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THE STRONG PERRON INTEGRAL

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Abstract. In this paper, we study the strong Perron integral, and show that the strong Perron integral is equivalent to the McShane integral.

1. Introduction

The major and minor functions are first defined using the upper and lower derivates, and then the Perron integral is defined using the major and minor functions.

It is well-known [4] that the Perron integral is equivalent to the Henstock integral.

In this paper, we change the definitions of major and minor functions by strong derivates rather than ordinary derivates, and then define the strong Perron integral using such major and minor functions. We also show that the strong Perron integral is equivalent to the McShane integral.

2. The strong Perron and McShane integrals

Let $F : [a, b] \to \mathbf{R}$ be a function. The *upper* and *lower derivates* of F at c are defined by

$$\overline{D}F(c) = \lim_{\delta \to 0^+} \sup \left\{ \frac{F(x) - F(c)}{x - c} : 0 < |x - c| < \delta \right\};$$

$$\underline{D}F(c) = \lim_{\delta \to 0^+} \inf \left\{ \frac{F(x) - F(c)}{x - c} : 0 < |x - c| < \delta \right\}.$$

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The function F is said to be *differentiable* at $c \in [a, b]$ if $\overline{D}F(c)$ and $\underline{D}F(c)$ are finite and equal. This common value is called the *derivative* of F at c and is denoted by $F'(c)$.

We first define the strong derivates of a function.

DEFINITION 2.1. Let $F : [a, b] \to \mathbf{R}$ be a function and let $c \in [a, b]$. The *upper* and *lower strong derivates* of F at c are defined by

$$\overline{SD}F(c) = \lim_{\delta \to 0^+} \sup \left\{ \frac{F(y) - F(x)}{y - x} : [x, y] \subseteq (c - \delta, c + \delta) \cap [a, b] \right\};$$

$$\underline{SD}F(c) = \lim_{\delta \to 0^+} \inf \left\{ \frac{F(y) - F(x)}{y - x} : [x, y] \subseteq (c - \delta, c + \delta) \cap [a, b] \right\}.$$

The function F is said to be *strongly differentiable at* c if $\overline{SD}F(c)$ and $\underline{SD}F(c)$ are finite and equal. This common value is called the *strong derivative* of F at c and is denoted by $F'_s(c)$.

Note that the interval $[x, y]$ does not have to contain the point c in the above definition. From definition, it is clear that

$$\underline{SD}F \leq \underline{D}F \leq \overline{D}F \leq \overline{SD}F.$$

From this relation, it is obvious that if F is strongly differentiable at c, then it is differentiable at c and $F'_s(c) = F'(c)$.

The derivative F' of a differentiable function $F : [a, b] \to R$ may not be continuous on $[a, b]$. But the following theorem shows that the strong derivative F'_s of a strongly differentiable function F is in fact continuous on $[a, b]$.

THEOREM 2.2. *Let $F : [a, b] \to R$ be a function. If F is strongly differentiable on $[a, b]$, then F'_s is continuous on $[a, b]$.*

Proof. Let $c \in [a, b]$ and let $\varepsilon > 0$ be given. Since F is strongly differentiable at c, there exists $\delta > 0$ such that

$$\left| \frac{F(y) - F(x)}{y - x} - F'_s(c) \right| < \varepsilon$$

for every interval $[x, y] \subseteq (c - \delta, c + \delta) \cap [a, b]$. If $|z - c| < \delta$ and $z \in [a, b]$, then there exists $\delta_1 > 0$ such that $(z - \delta_1, z + \delta_1) \cap [a, b] \subseteq (c - \delta, c + \delta) \cap [a, b]$ and $\left| \frac{F(q) - F(p)}{q - p} - F'_s(z) \right| < \varepsilon$ for every interval $[p, q] \subseteq (z - \delta_1, z + \delta_1) \cap [a, b]$, since F is strongly differentiable at z. Choose an interval $[p_0, q_0]$

such that $[p_0, q_0] \subseteq (z - \delta_1, z + \delta_1) \cap [a, b]$. Then we have

$$\begin{aligned} |F_s'(z) - F_s'(c)| &\le \left| F_s'(z) - \frac{F(q_0) - F(p_0)}{q_0 - p_0} \right| + \left| \frac{F(q_0) - F(p_0)}{q_0 - p_0} - F_s'(c) \right| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Hence F_s' is continuous at c. This completes the proof. □

EXAMPLE 2.3. Let $f : [a, b] \to \mathbf{R}$ be a continuous function and let $F(x) = \int_a^x f d\mu$ be the indefinite Lebesgue integral of f for each $x \in [a, b]$. Then F is strongly differentiable on $[a, b]$ and $F_s' = f$ on $[a, b]$.

To show this, let $c \in [a, b]$ and let $\varepsilon > 0$ be given. Since f is continuous at c, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ if $|x - c| < \delta$ and $x \in [a, b]$. Let $[s, t] \subseteq (c - \delta, c + \delta) \cap [a, b]$. Then we have

$$\int_s^t \{f(c) - \varepsilon\}\, d\mu < \int_s^t f d\mu < \int_s^t \{f(c) + \varepsilon\}\, d\mu;$$

$$f(c) - \varepsilon < \frac{F(t) - F(s)}{t - s} < f(c) + \varepsilon;$$

and it follows that $\left| \frac{F(t) - F(s)}{t - s} - f(c) \right| < \varepsilon$. Hence F is strongly differentiable at c and $F_s'(c) = f(c)$.

Let $f : [a, b] \to \mathbf{R}_e$ be a function, where $\mathbf{R}_e = \mathbf{R} \cup \{\pm\infty\}$. A measurable function $U : [a, b] \to \mathbf{R}$ is called a *major function* of f on $[a, b]$ if $\underline{D}U(x) > -\infty$ and $\underline{D}U(x) \ge f(x)$ for all $x \in [a, b]$. A measurable function $V : [a, b] \to \mathbf{R}$ is called a *minor function* of f on $[a, b]$ if $\overline{D}V(x) < \infty$ and $\overline{D}V(x) \le f(x)$ for all $x \in [a, b]$.

Recall that a function $f : [a, b] \to \mathbf{R}_e$ is *Perron integrable* on $[a, b]$ if f has at least one major function and one minor function on $[a, b]$ and the numbers

$$\inf \left\{ U_a^b : U \text{ is a major function of } f \text{ on } [a, b] \right\};$$
$$\sup \left\{ V_a^b : V \text{ is a minor function of } f \text{ on } [a, b] \right\}$$

are equal, where $U_a^b = U(b) - U(a)$ and $V_a^b = V(b) - V(a)$.

Using upper and lower strong derivates, we define the strong major and strong minor functions.

DEFINITION 2.4. Let $f : [a, b] \to R_e$ be a function.

(1) A measurable function $U : [a, b] \to \mathbf{R}$ is a *strong major function* of f on $[a, b]$ if $\underline{SD}U(x) > -\infty$ and $\underline{SD}U(x) \ge f(x)$ for all $x \in [a, b]$.

(2) A measurable function $V : [a,b] \to \mathbf{R}$ is a *strong minor function* of f on $[a,b]$ if $\overline{SD}V(x) < \infty$ and $\overline{SD}V(x) \leq f(x)$ for all $x \in [a,b]$.

Now we define the strong Perron integral.

DEFINITION 2.5. A function $f : [a,b] \to R_e$ is *strongly Perron integrable* on $[a,b]$ if f has at least one strong major function and one strong minor function $[a,b]$ and the numbers

$$\inf \left\{ U_a^b : U \text{ is a strong major function of } f \text{ on } [a,b] \right\};$$
$$\sup \left\{ V_a^b : V \text{ is a strong minor function of } f \text{ on } [a,b] \right\}$$

are equal. This common value is called the *strong Perron integral* of f on $[a,b]$ and is denoted by $(SP)\int_a^b f d\mu$. The function f is strongly Perron integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is strongly Perron integrable on $[a,b]$.

It follows easily from definition that every strongly Perron integrable function is Perron integrable.

The following theorem is an immediate consequence of the definition.

THEOREM 2.6. *A function $f : [a,b] \to R_e$ is strongly Perron integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exist a strong major function U and a strong minor function V of f on $[a,b]$ such that $U_a^b - V_a^b < \varepsilon$.*

Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [c,d]$, and a free tagged interval $(x,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [a,b]$. The (free) tagged interval $(x,[c,d])$ is said to be *subordinate* to δ if

$$[c,d] \subseteq (x - \delta(x), x + \delta(x)).$$

Let $\mathcal{P} = \{(x_i,[c_i,d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping (free) tagged intervals in $[a,b]$. If $(x_i,[c_i,d_i])$ is subordinate to δ for each i, then we say that $\mathcal{P}$ is subordinate to δ. If $\mathcal{P}$ is subordinate to δ and $[a,b] = \cup_{i=1}^n [c_i,d_i]$, then we say that $\mathcal{P}$ is a (free) tagged partition of $[a,b]$ that is subordinate to δ.

Recall that a function $f : [a,b] \to \mathbf{R}$ is *McShane integrable* on $[a,b]$ if there exists a real number A with the following property: for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P}) - A| < \varepsilon$, whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to δ, where $f(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)$ if $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ is a free tagged partition of $[a, b]$. The real number A is called the *McShane integral* of f on $[a, b]$ and is denoted by $(M)\int_a^b f\,d\mu$. The function $f : [a, b] \to \mathbf{R}$ is said to be *Henstock integrable* on $[a, b]$ if we replace 'free tagged partition' by 'tagged partition' in the definition of the McShane integral.

The following two theorems show that the strong Perron integral is equivalent to the McShane integral.

THEOREM 2.7. *If $f : [a, b] \to \mathbf{R}$ is strongly Perron integrable on $[a, b]$, then f is McShane integrable on $[a, b]$ and the integrals are equal.*

Proof. Let $\varepsilon > 0$ be given. By the definition of strong Perron integrability, there exist a strong major function U and a strong minor function V of f on $[a, b]$ such that

$$-\varepsilon < V_a^b - (SP)\int_a^b f\,d\mu \le 0 \le U_a^b - (SP)\int_a^b f\,d\mu < \varepsilon.$$

Since $\overline{SD}V \le f \le \underline{SD}U$ on $[a, b]$, for each $c \in [a, b]$ there exists $\delta(c) > 0$ such that

$$\frac{U(y) - U(x)}{y - x} \ge f(c) - \varepsilon \quad \text{and} \quad \frac{V(y) - V(x)}{y - x} \le f(c) + \varepsilon,$$

whenever $[x, y] \subseteq (c - \delta(c), c + \delta(c)) \cap [a, b]$. Now let

$$\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$$

be a free tagged partition of $[a, b]$ that is subordinate to δ. Then we have

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i)(d_i - c_i) - (SP)\int_a^b f\,d\mu \\
= &\sum_{i=1}^{n} \Big(f(x_i)(d_i - c_i) - U_{c_i}^{d_i}\Big) + U_a^b - (SP)\int_a^b f\,d\mu \\
< &\sum_{i=1}^{n} \varepsilon(d_i - c_i) + \varepsilon \\
= &\ \varepsilon(b - a + 1).
\end{aligned}$$

Similarly, using the strong minor function V,

$$\sum_{i=1}^{n} f(x_i)(d_i - c_i) - (SP)\int_a^b f d\mu > -\varepsilon(b-a+1).$$

Since $\left|f(\mathcal{P}) - (SP)\int_a^b f d\mu\right| < \varepsilon(b-a+1)$, f is McShane integrable on $[a,b]$ and $(M)\int_a^b f d\mu = (SP)\int_a^b f d\mu$. □

Let $\omega(F,[c,d]) = \sup\{|F(y)-F(x)| : c \le x < y \le d\}$ denote the oscillation of the function F on the interval $[c,d]$. The function F is BV_* on E if $\sup\{\sum_{i=1}^{n}\omega(F,[c_i,d_i])\}$ is finite, where the supremum is over all finite collections $\{[c_i,d_i] : 1 \le i \le n\}$ of non-overlapping intervals in $[a,b]$. And the function F is said to be BVG_* on E if E can be written as a countable union of sets on each of which F is BV_*.

THEOREM 2.8. *If $f : [a,b] \to R$ is McShane integrable on $[a,b]$, then f is strongly Perron integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$ be given. By the definition of McShane integrability, there exists a positive function δ on $[a,b]$ such that $\left|f(\mathcal{P}) - (M)\int_a^b f d\mu\right| < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ. For each $x \in (a,b]$, let

$$\begin{aligned} U(x) = &\ \sup\Big\{f(\mathcal{P}) : \mathcal{P} \text{ is a free tagged partition of } [a,x] \text{ that is} \\ &\qquad\qquad \text{subordinate to } \delta\Big\}; \\ V(x) = &\ \inf\Big\{f(\mathcal{P}) : \mathcal{P} \text{ is a free tagged partition of } [a,x] \text{ that is} \\ &\qquad\qquad \text{subordinate to } \delta\Big\}; \end{aligned}$$

and let $U(a) = 0 = V(a)$. By the Saks-Henstock Lemma [4], the functions U and V are finite-valued on $[a,b]$. We prove that U is a strong major function of f on $[a,b]$; the proof that V is a strong minor function of f on $[a,b]$ is quite similar.

Fix a point $c \in [a,b]$ and let $[x,y]$ be any interval such that $[x,y] \subseteq (c-\delta(c), c+\delta(c)) \cap [a,b]$. For each free tagged partition $\mathcal{P}$ of $[a,x]$ that is subordinate to δ, we find that

$$U(y) \ge f(\mathcal{P}) + f(c)(y-x)$$

and it follows that $U(y) \geq U(x)+f(c)(y-x)$. This shows that $\frac{U(y)-U(x)}{y-x} \geq f(c)$ and hence $\underline{SD}U(c) \geq f(c) > -\infty$. Since $-\infty < \underline{SD}U \leq \underline{D}U$ on $[a,b]$, U is BVG_* on $[a,b]$ by [4, Theorem 6.21] and it follows that U is measurable on $[a,b]$ by [4, Corollary 6.9]. Hence, U is a strong major function of f.

Since $|f(\mathcal{P}_1) - f(\mathcal{P}_2)| < 2\varepsilon$ for any two free tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ that are subordinate to δ, it follows that $U_a^b - V_a^b \leq 2\varepsilon$. By Theorem 2.6, the function f is strongly Perron integrable on $[a,b]$. □

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