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SHAPE OPERATOR OF SLANT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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ABSTRACT. In this article, we establish relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for a slant submanifold in a Sasakian space form of constant φ- sectional curvature with arbitrary codimension.

1. Introduction

Nash's theorem enables us to view any Riemannian manifold as a submanifold of Euclidean space. This gives us a natural motivation for the study of submanifolds of Riemannian manifolds. In this case, we have intrinsic invariants as well as extrinsic invariants. Among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones. Gauss-Bonnet Theorem, Isoperimetric inequality and Chern-Lashof Theorem provide relations between intrinsic invariants and extrinsic invariants for a submanifold in a Euclidean space.

Recently, B. -Y. Chen ([5, 6]) established an inequality relating intrinsic quantities and extrinsic ones for submanifolds in a space form with arbitrary codimension. In particular, in ([5]) he investigated a relation between the sectional curvature and the shape operator for submanifolds in real space forms. And, in ([6]) he established a sharp relation between the k-Ricci curvature and the shape operator. On the other hand, for the above mentioned contents K. Matsumoto, I. Mihai and A. Oiaga ([7]) studied these relations of slant submanifolds in complex space forms.

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In this paper, we study slant submanifolds of Sasakian space forms of constant φ-sectional curvature with arbitrary codimension and establish relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for slant submanifolds in Sasakian space forms.

2. Preliminaries

Let $\widetilde{M}$ be an odd-dimensional Riemannian manifold with $\langle\cdot,\cdot\rangle$ the metric tensor satisfying

$$\eta(\xi)=1,\quad \varphi^2=-I+\eta\otimes\xi,\quad \eta(X)=\langle X,\xi\rangle,$$
$$\langle\varphi X,\varphi Y\rangle=\langle X,Y\rangle-\eta(X)\eta(Y).$$

Then $(\varphi,\xi,\eta,\langle\ ,\rangle)$ is called *the almost contact metric structure* on $\widetilde{M}$. Let Φ denote the *fundamental 2-form* in $\widetilde{M}$ given by $\Phi(X,Y)=\langle X,\varphi Y\rangle$ for all $X,Y\in T\widetilde{M}$, the set of vector fields of $\widetilde{M}$. If $\Phi=d\eta$, then $\widetilde{M}$ is said to be a *contact metric manifold.* Moreover, if ξ is a Killing vector field with respect to $\langle\ ,\rangle$, the contact metric structure is called a *K-contact structure.* It is easy to prove that a contact metric manifold is K-contact if and only if

$$(2.1)\qquad \widetilde{\nabla}_X\xi=-\varphi X$$

for any $X\in T\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$. The structure of $\widetilde{M}$ is said to be *normal* if $[\varphi,\varphi]+2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. A *Sasakian manifold* is a normal contact metric manifold. In fact, an almost contact metric structure is Sasakian if and only if

$$(\widetilde{\nabla}_X\varphi)Y=\langle X,Y\rangle\xi-\eta(Y)X$$

for all vector fields X and Y. Every Sasakian manifold is a K-contact manifold.

Given a Sasakian manifold $\widetilde{M}$, a plane section π in $T_p\widetilde{M}$ is called a *φ-section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\widetilde{K}(\pi)$ of a φ-section π is called *φ-sectional curvature.* If a Sasakian manifold $\widetilde{M}$ has constant φ-sectional curvature c, $\widetilde{M}$ is called a *Sasakian-space-form* and it is denoted by $\widetilde{M}(c)$. For more details and background, refer to the reference [1].

Now let M be a submanifold immersed in $(\widetilde{M},\varphi,\xi,\eta,\langle\ ,\rangle)$. We also denote by $\langle\ ,\rangle$ the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to

M. We denote by h the second fundamental form of M and by A_v the Weingarten endomorphism associated with any $v \in T^{\perp}M$. We put $h^r_{ij} = \langle h(e_i, e_j), e_r \rangle$ for any orthonormal vectors e_i , e_j $\in TM$ and $e_r \in T^{\perp}M$. The mean curvature vector field H is defined by $H = \frac{1}{\dim M}\mathrm{trace}h$. M is said to be *totally geodesic* if the second fundamental vanishes identically.

From now on, we assume that the dimension of M is $n+1$ and that of the ambient manifold $\widetilde{M}$ is $2m+1$ $(m \geq 2)$. We also assume that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we have the orthogonal direct decomposition of TM by $TM = D \oplus \mathrm{Span}\{\xi\}$. For any $X \in TM$, we write $\varphi X = TX + NX$, where TX(resp. NX) is the tangential (resp. normal) component of φX. If $\widetilde{M}$ is a K-contact manifold, (2.1) gives

$$h(X, \xi) = -NX \tag{2.2}$$

for any X in TM. Given a local orthonormal frame $\{e_1, \cdots, e_n\}$ of D, we can define the squared norms of T and N by

$$\|T\|^2 = \sum_{i,j=1}^{n} \langle e_i, Te_j \rangle^2, \quad \|N\|^2 = \sum_{i,j=1}^{n} \langle e_i, Ne_j \rangle^2, \tag{2.3}$$

respectively. It is easy to show that both $\|T\|^2$ and $\|N\|^2$ are independent of the choice of the orthonormal frame. The submanifold M is said to be *invariant* if N is identically zero, that is $\varphi X \in TM$ for any $X \in TM$. On the other hand M is said to be an *anti-invariant submanifold* if T is identically zero, that is, $\varphi X \in T^{\perp}M$ for any $X \in TM$. For each nonzero vector X tangent to M at a point p in M such that X is not proportional to ξ_p, we denote by $\theta(X)$ the angle between φX and T_pM. Then M is said to be *slant* if the angle $\theta(X)$ is a constant which is independent of the choice of $p \in M$ and $X \in D_p$ ([2]). The angle θ of a slant immersion is called *the slant angle of the immersion.* Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called *proper.* Also, a slant submanifold with the slant angle θ is called θ-slant ([2, 4]). It is easily proved that a θ-slant submanifold M of an almost contact metric manifold $\widetilde{M}$ satisfies

$$\sum_{j=1}^{n} \langle e_i, \varphi e_j \rangle^2 = \cos^2 \theta \tag{2.4}$$

for any $i = 1, 2, \cdots, n$ where $\{e_1, \cdots, e_n, \xi\}$ is a local orthonormal frame of TM (cf. [3]). On the other hand, the Gauss' equation gives rise to the curvature tensor R of the submanifold M of a Sasakian space form $\widetilde{M}(c)$ satisfies

$$
\begin{aligned}
\langle R(X,Y)Z,W\rangle =& \frac{c+3}{4}\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} \\
&+ \frac{c-1}{4}\{\eta(X)\eta(Z)\langle Y,W\rangle - \eta(Y)\eta(Z)\langle X,W\rangle \\
&+ \langle X,Z\rangle\eta(Y)\eta(W) - \langle Y,Z\rangle\eta(X)\eta(W) \\
&+ \langle Z,\varphi Y\rangle\langle\varphi X,W\rangle - \langle Z,\varphi X\rangle\langle\varphi Y,W\rangle \\
&+ 2\langle X,\varphi Y\rangle\langle\varphi Z,W\rangle\} \\
&+ \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle
\end{aligned} \tag{2.5}
$$

for any $X,Y,Z,W \in TM$.

For an orthonormal basis $\{e_1,\cdots,e_{n+1}\}$ of the tangent space T_pM, $p \in M$, the scalar curvature τ at p is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \tag{2.6}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of M associated with the plane section spanned by e_i and e_j. In particular, if we put $e_{n+1} = \xi_p$, then (2.6) implies

$$2\tau = \sum_{i\neq j}^{n} K(e_i \wedge e_j) + 2\sum_{i=1}^{n} K(e_i \wedge \xi). \tag{2.7}$$

From (2.3), (2.5) and (2.7) we obtain the following relationship between the scalar curvature and the mean curvature of M,

$$2\tau = (n+1)^2\|H\|^2 - \|h\|^2 + n(n+1)\frac{c+3}{4} + 2n + \frac{3(c-1)}{4}\|T\|^2. \tag{2.8}$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1,\cdots,e_k\}$ of L such that $e_1 = X$. Define the Ricci curvature Ric_L of L at X by

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{2.9}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j. We simply call such a curvature a *k-Ricci curvature*. The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{2.10}$$

Suppose L is a k-plane section of D_p and X a unit vector in L. The scalar curvature τ^D of the k-plane section L is given by

$$\tau^D(L) = \sum_{1\le i<j\le k} K_{ij}. \tag{2.11}$$

For each point $p \in M$, we put

$$\inf_D K(p) = \inf\{K(\pi) : \pi \text{ is a plane section in } D_p\}.$$

For each integer k $(2 \le k \le n)$ the Riemannian invariants θ_k and θ_k^D on an n-dimensional Riemannian manifold M are defined by

$$\theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{2.12}$$

where L runs over all k-plane sections in T_pM and X runs over unit vector in L, and

$$\theta_k^D(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{2.13}$$

where L runs over all k-plane sections in D_p and X runs over unit vector in L, respectively. For a submanifold M in a Riemannian manifold, the relative ξ-null space of M at a point $p \in M$ is defined by

$$N_\xi(p) = \{X \in D_p |\ h(X,Y) = 0 \quad \text{for all} \quad Y \in D_p\}.$$

3. Sectional curvature and shape operator

B. -Y. Chen ([5]) established a relation between the sectional curvature and the shape operator for submanifolds in real space forms. Also, K. Matsumoto, I. Mihai and A. Oiaga ([7]) have recently investigated these relations for slant submanifolds into complex space forms. We prove a similar inequality for an $(n+1)$-dimensional slant submanifold M into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c.

THEOREM 3.1. *Let $x : M \longrightarrow \widetilde{M}(c)$ be an isometric immersion of an $(n+1)$-dimensional θ-slant submanifold M into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c whose structure vector field ξ is tangent to M. If there exist a point $p \in M$ and a number $b > \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)} \cos^2\theta$ such that $\inf_D K(p) = K \ge b$ at* *p. Then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left\{b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\} I_n \quad \text{at} \quad p, \tag{3.1}$$

where I_n denotes the identity map in D_p which is naturally identified with an $(n+1)\times(n+1)$-matrix $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. Assume that M is a slant submanifold in $\widetilde{M}(c)$. Let $p \in M$ and a number $b > \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}\cos^2\theta$ such that $K \geq b$ at p. Choose an orthonormal basis $\{e_1, \cdots, e_{n+1}, \cdots, e_{2m+1}\}$ at p such that $e_{n+1} = \xi$, e_{n+2} is parallel to the mean curvature vector H and $e_1, \cdots, e_{n+1}$ diagonalize the shape operator A_{n+2}. Then we have

$$A_{n+2} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$A_r = (h_{ij}^r), \qquad \sum_{i=1}^{n+1}(h_{ii}^r) = 0, \quad 1 \leq i,j \leq n+1, \quad n+3 \leq r \leq 2m+1.$$

We put $u_{ij} = u_{ji} = a_i a_j$. From (2.5) we get

$$\begin{aligned} u_{ij} \geq\ & b - \frac{c+3}{4} - \frac{3(c-1)}{4}\langle e_i, \varphi e_j\rangle^2 \\ & - \sum_{r=n+3}^{2m+1}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\}, \quad 1 \leq i,j \leq n. \end{aligned} \tag{3.3}$$

We need the following lemmas in order to complete the proof of the theorem.

LEMMA 3.2. *The following statements hold.*

(1) *For any fixed $i \in \{1, \cdots, n\}$, we have $\sum_{j\neq i} u_{ij} \geq (n-1)\{b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\}$.*

(2) *$u_{ij} \neq 0$ for $i \neq j \in \{1, \cdots, n\}$.*

(3) *For distinct $i, j, k \in \{1, 2, \cdots, n\}$, we have $a_i^2 = u_{ij}u_{ik}u_{jk}^{-1}$.*

Proof. Together with (2.4), (3.2) and (3.3), we get

$$\begin{aligned}
&\sum_{j\neq i} u_{ij}\\
&\geq (n-1)\left\{b-\frac{c+3}{4}-\frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}-\sum_{r=n+3}^{2m+1}\left\{h_{ii}^r\sum_{j\neq i}h_{jj}^r-\sum_{i\neq j}(h_{ij}^r)^2\right\}\\
&=(n-1)\left\{b-\frac{c+3}{4}-\frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}+\sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2,
\end{aligned}$$

which yields statement (1).

For statement (2), if $u_{ij}=0$ for $i\neq j$, then $a_i=0$ or $a_j=0$. $a_i=0$ implies that $u_{it}=0$ for any $i\neq t$. Hence, $\sum_{i\neq t}u_{it}=0$ which contradicts the statement (1).

(3) follows from $u_{ij}u_{ik}=a_i^2a_ja_k=a_i^2u_{jk}$. □

We put $S_k=\{B\subset\{1,\cdots,n\}:|B|=k\}$. For any $B\in S_k$ we denote by $\bar{B}=\{1,\cdots,n\}\backslash B$.

LEMMA 3.3. *For a fixed k, $1\leq k\leq[\frac{n}{2}]$, and each $B\in S_k$, we have*

$$\sum_{j\in B}\sum_{t\in\bar{B}}u_{jt}\geq(n-k)k\{b-\frac{c+3}{4}-\frac{3(c-1)}{4(n-k)k}\cos^2\theta\}.$$

Proof. Without loss of generality, we may assume $B=\{1,\cdots,k\}$. From (3.3) together with the last equation of (3.2) we find

$$\begin{aligned}
\sum_{j\in B}\sum_{t\in\bar{B}}u_{jt}&\geq(n-k)k\{b-\frac{c+3}{4}-\frac{3(c-1)}{4(n-k)k}\cos^2\theta\}\\
&\quad-\sum_{r=n+3}^{2m+1}\sum_{j=1}^{k}\sum_{t=k+1}^{n+1}\{h_{jj}^rh_{tt}^r-(h_{jt}^r)^2\}\\
&=(n-k)k\{b-\frac{c+3}{4}-\frac{3(c-1)}{4(n-k)k}\cos^2\theta\}\\
&\quad+\sum_{r=n+3}^{2m+1}\{\sum_{j=1}^{k}\sum_{t=k+1}^{n+1}(h_{jt}^r)^2+\sum_{j=1}^{k}(h_{jj}^r)^2\},
\end{aligned}$$

which implies the lemma. □

LEMMA 3.4. *For any $1\leq i\neq j\leq n$, we have $u_{ij}>0$.*

Proof. Assume $u_{1n} < 0$. Then, by statement (3) of Lemma 3.2, we get $u_{1i}u_{in} < 0$ for $1 < i < n$. Without loss of generality, we may assume

$$\begin{cases} u_{12}, \cdots, u_{1l}, u_{(l+1)n}, \cdots, u_{(n-1)n} > 0 \\ u_{1(l+1)}, \cdots, u_{1n}, u_{2n}, \cdots, u_{ln} < 0 \end{cases} \tag{3.4}$$

for some $[\frac{n+1}{2}] \le l \le n-1$.

If $l = n-1$, then $u_{1(n)} + u_{2(n)} + \cdots + u_{(n-1)n} < 0$ which contradicts to statement (1) of Lemma 3.2. Thus, $l < n-1$. From statement (3) of Lemma 3.2, we get

$$a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \tag{3.5}$$

where $2 \le i \le l$ and $l+1 \le t \le n-1$. By (3.4) and (3.5), we have $u_{it} < 0$ which implies

$$\sum_{i=1}^{l} \sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.$$

This contradicts to Lemma 3.3. □

Now, we return to the proof of Theorem 3.1. From Lemma 3.4, it follows that $a_1, \cdots, a_n$ are of the same sign. Assume $a_j > 0$ for all $j \in \{1, \cdots, n\}$. Then from the statement (1) of Lemma 3.2, we get

$$\begin{aligned} a_i n\|H\| - a_i^2 &= a_i(a_1 + \cdots + a_n) - a_i^2 \\ &= a_i \sum_{i\neq j} a_j = \sum_{i\neq j} a_i a_j = \sum_{i\neq j} u_{ij} \\ &\ge (n-1)\left\{b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}. \end{aligned}$$

This inequality implies that

$$a_i\|H\| > \frac{n-1}{n}\left\{b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\},$$

and consequently (3.1) is established. This completes the proof of the theorem. □

Let I_n be the same matrix in Theorem 3.1. Then, we have

COROLLARY 3.5. *Let M be an isometric immersion of an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c whose structure vector field ξ is tangent to M. If there exist a point $p \in M$ and a number $b > \frac{c+3}{4}$ such that*

$inf_D K(p) = K \geq b$ at p. Then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}\left\{b - \frac{c+3}{4}\right\} I_n.$$

COROLLARY 3.6. *Let M be an isometric immersion of an $(n+1)$-dimensional invariant submanifold of a Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c whose structure vector field ξ is tangent to M. If there exist a point $p \in M$ and a number $b > \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$ such that $inf_D K(p) = K \geq b$ at p. Then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left\{b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\right\} I_n.$$

4. k-Ricci curvature and shape operator

In this section, we establish a relation between the k-Ricci curvature and the shape operator for an $(n+1)$-dimensional slant submanifold M into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c.

THEOREM 4.1. *Let $x : M \longrightarrow \widetilde{M}(c)$ be an isometric immersion of an $(n+1)$-dimensional θ-slant submanifold M into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c whose structure vector field ξ is tangent to M. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

(1) *If $\theta_k^D(p) \neq \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}\cos^2\theta$, then shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\} I_n \quad \text{at} \quad p, \tag{4.1}$$

where I_n denotes the identity map of D_p identified with the $(n+1) \times (n+1)$-matrix $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

(2) *If $\theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}\cos^2\theta$, then $A_H \geq 0$ at p.*

(3) *A unit vector $X \in D_p$ satisfies*

$$A_H X = \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\} X \tag{4.2}$$

if and only if $\theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}\cos^2\theta$ and $X \in N_\xi(p)$.

Proof. Let $\{e_1, \cdots, e_n, e_{n+1} = \xi\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1\cdots i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$ in D_p. It follows from (2.9) and (2.10) that

$$\tau^D(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i\in\{i_1,\cdots,i_k\}} Ric_{L_{i_1\cdots i_k}}(e_i), \tag{4.3}$$

$$\tau^D(p) = \frac{1}{\binom{n-2}{k-2}} \sum_{1\leq i_1<\cdots<i_k\leq n} \tau^D(L_{i_1\cdots i_k}). \tag{4.4}$$

Combining (2.14), (4.3) and (4.4), we find

$$\tau^D(p) \geq \frac{n(n-1)}{2}\theta_k^D(p). \tag{4.5}$$

From the equation of Gauss (2.5) for $X = Z = e_i, Y = W = e_j$, by summing over $\{1, 2, \cdots, n\}$ with respect to i and j $(i \neq j)$, we obtain

$$n^2 || H||^2 = 2\tau^D + || h||^2 - n(n-1)\frac{c+3}{4} - \frac{3(c-1)}{4}||T||^2. \tag{4.6}$$

We choose an orthonormal basis $\{e_1, \cdots, e_n, e_{n+1} = \xi, e_{n+2}, \cdots, e_{2m+1}\}$ at p such that e_{n+2} is parallel to the mean curvature vector $H(p)$ and $e_1, \cdots, e_{n+1} = \xi$ diagonalize the shape operator A_{n+2}. Then we have the relations (3.2) and (3.3). From (4.6) we get
(4.7)

$$n^2|| H||^2 = 2\tau^D + \sum_{i=1}^n a_i^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^n (h_{ij}^r)^2 - n(n-1)\frac{c+3}{4} - \frac{3(c-1)}{4}||T||^2.$$

On the other hand, since

$$0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^n a_i^2 - 2\sum_{i<j} a_i a_j,$$

which implies

$$n^2|| H||^2 = (\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i<j} a_i a_j \leq n \sum_{i=1}^n a_i^2. \tag{4.8}$$

We have from (4.7) and (4.8)

$$n^2|| H||^2 \geq 2\tau^D + n|| H||^2 - n(n-1)\frac{(c+3)}{4} - \frac{3(c-1)}{4}||T||^2, \tag{4.9}$$

or equivalently

$$|| H||^2 \geq \frac{2\tau^D}{n(n-1)} - \frac{c+3}{4} - \frac{3(c-1)}{4n(n-1)}||T||^2. \tag{4.10}$$

From (4.5) and (4.10), we have

$$\begin{aligned} \|H\|^2(p) &\geq \theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4n(n-1)}\|T\|^2 \\ &= \theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta. \end{aligned} \tag{4.11}$$

This show that $H(p) = 0$ may occur only when $\theta_k^D(p) \leq \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$ $\cos^2\theta$. Consequently, if $H(p) = 0$, statements (1) and (2) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$. From the Gauss' equation we get

$$a_i a_j = K_{ij} - \frac{c+3}{4} - \frac{3(c-1)}{4}\langle e_i, \varphi e_j\rangle^2 - \sum_{r=n+3}^{2m+1}\{h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\}. \tag{4.12}$$

By (4.12) we have

$$\begin{aligned} &a_1(a_{i_2} + \cdots + a_{i_k}) \\ &= Ric_{L_{1i_2\cdots i_k}}(e_1) - \frac{(k-1)(c+3)}{4} - \frac{3(c-1)}{4}\sum_{j=2}^{k}\langle e_1, \varphi e_{i_j}\rangle^2 \\ &\quad + \sum_{r=n+3}^{2m+1}\sum_{j=2}^{k}\{(h_{1i_j}^r)^2 - h_{11}^r h_{i_j i_j}^r\}, \end{aligned} \tag{4.13}$$

which yields

$$\begin{aligned} &a_1(a_2 + \cdots + a_n) \\ &= \frac{1}{\binom{n-2}{k-2}} \sum_{2\leq i_2 < \cdots < i_k \leq n} Ric_{L_{1i_2\cdots i_k}}(e_1) \\ &\quad - \frac{(n-1)(c+3)}{4} - \frac{3(c-1)}{4}\sum_{j=2}^{n}\langle e_1, \varphi e_j\rangle^2 + \sum_{r=n+3}^{2m+1}\sum_{j=1}^{n}(h_{1j}^r)^2. \end{aligned} \tag{4.14}$$

From (2.4), (2.14) and (4.14) we have

$$a_1(a_2 + \cdots + a_n) \geq (n-1)\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}. \tag{4.15}$$

Then
(4.16)
$$\begin{aligned} a_1(a_1+\cdots+a_n) &= a_1^2 + a_1(a_2+\cdots+a_n) \\ &\geq a_1^2 + (n-1)\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\} \\ &\geq (n-1)\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}. \end{aligned}$$

Similar inequalities hold when 1 were replaced by $j \in \{2,\cdots,n\}$. Hence, we have

$$\begin{aligned} & a_j(a_1+\cdots+a_n) \\ \geq\ & (n-1)\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\}, \quad j \in \{1,\cdots,n\}, \end{aligned}$$

which yields

$$A_H \geq \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right\} I_n.$$

The equality does not hold because $H(p) \neq 0$. Thus, (4.1) is valid. The statement (2) is obvious.

(3) Let X be a unit vector in D_p satisfying (4.2). By (4.16) and (4.14) one has $a_1 = 0$ and $h_{1j}^r = 0$, for all $j \in \{1,\cdots,n\}, r \in \{n+3,\cdots,2m+1\}$, respectively. It follows that $\theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}\cos^2\theta$ and $X \in N_\xi(p)$. The converse is clear. This completes the proof of the theorem. □

COROLLARY 4.2. *Let $x : M \longrightarrow \widetilde{M}(c)$ be an isometric immersion of an $(n+1)$-dimensional invariant submanifold M into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c whose structure vector field ξ is tangent to M. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

(1) *If $\theta_k^D(p) \neq \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$, then shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\right\} I_n \quad \text{at} \quad p,$$

where I_n denotes the identity map of D_p identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

(2) *If $\theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$ then $A_H \geq 0$ at p.*

(3) *A unit vector* $X \in D_p$ *satisfies*

$$A_H X = \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\right\}X$$

if and only if $\theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$ *and* $X \in N_\xi(p)$.

(4)

$$A_H = \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\right\}I_n \quad \text{at} \quad p$$

if and only if p is a totally geodesic point.

COROLLARY 4.3. *Let* $x : M \longrightarrow \widetilde{M}(c)$ *be an isometric immersion of an* $(n+1)$*-dimensional anti-invariant submanifold* M *into a* $(2m+1)$*-dimensional Sasakian space form* $\widetilde{M}(c)$ *of constant* φ*-sectional curvature* c *whose structure vector field* ξ *is tangent to* M. *Then, for any integer* k, $2 \le k \le n$, and any point $p \in M$, we have

(1) *If* $\theta_k^D(p) \neq \frac{c+3}{4}$, *then shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4}\right\}I_n \quad \text{at} \quad p,$$

where I_n *denotes the identity map of* D_p *identified with* $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

(2) *If* $\theta_k^D(p) = \frac{c+3}{4}$ *then* $A_H \ge 0$ *at* p.

(3) *A unit vector* $X \in D_p$ *satisfies*

$$A_H X = \frac{n-1}{n}\left\{\theta_k^D(p) - \frac{c+3}{4}\right\}X$$

if and only if $\theta_k^D(p) = \frac{c+3}{4}$ *and* $X \in N_\xi(p)$.

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