n \bigotimes m').$$

These conditions insure that the set T of generalized matrices

$$T = \left\{ \begin{pmatrix} a & n \\ m & b \end{pmatrix} \mid a \in A, b \in B, m \in M, n \in N \right\}$$

forms a ring, called the ring of the context (A, B, M, N, ψ, ϕ). In [8], A. Haghany and K. Varadarajan studied Morita contexts with all $N = 0$ (i.e., formal triangular rings). In [6], A. Haghany investigated hopficity and co-hopficity for Morita contexts with zero pairings. He showed that if T is the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings then T is strongly π-regular provided that A and B are strongly π-regular, and that zero divisors in A and B annihilate M and N.

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Following a new route, we now investigate the conditions under which the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings is strongly π-regular. We prove that T is a strongly π-regular ring of bounded index if and only if so are A and B. Furthermore, we extend this result to right (left) quasi-duo strongly π-regular rings.

Throughout, rings are associative with identity. $\mathrm{U}(R)$ denotes the set of units of R and $\mathrm{J}(R)$ denotes the Jacobson radical of R. We always use T to denote the ring of a Morita context (A, B, M, N, ψ, ϕ).

LEMMA 1. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. Then $T/J(T) \cong A/J(A) \oplus B/J(B)$.*

Proof. One easily checks that $\mathrm{J}(T) = \begin{pmatrix} \mathrm{J}(A) & N \\ M & \mathrm{J}(B) \end{pmatrix}$. We construct a map $\theta : T \to \begin{pmatrix} A/\mathrm{J}(A) & 0 \\ 0 & B/\mathrm{J}(B) \end{pmatrix}$ given by $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \mapsto \begin{pmatrix} a + \mathrm{J}(A) & 0 \\ 0 & b + \mathrm{J}(B) \end{pmatrix}$ for any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Because of zero pairings, we claim that θ is a ring epimorphism. Therefore $T/\mathrm{J}(T) \cong T/\mathrm{Ker}(\theta) \cong A/\mathrm{J}(A) \oplus B/\mathrm{J}(B)$, as asserted. □

Recall that a ring R is of bounded index provided that there exists some positive integer n such that $a^n = 0$ for all nilpotent $a \in R$. It is well known that every regular ring (or weakly P-exchange ring) of bounded index is strongly π-regular ring. For the Morita contexts over strongly π-regular rings of bounded index, we derive the following.

THEOREM 2. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. Then T is strongly π-regular of bounded index if and only if so are A and B.*

Proof. Suppose that T is a strongly π-regular ring of bounded index. Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $A \cong eTe$ is also a strongly π-regular ring of bounded index. Likewise, B is a strongly π-regular ring of bounded index, as required.

Conversely, assume now that A and B are both strongly π-regular rings of bounded index. Then $A/\mathrm{J}(A)$ and $B/\mathrm{J}(B)$ are also strongly π-regular. It follows by Lemma 1 that $T/\mathrm{J}(T)$ is strongly π-regular.

Using Lemma 1 again, we have $\mathrm{J}(T) \cong \begin{pmatrix} \mathrm{J}(A) & N \\ M & \mathrm{J}(B) \end{pmatrix}$. Assume that the bounded indices of A and B are s and t respectively. Since A and B are strongly π-regular, $\mathrm{J}(A)$ and $\mathrm{J}(B)$ are nil. Given any

$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in \mathrm{J}(T)$, then $a^{s+t} = 0$ and $b^{s+t} = 0$. So there exist $m_1, m_2 \in M$ and $n_1, n_2 \in N$ such that

$$\begin{aligned} & \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} \\ = & \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{s+t} \begin{pmatrix} a & n \\ m & b \end{pmatrix}^{s+t} \\ = & \begin{pmatrix} a^{s+t} & n_1 \\ m_1 & b^{s+t} \end{pmatrix} \begin{pmatrix} a^{s+t} & n_2 \\ m_2 & b^{s+t} \end{pmatrix} \\ = & \begin{pmatrix} 0 & n_1 \\ m_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & n_2 \\ m_2 & 0 \end{pmatrix} \\ = & \ 0. \end{aligned}$$

Therefore $\mathrm{J}(T)$ is a nil ideal of bounded index.

Assume that the bounded index of $\mathrm{J}(T)$ is k. Given any $x \in T$, we have a positive integer l such that

$$\begin{aligned} & \big(x + \mathrm{J}(T)\big)^l \big(T/\mathrm{J}(T)\big) \\ = & \big(x + \mathrm{J}(T)\big)^{l+1} \big(T/\mathrm{J}(T)\big) \\ = & \big(x + \mathrm{J}(T)\big)^{kl+1} \big(T/\mathrm{J}(T)\big). \end{aligned}$$

So we have a $y+\mathrm{J}(T) \in T/\mathrm{J}(T)$ such that $\big(x+\mathrm{J}(T)\big)^l = \big(x+\mathrm{J}(T)\big)^{kl+1}\big(y+\mathrm{J}(T)\big)$. Hence $x^l - x^{kl+1}y \in \mathrm{J}(T)$. Therefore $(x^l - x^{kl+1}y)^k = 0$. Thus we can find some $z \in T$ such that $x^{kl} = x^{kl+1}z$. That is, T is a strongly π-regular ring.

Suppose that $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^p = 0$ for some $p \geq 1$. One easily checks that $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^p = \begin{pmatrix} a^p & n_3 \\ m_3 & b^p \end{pmatrix}$ for some $m_3 \in M, n_3 \in N$. So $a^p = 0$ in A and $b^p = 0$ in B. Hence $a^s = 0$ and $b^t = 0$. Analogously to the consideration above, we claim that $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0$. Therefore T is of bounded index, as asserted. □

Let $A = B = k[x]/(x^2) = \{a + bt \mid a, b \in k, t^2 = 0\}$, where k is a field of characteristic 2. Take $M = N = k$ made into an A-module by $\alpha * (a + bt) = \alpha a$ with $\alpha, a, b \in k$. By [6, p.488], we know that A and B are both strongly π-regular rings. Assume that $(a + bt)^n = 0$ in A. Then $(a + bt)^{2n} = 0$, hence $a^{2n} = ((a + bt)^2)^n = 0$. So $a = 0$.

Therefore $(a+bt)^2 = a^2 = 0$. That is, $A = B$ is a strongly π-regular ring of bounded index 2. Then with the zero pairings, all the conditions in Theorem 2 hold.

COROLLARY 3. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. If A and B are regular rings of bounded index, then T is a strongly π-regular ring.*

Proof. Since A and B are regular rings of bounded index, they are strongly π-regular. Hence we get the result by Theorem 2. □

COROLLARY 4. *A ring R is a strongly π-regular ring of bounded index if and only if so is the ring of all $n \times n$ lower triangular matrices over R.*

Proof. Suppose that the ring T of all $n \times n$ lower triangular matrices over R is a strongly π-regular ring of bounded index. Then we have an idempotent $e \in T$ such that $R \cong eTe$. Thus we easily check that R is a strongly π-regular ring of bounded index as well.

Conversely, assume that R is a strongly π-regular ring of bounded index. Applying Theorem 2, the triangular matrix ring $\begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is a strongly π-regular rings of bounded index if and only if so are A and B. By induction, we obtain the result. □

Similarly, we deduce that a ring R is a strongly π-regular ring of bounded index if and only if so is the ring of all $n \times n$ upper triangular matrices over R.

Let A_1, A_2, A_3 be rings with identities, and let M_{21}, M_{31}, M_{32} be (A_2, A_1)-, (A_3, A_1)-, (A_3, A_2)-bimodules, respectively. Let

$$\phi : M_{32} \bigotimes_{A_2} M_{21} \to M_{31}$$

be an (A_3, A_1)-homomorphism, and let $A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ with usual matrix operations. Now we generalize Corollary 4 to formal triangular matrix rings.

THEOREM 5. *The following are equivalent:*

(1) *A_1, A_2 and A_3 are strongly π-regular rings of bounded index.*

(2) *The formal triangular matrix ring $A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ is strongly π-regular rings of bounded index.*

Proof. (1) $\Rightarrow$ (2) Let $B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}$ and $M = \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}$. Because A_2 and A_3 are strongly π-regular rings of bounded index, so is the ring B by Theorem 2. In addition, A_1 is a strongly π-regular rings of bounded index. By using Theorem 2 again, we have $A = \begin{pmatrix} A_1 & 0 \\ M & B \end{pmatrix}$ is also a strongly π-regular rings of bounded index, as required.

(2) $\Rightarrow$ (1) Suppose that the ring A is a strongly π-regular ring of bounded index. Then we have an idempotent $e \in T$ such that $R \cong eAe$. Therefore we conclude that R is a strongly π-regular ring of bounded index. □

COROLLARY 6. *Let A_1, A_2 and A_3 be regular rings of bounded index. Then the formal triangular matrix ring* $A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ *is strongly π-regular rings of bounded index.*

Proof. Since every regular ring of bounded index is a strongly π-regular ring, by Theorem 5, the result follows. □

Let I be an ideal of R. If there exists a positive integer p such that $I^p = 0$, then we call I a nilpotent ideal of R. By an argument of J. Stock (cf. [12, p.451]), one can construct a strongly π-regular ring R of bounded index 2, while $\mathrm{J}(R)$ is not T-nilpotent. Moreover, $\mathrm{J}(R)$ is not a nilpotent ideal. Let D be a division ring and let $R = \{(x_1, \cdots, x_n, y, y, \cdots) \mid x_i \in M_i(D), n \in \mathbb{N}, y \in D\}$ where y is treated as a scalar matrix of proper size when multiplied with x_i. By [14, Example 2.3], R is a strongly π-regular ring not of bounded index, while its Jacobson radical is nilpotent. For Morita context over strongly π-regular rings with nilpotent Jacobson radicals, we now observe the following fact.

THEOREM 7. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. Then T is a strongly π-regular ring with nilpotent Jacobson radical if and only if so are A and B.*

Proof. One direction is obvious. Conversely, assume now that A and B are strongly π-regular rings with nilpotent Jacobson radicals. In view of Lemma 1, $\mathrm{J}(T) \cong \begin{pmatrix} \mathrm{J}(A) & N \\ M & \mathrm{J}(B) \end{pmatrix}$. Suppose that $\mathrm{J}(A)^s = 0$ and $\mathrm{J}(B)^t = 0$ for some $s, t > 0$. Given any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in \mathrm{J}(T)$, similarly to the consideration in Theorem 2, we have $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0$. Hence $J(T)^{2(s+t)} = 0$. So the Jacobson radical of T is nilpotent. Given any $x \in T$, there is a positive integer k such that

$$\begin{aligned} &\big(x + J(T)\big)^k \big(T/J(T)\big) \\ &= \big(x + J(T)\big)^{k+1} \big(T/J(T)\big) \\ &= \big(x + J(T)\big)^{2(s+t)k+1} \big(T/J(T)\big). \end{aligned}$$

So we have a $y + J(T) \in T/J(T)$ such that

$$\big(x + J(T)\big)^k = \big(x + J(T)\big)^{2(s+t)k+1} \big(y + J(T)\big),$$

whence $x^k - x^{2(s+t)k+1}y \in J(T)$. Therefore $(x^k - x^{2(s+t)k+1}y)^{2(s+t)} = 0$. Consequently, $x^{2(s+t)k} = x^{2(s+t)k+1}z$ for a $z \in T$. This yields that T is a strongly π-regular ring, as required. □

COROLLARY 8. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. If A and B are right (left) artinian, then T is strongly π-regular.*

Proof. Inasmuch as A and B are right (left) artinian, they are strongly π-regular rings with nilpotent Jacobson radicals. The proof is completed by Theorem 7. □

COROLLARY 9. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. If A and B are regular P.I. rings, then T is a strongly π-regular ring.*

Proof. Since A is a regular ring, we claim that every projective right A-module has the finite exchange property. By [12, Corollary 4.12], A is strongly π-regular rings. Likewise, B is strongly π-regular. Clearly, $J(A) = 0$ and $J(B) = 0$. Thus the result follows from Theorem 7. □

A ring R is said to be right (left) quasi-duo if every maximal right (left) ideal is two-sided. Clearly, right (left) duo rings and weakly right (left) duo rings are all right (left) quasi-duo. By [13, Proposition 4.3], every P-exchange ring with all idempotents central is right (left) quasi-duo.

THEOREM 10. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. Then T is a right (left) quasi-duo strongly π-regular ring if and only if so are A and B.*

Proof. It suffices to show that the result holds for right quasi-duo rings. Suppose that T is a right quasi-duo strongly π-regular ring. Now we construct a map $\theta : T \to A$ given by $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \mapsto a$ for any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Because of zero pairings, we claim that θ is a ring epimorphism. Since every factor ring of right quasi-duo strongly π-regular ring is again a right quasi-duo strongly π-regular ring, $A \cong T/\mathrm{Ker}(\theta)$ is a right quasi-duo strongly π-regular ring. Likewise, B is also a right quasi-duo strongly π-regular ring.

Conversely, assume that A and B are both right quasi-duo strongly π-regular rings. It is well known that a ring R is right quasi-duo if and only if so is $R/\mathrm{J}(R)$. Thus $A/\mathrm{J}(A)$ and $B/\mathrm{J}(B)$ are both right quasi-duo rings. By using Lemma 1, we see that $T/\mathrm{J}(T)$ is right quasi-duo. Furthermore, T is also a right quasi-duo ring.

In view of [9, Lemma 6], $A/\mathrm{J}(A)$ and $B/\mathrm{J}(B)$ are both regular rings. Hence it follows by [13, Corollary 2.4] that they are abelian regular rings. This yields that $T/\mathrm{J}(T)$ is an abelian regular ring, so it is unit-regular. Thus for any $x + \mathrm{J}(T) \in T/\mathrm{J}(T)$, we have an idempotent $e \in T/\mathrm{J}(T)$ and unit $u \in T/\mathrm{J}(T)$ such that $x + \mathrm{J}(T) = eu$. Since T is an exchange ring, idempotents can be lifted modulo $\mathrm{J}(T)$. On the other hand, units can be lifted modulo $\mathrm{J}(T)$. Therefore we have idempotent $f \in T$ and unit $v \in T$ such that $x = fv + r$ for some $r \in \mathrm{J}(T)$.

Given any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in \mathrm{J}(T)$, then $a \in \mathrm{J}(A)$ and $b \in \mathrm{J}(B)$ by Lemma 1. As A and B are both strongly π-regular rings, there are positive integers s, t such that $a^s = 0$ and $b^t = 0$. Analogously to the discussion in Theorem 2, we have $\begin{pmatrix} a & n \\ m & b \end{pmatrix}^{2(s+t)} = 0$. That is, $\mathrm{J}(T)$ is nil. According to [9, Corollary 14], we conclude that T is a strongly π-regular ring. □

COROLLARY 11. *Let T be the ring of a Morita context (A, B, M, N, ψ, ϕ) with zero pairings. If A and B are right (left) quasi-duo rings with all prime ideals right (left) primitive, then T is a strongly π-regular ring.*

Proof. By [13, Theorem 2.5], A and B are strongly π-regular rings. Thus we complete the proof by Theorem 10. □

COROLLARY 12. *A ring R is a right (left) quasi-duo strongly π-regular ring if and only if so is the ring of all $n \times n$ lower triangular matrices over R.*

Proof. Suppose that the ring T of all $n \times n$ lower triangular matrices over R is a right (left) quasi-duo strongly π-regular ring. Then we have an idempotent $e \in T$ such that $R \cong eTe$. Thus R is a strongly π-regular ring. According to [13, Proposition 2.1], R is a right (left) quasi-duo ring, as required.

Conversely, assume now that R is a right (left) quasi-duo strongly π-regular ring. Using Theorem 10, we show that the triangular matrix ring $\begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ is a right (left) quasi-duo strongly π-regular ring if and only if so are A and B. By induction, we get the result. □

Analogously, we deduce that a ring R is a right (left) quasi-duo strongly π-regular ring if and only if so is the ring of all $n \times n$ upper triangular matrices over R.

THEOREM 13. *The following are equivalent:*

(1) *A_1, A_2 and A_3 are right (left) quasi-duo strongly π-regular rings.*

(2) *The formal triangular matrix ring $A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ is right (left) quasi-duo strongly π-regular ring.*

Proof. (2) $\Rightarrow$ (1) Clearly, A_1, A_2 and A_3 are all strongly π-regular rings. Since A is right (left) quasi-duo, so is $A/\mathrm{J}(A)$. One easily checks that $\mathrm{J}(A) = \begin{pmatrix} \mathrm{J}(A_1) & 0 & 0 \\ M_{21} & \mathrm{J}(A_2) & 0 \\ M_{31} & M_{32} & \mathrm{J}(A_3) \end{pmatrix}$; hence, $A/\mathrm{J}(A) \cong A_1/\mathrm{J}(A_1) \oplus A_2/\mathrm{J}(A_2) \oplus A_3/\mathrm{J}(A_3)$. It is straightforward that $A_1/\mathrm{J}(A_1) \oplus A_2/\mathrm{J}(A_2) \oplus A_3/\mathrm{J}(A_3)$ is right (left) quasi-duo if and only if so are $A_1/\mathrm{J}(A_1), A_2/\mathrm{J}(A_2)$ and $A_3/\mathrm{J}(A_3)$. Therefore A_1, A_2 and A_3 are right (left) quasi-duo.

(1) $\Rightarrow$ (2) Set $B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}$ and $M = \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}$. By Theorem 10, B is a right (left) quasi-duo strongly π-regular rings. Using Theorem 10 again, we get the result. □

COROLLARY 14. *Let A_1, A_2 and A_3 be right (left) quasi-duo regular rings. Then the formal triangular matrix ring $A = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ is a strongly π-regular rings.*

Proof. By [13, Theorem 2.7], every right (left) quasi-duo regular ring is strongly π-regular. It follows by Theorem 13 that A is a strongly π-regular ring. □

References

[1] P. Ara, *Strongly π-regular rings have stable range one*, Proc. Amer. Math. Soc. **124** (1996), 3293–3298.

[2] P. Ara, M.G. Lozano and M.S. Molina, *Local rings of exchange rings*, Comm. Algebra **26** (1998), 4191–4205.

[3] A. Badawi, *On abelian π-regular rings*, Comm. Algebra **25** (1997), 1009–1021.

[4] W. D. Burgess and P. Menal, *On strongly π-regular rings and homomorphisms into them*, Comm. Algebra **16** (1988), 1701–1725.

[5] H. Chen, *Exchange rings with Artinian primitive factors*, Algebra and Represent. Theory **2** (1999), 201–207.

[6] A. Haghany, *Hopficity and co-hopficity for Morita contexts*, Comm. Algebra **27** (1999), 477–492.

[7] C. Y. Hong, N. K. Kim, T. K. Kwak and Y. Lee, *On weak π-regularity of rings whose prime ideals are maximal*, J. Pure. Appl. Algebra **146** (2000), 35–44.

[8] A. Haghany and K. Varadarajan, *Study of formal triangular matrix rings*, Comm. Algebra **27** (1999), 5507–5525.

[9] N. K. Kim and Y. Lee, *On right quasi-duo rings which are π-regular*, Bull Korean Math. Soc. **37** (2000), 217–227.

[10] Y. Lee and C. Huh, *A note on π-regular rings*, Kyungpook Math. J. **38** (1998), 157–161.

[11] Y. Lee, C. O. Kim and H. K. Kim, *On strongly π-regular rings*, Kyungpook Math. J. **37** (1997), 291–295.

[12] J. Stock, *On rings whose projective modules have the exchange property*, J. Algebra **103** (1986), 437–453.

[13] H. P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), 21–31.

[14] ______, *On the structure of exchange rings*, Comm. Algebra **25** (1997), 661–670.

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