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ON THE STABILITY OF THE JENSEN'S EQUATION IN A HILBERT MODULE

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ABSTRACT. We prove the generalized Hyers-Ulam-Rassias stability of the invertible mapping in a Banach module over a unital Banach algebra, and prove the generalized Hyers-Ulam-Rassias stability of the Jensen's functional equations in a Hilbert module over a unital C^*-algebra.

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th. M. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $x \in E_1$.

In this paper, let A be a unital Banach algebra with norm $|\cdot|$, $A_1 = \{a \in A \mid \ |a| = 1\}$, and ${}_A\mathcal{H}$ a left Banach A-module with norm $\|\cdot\|$. Throughout this paper, assume that $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are mappings such that $F(tx)$ and $G(tx)$ are continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$.

We are going to prove the generalized Hyers-Ulam-Rassias stability of the invertible mapping in a Banach module over a unital Banach algebra.

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LEMMA 1. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ such that*

$$\text{(i)} \qquad \widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\|2F(\frac{ax+ay}{2}) - aF(x) - aF(y)\| \leq \varphi(x,y)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{H}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(ii)} \qquad \|F(x) - F(0) - T(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x))$$

for all $x \in {}_A\mathcal{H}$.

Proof. By [2, Theorem 1], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii). The additive mapping T given in the proof of [2, Theorem 1] is similar to the additive mapping T given in the proof of [5, Theorem]. By the same reasoning as the proof of [5, Theorem], it follows from the assumption that $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{H}$ that the additive mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is $\mathbb{R}$-linear.

By the assumption, for each $a \in A_1$,

$$\|2F(3^n ax) - aF(2 \cdot 3^{n-1}x) - aF(4 \cdot 3^{n-1}x)\| \leq \varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)$$

for all $x \in {}_A\mathcal{H}$. Using the fact that there exists a $K > 0$ such that, for each $a \in A$ and each $z \in {}_A\mathcal{H}$, $\|az\| \leq K|a| \cdot \|z\|$, one can show that

$$\begin{aligned}
&\|\frac{1}{2}aF(2 \cdot 3^{n-1}x) + \frac{1}{2}aF(4 \cdot 3^{n-1}x) - aF(3^n x)\| \\
&\leq \frac{1}{2}K|a| \cdot \|2F(3^n x) - F(2 \cdot 3^{n-1}x) - F(4 \cdot 3^{n-1}x)\| \\
&\leq \frac{K}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)
\end{aligned}$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. So

$$\begin{aligned}
&\|F(3^n ax) - aF(3^n x)\| \\
&\leq \|F(3^n ax) - \frac{1}{2}aF(2 \cdot 3^{n-1}x) - \frac{1}{2}aF(4 \cdot 3^{n-1}x)\| \\
&\quad + \|\frac{1}{2}aF(2 \cdot 3^{n-1}x) + \frac{1}{2}aF(4 \cdot 3^{n-1}x) - aF(3^n x)\| \\
&\leq \frac{1}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x) + \frac{K}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x)
\end{aligned}$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F(3^n ax) - aF(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in A_1$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax) = \lim_{n\to\infty} \frac{F(3^n ax)}{3^n} = \lim_{n\to\infty} \frac{aF(3^n x)}{3^n} = aT(x)$$

for each $a \in A_1$. So

$$T(ax) = |a|T(\frac{a}{|a|}x) = |a|\frac{a}{|a|}T(x) = aT(x)$$

for all $a \in A \setminus \{0\}$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear mapping, as desired. □

THEOREM 2. *Let $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (i) *such that*

$$\|2F(\frac{ax + ay}{2}) - aF(x) - aF(y)\| \le \varphi(x, y),$$
$$\|2G(\frac{ax + ay}{2}) - aG(x) - aG(y)\| \le \varphi(x, y)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{H}$. Assume that $F(3^n x) = 3^n F(x)$ and $G(3^n x) = 3^n G(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear mappings. Furthermore, if the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfy the inequalities

$$\|F \circ G(x) - x\| \le \varphi(x, x),$$
$$\|G \circ F(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping G is the inverse of the mapping F.

Proof. By the same method as the proof of Lemma 1, one can show that there exists a unique A-linear mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\|G(x) - L(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in {}_A\mathcal{H}$.

By the assumption,

$$T(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = F(x),$$
$$L(x) = \lim_{n\to\infty} \frac{G(3^n x)}{3^n} = G(x)$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 1. Hence the A-linear mappings T and L are the mappings F and G, respectively. So the mappings $F, G : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are A-linear mappings.

Now by the assumption,

$$\|F \circ G(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x),$$
$$\|G \circ F(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$3^{-n}\|F \circ G(3^n x) - 3^n x\| \to 0,$$
$$3^{-n}\|G \circ F(3^n x) - 3^n x\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F \circ G(x) = \lim_{n\to\infty} \frac{F \circ G(3^n x)}{3^n} = x,$$
$$G \circ F(x) = \lim_{n\to\infty} \frac{G \circ F(3^n x)}{3^n} = x$$

for all $x \in {}_A\mathcal{H}$. So the mapping G is the inverse of the mapping F. □

From now on, let A be a unital C^*-algebra with norm $|\cdot|$, A_1^+ the set of positive elements in A_1, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $\|\cdot\|$.

Now we are going to prove the generalized Hyers-Ulam-Rassias stability of linear functional equations in a Hilbert module over a unital C^*-algebra.

LEMMA 3. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (i) *such that*

$$\|2F(\frac{ax + ay}{2}) - aF(x) - aF(y)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then there exists a unique A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii).

Proof. By the same reasoning as the proof of Lemma 1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii).

By the same method as the proof of Lemma 1, one can obtain that

$$T(ax) = \lim_{n\to\infty} \frac{F(3^n ax)}{3^n} = \lim_{n\to\infty} \frac{aF(3^n x)}{3^n} = aT(x)$$

for each $a \in A_1^+ \cup \{i\}$. So

$$\begin{aligned} T(ax) = |a|T(\frac{a}{|a|}x) =& |a|\frac{a}{|a|}T(x) = aT(x), \quad \forall a \in A^+ \setminus \{0\},\ \forall x \in {}_A\mathcal{H}, \\ T(ix) =& iT(x), \quad \forall x \in {}_A\mathcal{H}. \end{aligned}$$

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, where $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [1, Lemma 38.8]). So

$$\begin{aligned} &T(ax) \\ &= T((\frac{a+a^*}{2})^+x - (\frac{a+a^*}{2})^-x + i(\frac{a-a^*}{2i})^+x - i(\frac{a-a^*}{2i})^-x) \\ &= (\frac{a+a^*}{2})^+T(x) + (\frac{a+a^*}{2})^-T(-x) + (\frac{a-a^*}{2i})^+T(ix) \\ &\quad + (\frac{a-a^*}{2i})^-T(-ix) \\ &= (\frac{a+a^*}{2})^+T(x) - (\frac{a+a^*}{2})^-T(x) + i(\frac{a-a^*}{2i})^+T(x) \\ &\quad - i(\frac{a-a^*}{2i})^-T(x) \\ &= ((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x) = aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{H}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{H}$. So the unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator, as desired. □

THEOREM 4. *Let $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (i) *such that*

$$\|2F(\frac{ax+ay}{2}) - aF(x) - aF(y)\| \leq \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Assume that $F(3^n x) = 3^n F(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an A-linear operator. Furthermore,

(1) *if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|F(x) - F^*(x)\| \leq \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator,

(2) *if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|F \circ F^*(x) - F^* \circ F(x)\| \leq \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a normal operator,

(3) *if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|F \circ F^*(x) - x\| \leq \varphi(x, x),$$
$$\|F^* \circ F(x) - x\| \leq \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a unitary operator, and

(4) *if the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|F \circ F(x) - F(x)\| \leq \varphi(x, x),$$
$$\|F^*(x) - F(x)\| \leq \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a projection.

Proof. By the assumption,

$$T(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = F(x)$$

for all $x \in {}_A\mathcal{H}$, where the operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Lemma 3. So the A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is the mapping $F : {}_A\mathcal{H} \to {}_A\mathcal{H}$.

(1) By the assumption,

$$\|F(3^n x) - F^*(3^n x)\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F(3^n x) - F^*(3^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F(x) = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F^*(3^n x)}{3^n} = F^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a self-adjoint operator.

(2) By the assumption,

$$\|F \circ F^*(3^n x) - F^* \circ F(3^n x)\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus $3^{-n}\|F \circ F^*(3^n x) - F^* \circ F(3^n x)\| \to 0$ as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F \circ F^*(x) = \lim_{n\to\infty} \frac{F \circ F^*(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F^* \circ F(3^n x)}{3^n} = F^* \circ F(x)$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a normal operator.

(3) By the assumption,

$$\|F \circ F^*(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x),$$
$$\|F^* \circ F(3^n x) - 3^n x\| \le \varphi(3^n x, 3^n x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$3^{-n}\|F \circ F^*(3^n x) - 3^n x\| \to 0,$$
$$3^{-n}\|F^* \circ F(3^n x) - 3^n x\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$F \circ F^*(x) = \lim_{n\to\infty} \frac{F \circ F^*(3^n x)}{3^n} = x,$$
$$F^* \circ F(x) = \lim_{n\to\infty} \frac{F^* \circ F(3^n x)}{3^n} = x$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a unitary operator.

(4) By the assumption,

$$\begin{aligned} \|F \circ F(3^n x) - F(3^n x)\| &\le \varphi(3^n x, 3^n x), \\ \|F^*(3^n x) - F(3^n x)\| &\le \varphi(3^n x, 3^n x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} 3^{-n}\|F \circ F(3^n x) - F(3^n x)\| &\to 0, \\ 3^{-n}\|F^*(3^n x) - F(3^n x)\| &\to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$\begin{aligned} F \circ F(x) &= \lim_{n\to\infty} \frac{F \circ F(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = F(x), \\ F^*(x) &= \lim_{n\to\infty} \frac{F^*(3^n x)}{3^n} = \lim_{n\to\infty} \frac{F(3^n x)}{3^n} = F(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the A-linear mapping F is a projection. □

REMARK. When the inequalities

$$\|2F(\frac{ax + ay}{2}) - aF(x) - aF(y)\| \le \varphi(x, y)$$

in the statements of the above results are replaced by the inequalities

$$\|2aF(\frac{x + y}{2}) - F(ax) - F(ay)\| \le \varphi(x, y)$$

or the inequalities

$$\begin{aligned} \|2F(\frac{x + y}{2}) - F(x) - F(y)\| &\le \varphi(x, y), \\ \|F(ax) - aF(x)\| &\le \varphi(x, x), \end{aligned}$$

the results do also hold. The proofs are similar to the proofs of the results.

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