Bull. Korean Math. Soc. **40** (2003), No. 1, pp. 85–89

UNIFORMITY OF HOLOMORPHIC VECTOR BUNDLES ON INFINITE-DIMENSIONAL FLAG MANIFOLDS

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ABSTRACT. Let V be a localizing infinite-dimensional complex Banach space. Let X be a flag manifold of finite flags either of finite codimensional closed linear subspaces of V or of finite dimensional linear subspaces of V. Let E be a holomorphic vector bundle on X with finite rank. Here we prove that E is uniform, i.e. that for any two lines D, R in the same system of lines on X the vector bundles $E|D$ and $E|R$ have the same splitting type.

1. Introduction

Let V be a locally convex and Hausdorff topological vector space and r a positive integer. Let $\mathbf{P}(V)$ be the set of all one-dimensional linear subspaces of V ([2], §7). Let $\mathrm{Grass}(r,V)$ be the set of all r-codimensional closed linear subspaces of V. By Hahn - Banach any such subspace A has a closed supplement M. Fixing M and varying A among the closed supplements of M we obtain an open chart of $\mathrm{Grass}(r,V)$. Varying M we equip $\mathrm{Grass}(r,V)$ with a structure of complex manifold ([1], Chapter 2, or [3], Chapter III, §1). Let $Gr(r,V)$ be the set of all r-dimensional linear subspaces of V. Every finite-dimensional linear subspace of V is closed and complemented ([4], Proposition V.31). Hence choosing such complements we equip $Gr(r,V)$ with a structure of complex manifold. Hence $G(1,V) = \mathbf{P}(V)$, while $\mathrm{Grass}(1,V) = \mathbf{P}(V')$, where V' is the topological dual of V. All the lines of $\mathrm{Grass}(r,V)$ are described in the following way. Fix a closed $(r-1)$-codimensional linear subspace B of V and a closed two-codimensional linear subspace A of B. Let $D(A,B)$ be the set of all closed r-codimensional linear subspaces H of V such that

Received April 29, 2002.
2000 Mathematics Subject Classification: 32K05, 32L05.
Key words and phrases: flag manifold, infinite-dimensional flag manifold, holomorphic vector bundle, uniform vector bundle, splitting type.
The author was partially supported by MURST and GNSAGA of INdAM (Italy).

$A \subset H \subset B$. The set $D(A,B)$ is the line determined by the subspaces A and B. Notice that $D(A,B) \cong \mathbf{P}^1$. All the lines of $Gr(r,V)$ are described in the following way. Fix an $(r-1)$-dimensional linear subspace A of V and an $(r+1)$-dimensional linear subspace B of V such that $A \subset B$. Let $D(A,B)$ be the set of all r-dimensional linear subspaces H of V such that $A \subset H \subset B$.

Now we generalize this construction to the case of flags of linear subspaces of V. Fix a positive integer m and positive integers $r_1 > \cdots > r_m$. Let $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ be the set of all m-ples $(H_1, \cdots, H_m)$ of closed linear subspaces of V such that $H_i \subset H_j$ if $i < j$ and each H_i has codimension r_i. Let $Fl(m; r_1, \cdots, r_m; V)$ be the set of all m-ples $(A_1, \cdots, A_m)$ with A_i r_i-dimensional linear subspace of V and $A_j \subset A_i$ if $j < i$. The flag manifolds $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and $Fl(m; r_1, \cdots, r_m; V)$ are connected complex manifolds. If $m = 1$ the flag manifolds $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and $Fl(m; r_1, \cdots, r_m; V)$ are just the Grassmannian manifolds $\mathrm{Grass}(r_1, V)$ and $Gr(r_1, V)$. Now assume $m \geq 2$. There are morphisms $f_i : \mathrm{Flag}(m; r_1, \cdots, r_m; V) \to \mathrm{Grass}(r_i, V)$, $1 \leq i \leq m$, defined by $f_i((H_1, \cdots, H_m)) = H_i$. There are morphisms $g_i : Fl(m; r_1, \cdots, r_m; V) \to Gr(r_i, V)$, $1 \leq i \leq m$, defined by $g_i((A_1, \cdots, A_m)) = A_i$. Fix an integer i with $1 \leq i \leq m$ and codimension r_j linear subspaces H_j of V, $1 \leq j \leq m$, $j \neq i$. If $i = m$ let H''_m be a codimension $r_m - 1$ closed linear subspace of V and H'_m a closed codimension two linear subspace of H''_m. If $1 \leq i \leq m-1$ let H''_i be a closed codimension $r_{i+1} - r_i - 1$ linear subspace of H_{i+1} and H'_i a closed codimension two linear subspace of H'_i containing H_{i-1}. Let D be the set of all m-ples $(H_1, \cdots, H_m)$ such that H_i is a closed hyperplane of H'_i containing H''_i. Hence D is a closed analytic subset of $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and $D \cong \mathbf{P}^1$. We will say that D is a *line of type* i or an *i-line* of $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$. In a very similar way one can define the *lines of type* i or the *i-lines* of $Fl(m; r_1, \cdots, r_m; V)$. It is easy to see by induction on m that the linear group $GL(V)$ acts transitively on the set of all i-lines of $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and on the set of all i-lines of $Fl(m; r_1, \cdots, r_m; V)$. Let E be a holomorphic vector bundle on $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ (resp. $Fl(m; r_1, \cdots, r_m; V)$) with finite rank. Set $s := \mathrm{rank}(E)$. We will say that E is *i-uniform* if there are integers $a_1, \cdots, a_s$ such that $E|D$ has splitting type $a_1, \cdots, a_s$ for every i-line D. We will say that E is *totally uniform* if it is i-uniform for every integer i with $1 \leq i \leq m$. For the notion of localizing Banach space and of localizing complex manifold, see [2], p.509. Every Hilbert space is localizing. Hence Theorems 1 and 2 below are true for any infinite-dimensional

Hilbert space. By Remark 2 below if V' (resp. V) is localizing, then $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ (resp. $Fl(m; r_1, \cdots, r_m; V)$) is localizing. The aim of this paper is to prove the following results.

THEOREM 1. *Let V be an infinite-dimensional Banach space such that V' is localizing and E a holomorphic vector bundle with finite rank on $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$. Then E is totally uniform.*

THEOREM 2. *Let V be an infinite-dimensional and localizing Banach space and E a holomorphic vector bundle with finite rank on $Fl(m; r_1, \cdots, r_m; V)$. Then E is totally uniform.*

REMARK 1. Notice that for every integer i with $1 \leq i \leq m$ both $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and $Fl(m; r_1, \cdots, r_m; V)$ are covered by i-lines and hence Theorems 1 and 2 seem to capture a very strong property of $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and $Fl(m; r_1, \cdots, r_m; V)$ which does not hold for finite-dimensional V.

Holomorphic vector bundles with finite ranks on increasing unions of projective spaces (i.e. on $\mathbf{P}(V)$ with $V \cong \mathbf{C}^{(\mathbf{N})}$, i.e. V with countable algebraic dimension) are classified in [6] and [5]. A similar classification is known for holomorphic vector bundles with very low rank on $Gr(r, \mathbf{C}^{(\mathbf{N})})$ ([5], 4.7, 4.8, 4.12, 4.19, and 4.20). However, $\mathbf{C}^{(\mathbf{N})}$ is not a Banach space and for infinite-dimensional Banach spaces the geometry of $\mathbf{P}(V)$ seems to be quite different. L. Lempert proved that if V is an infinite-dimensional localizing Banach space, then every holomorphic vector bundle on $\mathbf{P}(V)$ is isomorphic to a direct sum of suitable line bundles ([2], Theorem 8.5) and in particular it is uniform. To prove Theorems 1 and 2 we will heavily use this theorem of Lempert.

2. The proofs

REMARK 2. Let V be a Banach space. V is localizing if and only if $V \oplus \mathbf{C}$ is localizing. If V is localizing, then for all integers $s \geq 1$ the Banach space $V^{\oplus s}$ is localizing.

Proofs of Theorems 1.1 and 1.2. We will write down only the proof of Theorem 1 because the proof of Theorem 2 requires only notational modifications.

Step 1) Here we will do the case $m = 1$ and set $r := r_1$. If $r = 1$, then the result is just a very particular case of [2], Theorem 8.5. Hence we may assume $r \geq 2$. Fix lines D, R of $\mathrm{Grass}(r, V)$, say represented by pairs (D', D'') (resp. (R', R'')) with D'' (resp. R'') codimension

$r-1$ closed linear subspace of V and D' (resp. R') codimension two closed linear subspace of D'' (resp. R''). There are natural inclusions of $\mathrm{Grass}(1, D'')$ and $\mathrm{Grass}(1, R'')$ into $\mathrm{Grass}(r, V)$. $\mathrm{Grass}(1, D'')$ (resp. $\mathrm{Grass}(1, R'')$) is isomorphic to the projective space over the topological dual of D'' (resp. R''). Since D'' and R'' have a closed supplement in V, by Remark 2 we may apply [2], Theorem 8.5, to $E|\mathrm{Grass}(1, D'')$ and $E|\mathrm{Grass}(1, R'')$. Thus there are integers $a_1 \geq \cdots \geq a_s$ and $b_1 \geq \cdots \geq b_s$, $s = \mathrm{rank}(E)$, such that $E|\mathrm{Grass}(1, D'') \cong \mathcal{O}_{\mathrm{Grass}(1,D'')}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathrm{Grass}(1,D'')}(a_s)$ and $E|\mathrm{Grass}(1, R'') \cong \mathcal{O}_{\mathrm{Grass}(1,R'')}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathrm{Grass}(1,R'')}(b_s)$. It is sufficient to prove that $a_i = b_i$ for every i and any choice of linear subspaces D'' and R''. Since $\dim(V')$ is infinite, there is a holomorphic family, say $\{B_\lambda\}_{\lambda\in\Delta}$, of $(s+1)$-dimensional projective subspaces of $\mathrm{Grass}(r, V)$ with Δ open disk of $\mathbf{C}$ and $a, b \in \Delta$ such that $B_a \subset \mathrm{Grass}(1, D'')$ and $B_b \subset \mathrm{Grass}(1, R'')$. All vector bundles are direct sums of line bundles. Since $s+1 \geq 2$, we may use local rigidity of direct sums of line bundles on any projective space of dimension at least two to obtain $a_i = b_i$ for every i; since $\dim(B_\lambda) = s+1 > \mathrm{rank}(E)$, it would be sufficient to take a continuous family $\{B_\lambda\}_{\lambda\in\Delta}$ of $(s+1)$-dimensional projective subspaces and compute the Chern classes of the decomposable vector bundle $E|B_\lambda$.

Step 2) Here we assume $m \geq 2$. Fix $i \in \{1, \cdots, m\}$ and two i-lines D, R contained in $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$. As in the proof of the case $m = 1$ given in Step 1 it is sufficient to show the existence of infinite-dimensional localizing Banach spaces A, B such that $D \subset \mathbf{P}(A) \subset \mathrm{Flag}(m; r_1, \cdots, r_m; V)$, $R \subset \mathbf{P}(B) \subset \mathrm{Flag}(m; r_1, \cdots, r_m; V)$, a connected continuous family of $(s+1)$-dimensional projective subspaces of $\mathrm{Flag}(m; r_1, \cdots, r_m; V)$ and two members of the family, one containing D and the other one containing R. These assertions are easily proved by induction on m using the projections f_j, $1 \leq j \leq m$, and the case $m = 1$ proved in Step 1. □

References

[1] A. Douady, *Le problème des modules pour les sous-espace analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier **16** (1966), 1–95.

[2] L. Lempert, *The Dolbeaut complex in infinite dimension. I*, J. Amer. Math. Soc. **11** (1998), 485–520.

[3] J. -P. Ramis, *Sous-ensembles analytiques d'une variété banachique complexe*, Erg. der Math. **53**, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[4] A. P. Robertson and W. Robertson, *Topological vector spaces*, Cambridge Tracts in Math. no. 53, Cambridge University Press, 1973.

[5] E. Sato, *On the decomposability of infinitely extendable vector bundles on projective spaces and Grassmann varieties*, J. Math. Kyoto Univ. **17** (1977), 127–150.
[6] A. N. Tyurin, *Vector bundles of finite rank over infinite varieties*, Math. USSR Izvestija **10** (1976), 1187–1204.

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