

**GL_n - DECOMPOSITION OF
THE SCHUR COMPLEX $S_r(\wedge^2\varphi)$**

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ABSTRACT. In this paper we construct a natural filtration associated to the plethysm $S_r(\wedge^2\varphi)$ over arbitrary commutative ring R . Let $\varphi : G \rightarrow F$ be a morphism of finite free R -modules. We construct the natural filtration of $S_r(\wedge^2\varphi)$ as a $GL(F) \times GL(G)$ -complex such that its associated graded complex is $\sum_{\lambda \in \Omega_r^-} L_{2\lambda}\varphi$, where Ω_r^- is a set of partitions such that $|\lambda| = r$ and 2λ is a partition of which i -th term is $2\lambda_i$. Specializing our result, we obtain the filtrations of $S_r(\wedge^2 F)$ and $D_r(D_2 G)$.

1. Introduction

Akin, Buchsbaum, Weyman [3] have introduced and studied the Schur and Weyl modules parametrized by Young diagrams, which turn out to be the natural generalizations to commutative rings of the constructions of the classical representations of the general linear group given by I. Schur [14] and H. Weyl [15], respectively. In fact, the more general notion of Schur complexes of a complex in the category of finitely generated projective modules is defined; the Schur and Weyl modules result as special cases of the Schur complexes, whose usefulness is abundant. For instance, Schur complexes play central roles in the resolutions of determinantal and pfaffian ideals [3], [7] and in the characteristic-free representation theory of the general linear group [1], [2], [8]. This forces us to further study Schur complexes. One way to study Schur complexes is to look for complex-theoretic versions of classical character relations for the general linear group. The formal character of the Schur module

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is a Schur function. D. E. Littlewood [10], [11] introduced plethysm as an operation on symmetric functions. So we have plethysm on Schur and Weyl modules. Let R be a commutative ring with identity and let $\varphi : G \rightarrow F$ be a morphism of finitely generated free R -modules. We denote by $\wedge F$, SF and DF , the exterior, the symmetric, and the divided power algebra of F , respectively. Over the last decade or so, plethysms such as $S(S_2F)$, $S(\wedge^2 F)$, $S_r F \otimes L_\lambda F$, $\wedge^r F \otimes K_\lambda F$, and $S_k \varphi \otimes L_\lambda \varphi$ have been studied mostly in connection with invariant theory, resolutions of determinantal ideals, and the characteristic-free representation theory of the general linear group [5], [8], [9], [4], [13].

The purpose of this paper is to provide the natural filtration associated to the plethysm $S_r(\wedge^2 \varphi)$ as a $GL(F) \times GL(G)$ -complex such that its associated graded complex is

$$\sum_{\lambda \in \Omega_r^-} L_{2\lambda} \varphi.$$

Section 2 covers the basic definitions and some of the important properties utilized in the main body of the paper. In Section 3 we construct a natural filtration of $S_r(\wedge^2 \varphi)$. And as Corollaries, we obtain the natural filtrations of $S_r(\wedge^2 F)$ and $D_r(D_2G)$ easily.

2. Preliminaries

In this section, we review some of the basic facts and notations that will be used throughout. Therefore all proofs are omitted. As for the proofs of Theorems, we refer to Akin, Buchsbaum, and Weyman [2], Hashimoto and Kurano [7], and Macdonald's book [12]. Throughout this chapter, R is a commutative ring and $\varphi : G \rightarrow F$ is an R -module homomorphism between free R -modules of rank n and m , respectively. We will denote by \mathbb{N} (resp. \mathbb{N}_0) the set of natural numbers (resp. non-negative integers) and by Ω^+ the set of sequence of elements of \mathbb{N} of finite support. For any elements λ and μ of Ω^+ and $k \in \mathbb{N}_0$, we define $\lambda + \mu$ to be $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$ and $k \cdot \lambda$ to be the sequence $(k\lambda_1, k\lambda_2, \dots)$.

DEFINITION 2.1. For any $\lambda = (\lambda_1, \lambda_2, \dots) \in \Omega^+$, the number of nonzero terms of λ is called the length of λ . The weight of λ , denoted by $|\lambda|$, is the sum $\sum \lambda_i$. We put $\Omega_n^+ = \{\lambda \in \Omega^+ \mid |\lambda| = n\}$, $\Omega^- = \{\lambda \in \Omega^+ \mid \forall i \in \mathbb{N} \lambda_i \geq \lambda_{i+1}\}$, and $\Omega_n^- = \Omega_n^+ \cap \Omega^-$. We call an element of Ω^- a *partition*. To each partition λ of weight n , we associate its transpose

$\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_t)$, where $\tilde{\lambda}_k$ is the number of integers λ_l such that $\lambda_l \geq k$. If $\mu = (\mu_1, \mu_2, \dots)$ is also a partition, we will say that μ is a sub-partition of λ , or that $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i .

We denote by 0 the element $(0, 0, \dots) \in \Omega_0^-$. Let $k \in \mathbb{N}$. We define the element ε_k of Ω_1^+ given by $\forall i \in \mathbb{N}, (\varepsilon_k)_i = \delta_{ki}$ (Kronecker's δ). We now set $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$. For $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$, we define the subset $S_{\square}(\lambda/\mu)$ of Ω^+ by

$$S_{\square}(\lambda/\mu) = \{\nu \in \Omega^+ \mid \exists t \in \mathbb{N}, \exists k \in \mathbb{N}_0 \quad k < \lambda_{t+1} - \mu_t \\ \text{and } \nu = \lambda - \mu + (\lambda_{t+1} - \mu_{t+1} - k) \cdot \alpha_t\}.$$

$S_{\square}(\lambda)$ stands for $S_{\square}(\lambda/0)$.

DEFINITION 2.2. The diagram (or shape) of an element $\lambda \in \Omega^+$ is the set $\{(i, j) \in \mathbb{N}^2 \mid j \leq \lambda_i\}$, and is denoted by Δ_{λ} . The skew-shape of a pair $\lambda, \mu \in \Omega^+$, such that $\lambda \supset \mu$, is $\Delta_{\lambda} - \Delta_{\mu}$ and is denoted by $\Delta_{\lambda/\mu}$.

Clearly, $\Delta_{\lambda/0} = \Delta_{\lambda}$.

DEFINITION 2.3. Let X be a totally ordered set and let $\lambda, \mu \in \Omega^+$. We define $\text{Tab}_{\lambda/\mu}(X)$ to be the set $\text{Hom}_{\text{set}}(\Delta_{\lambda/\mu}, X)$. An element of $\text{Tab}_{\lambda/\mu}(X)$ is called a tableau of shape λ/μ with values in the set X .

DEFINITION 2.4. For $\lambda, \mu \in \Omega^+$ with $\text{lg}(\lambda - \mu) \leq q$, we define $S_{\lambda/\mu}F, \wedge_{\lambda/\mu}F, D_{\lambda/\mu}F$ as follows :

$$S_{\lambda/\mu}F = S_{\lambda_1 - \mu_1}F \otimes \dots \otimes S_{\lambda_q - \mu_q}F \\ \wedge_{\lambda/\mu}F = \wedge^{\lambda_1 - \mu_1}F \otimes \dots \otimes \wedge^{\lambda_q - \mu_q}F \\ D_{\lambda/\mu}F = D_{\lambda_1 - \mu_1}F \otimes \dots \otimes D_{\lambda_q - \mu_q}F.$$

Since $S_0F = \wedge^0F = D_0F = R$, this definition does not depend on the choice of q . If $\lambda \not\supseteq \mu$, then $S_{\lambda/\mu}F = \wedge_{\lambda/\mu}F = D_{\lambda/\mu}F = 0$ by definition.

For $\lambda, \mu \in \Omega^-$, let $s = \tilde{\lambda}_1$ and $t = \lambda_1$ and (a_{ij}) be the $s \times t$ matrix given by $a_{ij} = 1$ if $(i, j) \in \Delta_{\lambda/\mu}$ and $a_{ij} = 0$ otherwise. We denote by

$d_{\lambda/\mu}(F)$ the composition of maps

$$\begin{aligned}
& \Lambda_{\lambda/\mu} F = \wedge^{\lambda_1 - \mu_1} F \otimes \cdots \otimes \wedge^{\lambda_q - \mu_q} F \\
& \xrightarrow{\Delta \otimes \cdots \otimes \Delta} (\wedge^{a_{11}} F \otimes \cdots \otimes \wedge^{a_{1t}} F) \otimes \cdots \otimes (\wedge^{a_{s1}} F \otimes \cdots \otimes \wedge^{a_{st}} F) \\
(1) \quad & \longrightarrow (S_{a_{11}} F \otimes \cdots \otimes S_{a_{1t}} F) \otimes \cdots \otimes (S_{a_{s1}} F \otimes \cdots \otimes S_{a_{st}} F) \\
& \longrightarrow (S_{a_{11}} F \otimes \cdots \otimes S_{a_{s1}} F) \otimes \cdots \otimes (S_{a_{1t}} F \otimes \cdots \otimes S_{a_{st}} F) \\
& \xrightarrow{m \otimes \cdots \otimes m} S_{\tilde{\lambda}_1 - \tilde{\mu}_1} F \otimes \cdots \otimes S_{\tilde{\lambda}_t - \tilde{\mu}_t} F = S_{\tilde{\lambda}/\tilde{\mu}} F,
\end{aligned}$$

where the second map is induced by identification $\wedge^{a_{ij}} F = S_{a_{ij}} F$ and the third map is the permutation according to the index a_{ij} . Similarly, we define the map $d'_{\lambda/\mu}(F)$ from $D_{\lambda/\mu} F$ to $\wedge_{\tilde{\lambda}/\tilde{\mu}} F$.

DEFINITION 2.5 (Schur functors and co-Schur functors). $\text{Im}(d_{\lambda/\mu}(F))$ (resp. $\text{Im}(d'_{\lambda/\mu}(F))$) is denoted by $L_{\lambda/\mu} F$ (resp. $K_{\lambda/\mu} F$). $L_{\lambda/\mu} F$ (resp. $K_{\lambda/\mu} F$) is called Schur (resp. co-Schur) functor with respect to the skew shape λ/μ .

If $R = \mathbb{Q}$, then $K_{\lambda/\mu} F$ is isomorphic to $L_{\tilde{\lambda}/\tilde{\mu}} F$ as a $GL(F)$ -module, and is irreducible for $\mu = (0)$.

The definition of Schur complex is quite similar to that of Schur functor. Let $\varphi : G \rightarrow F$ be a morphism of finite free R -modules. We associate $S\varphi$ and $\wedge\varphi$ with the morphism φ . $S\varphi = SF \otimes \wedge G$ and $\wedge\varphi = \wedge F \otimes DG$ are Hopf algebras, and their structures as Hopf algebras are not dependent on the morphism φ . They depend only on the modules F and G . We can also consider them as complexes. The boundary map of $S\varphi$ is the composition of the maps

$$\begin{aligned}
\partial^{S\varphi} : S\varphi &= SF \otimes \wedge G \xrightarrow{id_{SF} \otimes \Delta_{\wedge G}} SF \otimes \wedge^1 G \otimes \wedge G \\
&\xrightarrow{id_{SF} \otimes \varphi \otimes id_{\wedge G}} SF \otimes S_1 F \otimes \wedge G \xrightarrow{m_{SF} \otimes id_{\wedge G}} SF \otimes \wedge G = S\varphi,
\end{aligned}$$

where we identify $\wedge^1 G = G$ and $S_1 F = F$. Similarly, the boundary map of $\wedge\varphi$ is the composition of the maps

$$\begin{aligned}
\partial^{\wedge\varphi} : \wedge\varphi &= \wedge F \otimes DG \xrightarrow{id_{\wedge F} \otimes \Delta_{DG}} \wedge F \otimes D_1 G \otimes DG \\
&\xrightarrow{id_{\wedge F} \otimes \varphi \otimes id_{DG}} \wedge F \otimes \wedge^1 F \otimes DG \xrightarrow{m_{\wedge F} \otimes id_{DG}} \wedge F \otimes DG = \wedge\varphi.
\end{aligned}$$

For $i \in \mathbb{N}_0$, we denote by $S_i\varphi$ the subcomplex of $S\varphi$ given by

$$0 \rightarrow \wedge^i G \rightarrow F \otimes \wedge^{i-1} G \rightarrow \cdots \rightarrow S_{i-1} F \otimes G \rightarrow S_i F \rightarrow 0.$$

$S\varphi$ is the graded R -complex $\sum_{i \in \mathbb{N}_0} S_i\varphi$, and $\deg(S_i\varphi) = 2i$ in $S\varphi$. Similarly, we denote by $\wedge^i\varphi$ the subcomplex of $\wedge\varphi$ given by

$$0 \longrightarrow D_i G \longrightarrow F \otimes D_{i-1} G \longrightarrow \cdots \longrightarrow \wedge^{i-1} F \otimes G \longrightarrow \wedge^i F \longrightarrow 0$$

$\wedge\varphi$ is the graded R -complex $\sum_{i \in \mathbb{N}_0} \wedge^i\varphi$, and $\deg(\wedge^i\varphi) = i$ in $\wedge\varphi$. Note that $\wedge^i\varphi$ is isomorphic to $(S_i\varphi^*)^*$ as an R -complex. We always assume that $S_i\varphi = \wedge^i\varphi = 0$ if $i < 0$.

Now let $\lambda, \mu \in \Omega^+$ with $\text{lg}(\lambda - \mu) \leq q$.

DEFINITION 2.6. We define $S_{\lambda/\mu}\varphi$ and $\wedge_{\lambda/\mu}\varphi$ as follows:

$$\begin{aligned} S_{\lambda/\mu}\varphi &= S_{\lambda_1 - \mu_1}\varphi \otimes \cdots \otimes S_{\lambda_q - \mu_q}\varphi, \\ \wedge_{\lambda/\mu}\varphi &= \wedge^{\lambda_1 - \mu_1}\varphi \otimes \cdots \otimes \wedge^{\lambda_q - \mu_q}\varphi. \end{aligned}$$

Now let $\lambda, \mu \in \Omega^-$. We define $d_{\lambda/\mu}(\varphi)$ as the similar composite map to $d_{\lambda/\mu}(F)$. We have only to replace every F by φ in the definition (1). The Schur complex of φ with respect to the skew shape λ/μ is $\text{Im}(d_{\lambda/\mu}\varphi)$. If $G = 0$, then $L_{\lambda/\mu}\varphi = L_{\lambda/\mu}F$. If $F = 0$, then $L_{\lambda/\mu}\varphi = K_{\lambda/\mu}G$ in degree $|\lambda - \mu|$.

Let us fix an ordered basis $X = \{x_1 < \cdots < x_m\}$ of F and an ordered basis $Y = \{y_1 < \cdots < y_n\}$ of G . We put $Z = X \cup Y$ and let Z be a totally ordered basis for which $X < Y$, that is, for which $x_i < y_j$ for any i and j .

DEFINITION 2.7. A tableau $S \in \text{Tab}_{\lambda/\mu}(X)$ is called row-standard if the rows of S are strictly increasing, i.e., if for all $i = 1, \dots, q$ we have $S(i, \mu_i + 1) < S(i, \mu_i + 2) < \cdots < S(i, \lambda_i)$. The tableau S is called column-standard if the columns of S are non-decreasing, i.e., if for all $j = 1, \dots, t$ ($t = \lambda_1$) we have $S(i, j) \leq S(i + 1, j)$ where (i, j) and $(i + 1, j)$ are both in $\Delta_{\lambda/\mu}$. S is called standard if it is both row- and column-standard.

DEFINITION 2.8. A tableau $T \in \text{Tab}_{\lambda/\mu}(Y)$ is called co-row-standard if the rows of T are non-decreasing, and co-column-standard if the columns of T are strictly increasing. T is called co-standard if it is co-row- and co-column-standard.

Clearly, the set $\{X_S \mid S \in \text{Tab}_{\lambda/\mu}(X) \text{ and } S \text{ is row-standard}\}$ is a free basis of $\wedge_{\lambda/\mu}F$ and the set $\{Y_T \mid T \in \text{Tab}_{\lambda/\mu}(Y) \text{ and } T \text{ is co-row-standard}\}$ is a free basis of $D_{\lambda/\mu}G$.

For $U \in \text{Tab}_{\lambda/\mu}(Z)$, we obtain an element $Z_U = z_{U_1} \otimes \cdots \otimes z_{U_q}$, with

$$z_{U_i} = \varepsilon_{U_i} \cdot U(i, \alpha_1) \wedge \cdots \wedge U(i, \alpha_l) \otimes y_1^{(t_1^i)} \cdots y_n^{(t_n^i)},$$

where

$$\begin{aligned} \varepsilon_{U_i} &= (-1)^{\#\{(\alpha, \beta) | \alpha > \beta, U(i, \alpha) \in X, U(i, \beta) \in Y\}}, \\ \{\alpha_1, \cdots, \alpha_l\} &= \{j | (i, j) \in \Delta_{\lambda/\mu} \text{ and } U(i, j) \in X\} \end{aligned}$$

with $\alpha_1 < \cdots < \alpha_l$ and $t_j^i = n_i(U, y_j)$ for each j .

DEFINITION 2.9. A tableau $U \in \text{Tab}_{\lambda/\mu}(Z)$ is called row-standard mod Y if each row of U is non-decreasing, and if, when repeats occur in a row, they occur only among elements of Y . U is column-standard mod Y if each column is non-decreasing, and if, when repeats occur in a column, they occur only among elements in $Z - Y = X$. U is standard mod Y if U is row- and column-standard mod Y .

We let $\text{Row}_{\lambda/\mu}(Z, Y) = \{U \in \text{Tab}_{\lambda/\mu}(Z) | U \text{ is row-standard mod } Y\}$. It is easy to see that $\{Z_U \in \wedge_{\lambda/\mu} \varphi | U \in \text{Row}_{\lambda/\mu}(Z, Y)\}$ is a basis of $\wedge_{\lambda/\mu} \varphi$. Before stating the standard basis theorem, we introduce the notion of universally free functor.

DEFINITION 2.10. Let $T_R(F_1, \cdots, F_n)$ be a functor to be the category of R -modules defined for all commutative rings R and all n -tuples of (finite) free R -modules F_1, \cdots, F_n . T_R is called universal if $T_S(S \otimes -, \cdots, S \otimes -)$ is naturally equivalent to $S \otimes T_R(-, \cdots, -)$ for any ring morphism $\phi : R \rightarrow S$. T_R is called universally free if T_R is universal and $T_R(F_1, \cdots, F_n)$ is free for any n -tuple F_1, \cdots, F_n . Let $f_R : T_R \rightarrow T'_R$ be a natural transformation of universal functors defined for all commutative rings R . f_R is called universal if for any ring morphism $\phi : R \rightarrow S$ and any n -tuple of R -modules F_1, \cdots, F_n the diagram

$$\begin{array}{ccc} S \otimes T_R(F_1, \cdots, F_n) & \xrightarrow{S \otimes f_R(F_1, \cdots, F_n)} & S \otimes T'_R(F_1, \cdots, F_n) \\ \simeq \downarrow & & \simeq \downarrow \\ T_S(S \otimes F_1, \cdots, S \otimes F_n) & \xrightarrow{f_S(S \otimes F_1, \cdots, S \otimes F_n)} & T'_S(S \otimes F_1, \cdots, S \otimes F_n) \end{array}$$

is commutative.

For example, $SF, \wedge F$ and DF are universally free. Their structure morphisms as Hopf algebras are universal. Tensor product and direct

sum of universally free functors are universally free. If $f_R : T_R \rightarrow T'_R$ is universal, then $\text{Coker}(f_R)$ is universal functor and the map $\text{coker } f_R : T'_R \rightarrow \text{Coker}(f_R)$ is universal, since $S \otimes -$ preserves cokernel.

THEOREM 2.11. (The Standard Basis Theorem for Schur Complexes)
Let $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$, and let φ, X, Y , and Z be as above.

$$\{d_{\lambda/\mu}(Z_T) \mid T \text{ is standard tableau mod } Y \text{ in } \text{Tab}_{\lambda/\mu}(Z)\}$$

is a free basis for $L_{\lambda/\mu}\varphi$, and the sequence

$$\sum_{\nu \in S_{\square}(\lambda/\mu)} \wedge_{\nu} \varphi \xrightarrow{\square_{\lambda/\mu}} \wedge_{\lambda/\mu} \varphi \xrightarrow{d_{\lambda/\mu} \varphi} L_{\lambda/\mu} \varphi \rightarrow 0$$

is exact. Hence $L_{\lambda/\mu}\varphi$ is universally free and $d_{\lambda/\mu}(\varphi)$ is universal. In particular,

$$\{d_{\lambda/\mu}(X_T) \mid T \text{ is a standard tableau in } \text{Tab}_{\lambda/\mu}(X)\}$$

$$\text{(resp. } \{d'_{\lambda/\mu}(Y_T) \mid T \text{ is a co-standard tableau in } \text{Tab}_{\lambda/\mu}(Y)\})$$

is a free basis of $L_{\lambda/\mu}F$ (resp. $K_{\lambda/\mu}G$), and the sequence

$$\sum_{\nu \in S_{\square}(\lambda/\mu)} \wedge_{\nu} F \xrightarrow{\square_{\lambda/\mu}} \wedge_{\lambda/\mu} F \xrightarrow{d_{\lambda/\mu}(F)} L_{\lambda/\mu} F \rightarrow 0$$

$$\text{(resp. } \sum_{\nu \in S_{\square}(\lambda/\mu)} D_{\nu} G \xrightarrow{\square_{\lambda/\mu}} D_{\lambda/\mu} G \xrightarrow{d'_{\lambda/\mu}(G)} K_{\lambda/\mu} G \rightarrow 0)$$

is exact. Hence, $L_{\lambda/\mu}F$ and $K_{\lambda/\mu}G$ are universally free.

The rest of this section is devoted to review the theory of symmetric functions is deeply related to the representation theory of general linear groups.

Let $x = x_1, \dots, x_n, \dots$ be an infinite sequence of indeterminates. For $n \in \mathbb{N}_0$, we denote by R_n the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$. The symmetric group \mathfrak{S}_n acts on R_n via the permutations of indeterminates. We denote by Λ_n the invariant subring $R_n^{\mathfrak{S}_n}$. Λ_n is decomposed into direct sums: $\Lambda_n = \sum_{k \geq 0} \Lambda_n^k$, where Λ_n^k is the module of homogeneous invariants of degree k . For integers m, n with $m \geq n \geq 0$, we define $\rho_{n,m} : R_m \rightarrow R_n$ to be the map given by $\rho_{n,m}(x_i) = x_i$ ($i \leq n$) and $\rho_{n,m}(x_i) = 0$ ($i > n$). It is easy to see that $\rho_{n,m}(\Lambda_m^k) = \Lambda_n^k$. We define $\Lambda^k = \varinjlim_n \Lambda_n^k$, and $\Lambda = \sum_{k \geq 0} \Lambda^k$. $\Lambda_x = \Lambda$ has the structure of a graded ring. We call an element of Λ a *symmetric function* (of x). Λ is not dependent on the order of the sequence x . It only depends on x as a set.

DEFINITION 2.12. Let $r \in \mathbb{N}_0$. We define e_r and h_r to be the symmetric functions given by

$$e_r = e_r(x) = \sum_{\lambda \in \Omega_r^+, \bar{\lambda}_1=r} x^\lambda = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} \cdot x_{i_2} \cdots x_{i_r} \in \Lambda^r,$$

$$h_r = h_r(x) = \sum_{\lambda \in \Omega_r^+} x^\lambda \in \Lambda^r, \text{ where } x^\lambda = \prod_{i \in \mathbb{N}} x_i^{\lambda_i}.$$

Furthermore, for a partition λ , we define

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq \text{lg}(\lambda)} \quad (h_l = 0 \text{ for } l \leq 0)$$

and call S_λ the Schur function corresponding to the partition λ .

LEMMA 2.13 [12, I.(2.4)]. $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$ and the e_i are algebraically independent over \mathbb{Z} .

By Lemma 2.13, we can define a ring homomorphism $\omega (= \omega_x) : \Lambda_x \longrightarrow \Lambda_x$ given by $\omega(e_r) = h_r$.

LEMMA 2.14 [12, I.(2,7)]. ω is an involution (i.e., $\omega^2 = \text{id}_\Lambda$).

LEMMA 2.15 [12, I.(3,3)]. $\{s_\lambda \mid \lambda \in \Omega_k^-\}$ is a \mathbb{Z} -basis of Λ^k .

Now let $\mu \in \Omega_k^-$ and $\nu \in \Omega_l^-$ ($k, l \in \mathcal{N}_0$). We can write

$$s_\mu \cdot s_\nu = \sum_{\lambda \in \Omega^-} c_{\mu, \nu}^\lambda \cdot s_\lambda \quad (c_{\mu, \nu}^\lambda \in \mathbb{Z}).$$

Since $s_\mu \cdot s_\nu \in \Lambda^{k+l}$, $|\lambda| \neq k+l$ implies $c_{\mu, \nu}^\lambda = 0$. For arbitrary partitions λ and μ , we define

$$s_{\lambda/\mu} = \sum_{\nu \in \Omega^-} c_{\mu, \nu}^\lambda \cdot s_\nu$$

and call $s_{\lambda/\mu}$ a skew Schur function corresponding to λ and μ .

LEMMA 2.16 [12, I.(5,4),(5,5),(5,6)].

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq \bar{\lambda}_1} = \det(e_{\bar{\lambda}_i - \bar{\mu}_j - i + j})_{1 \leq i, j \leq \lambda_1}$$

so that $\omega(s_{\lambda/\mu}) = s_{\bar{\lambda}/\bar{\mu}}$.

LEMMA 2.17 [12, I.(4.10)]. *The involution ω is an isometry, i.e., $\langle \omega u, \omega v \rangle = \langle u, v \rangle$.*

Now we consider two sets of indeterminates $x = x_1, x_2, \dots$ and $y = y_1, y_2, \dots$.

LEMMA 2.18 [12, I.Example 5 (a), (b)]. *We have two equations:*

$$\sum_{\mu: \text{ even}} s_\mu = \prod_i (1 - x_i^2)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1},$$

where the sum on the left is over all even partitions μ (i.e., with all parts μ_i even). And

$$\sum_{\tilde{\nu}: \text{ even}} s_{\tilde{\nu}} = \prod_{i < j} (1 - x_i x_j)^{-1}.$$

LEMMA 2.19 [12, I.Example 27 (a)].

$$\sum_{\rho: \text{ even}} s_{\rho/\lambda} = \prod_{i \leq j} (1 - x_i x_j)^{-1} \cdot \sum_{\tau: \text{ even}} s_{\lambda/\tau}$$

and

$$\sum_{\tilde{\rho}: \text{ even}} s_{\tilde{\rho}/\lambda} = \prod_{i < j} (1 - x_i x_j)^{-1} \cdot \sum_{\tilde{\tau}: \text{ even}} s_{\lambda/\tilde{\tau}}.$$

LEMMA 2.20 [7, Lemma I.4.10]. *Let $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$. We have*

$$\text{rank } L_{\lambda/\mu} F = \text{rank } K_{\tilde{\lambda}/\tilde{\mu}} F = s_{\tilde{\lambda}/\tilde{\mu}}(1_m),$$

where $s_{\tilde{\lambda}/\tilde{\mu}}(1_m)$ is the value of skew Schur function in m variables $S_{\tilde{\lambda}/\tilde{\mu}}(x_1, \dots, x_m)$ at $x_1 = \dots = x_m = 1$.

Now for a morphism of finite free R -modules $\varphi : G \rightarrow F$ with $\text{rank } F = m$ and $\text{rank } G = n$, and for partitions $\lambda, \mu \in \Omega^-$ with $\lambda \supset \mu$, the rank of the underlying module of $L_{\lambda/\mu}\varphi$ is calculated as follows:

$$\text{rank } L_{\lambda/\mu}\varphi = \sum_{\mu \subset \gamma \subset \lambda} \text{rank}[L_{\gamma/\mu} F \otimes K_{\lambda/\gamma} G] = \sum_{\mu \subset \gamma \subset \lambda} s_{\lambda/\gamma}(1_n) \cdot s_{\tilde{\gamma}/\tilde{\mu}}(1_m).$$

3. The decomposition of $\mathcal{S}(\wedge^2\varphi)$

For a morphism of finite free R -modules $\varphi : G \rightarrow F$, we have a complex

$$\wedge^2\varphi : 0 \rightarrow D_2G \xrightarrow{\alpha} G \otimes F \xrightarrow{\beta} \wedge^2F \rightarrow 0,$$

where $\alpha = \beta = \partial^{\wedge\varphi} : D_iG \otimes \wedge^jF \xrightarrow{\Delta \otimes 1} D_{i-1}G \otimes G \otimes \wedge^jF \xrightarrow{1 \otimes \varphi \otimes 1} D_{i-1}G \otimes F \otimes \wedge^jF \xrightarrow{1 \otimes m_{\wedge^jF}} D_{i-1}G \otimes \wedge^{j+1}F$.

Now by applying a functor S to $\wedge^2\varphi$, we obtain a new complex

$$S(\wedge^2\varphi) = S(\wedge^2F) \otimes \wedge(F \otimes G) \otimes D(D_2G) = \oplus S_r(\wedge^2\varphi),$$

where $S_r(\wedge^2\varphi) = \sum_{a+b+c=r} S_a(\wedge^2F) \otimes \wedge^b(F \otimes G) \otimes D_c(D_2G)$. Its boundary map is

$$\begin{aligned} \partial_l^{S(\wedge^2\varphi)} : (S(\wedge^2\varphi))_l &= \sum_{b+2c=l} S(\wedge^2F) \otimes \wedge^b(F \otimes G) \otimes D_c(D_2G) \\ &\quad \downarrow \partial^{S(\beta)} \otimes id + (-1)^b id \otimes \partial^{\wedge(\alpha)} \\ (S(\wedge^2\varphi))_{l-1} &= \sum_{b'+2c'=l-1} S(\wedge^2F) \otimes \wedge^{b'}(F \otimes G) \otimes D_{c'}(D_2G). \end{aligned}$$

For a nonnegative integer r , in order to construct a natural filtration of $S_r(\wedge^2\varphi)$, we need to define some maps.

DEFINITION 3.1. We define a map $\theta_k = \theta_k(\varphi)$ as the composite map

$$\theta_k = \theta_k(\varphi) : \wedge^{2k}\varphi \xrightarrow{\Delta \otimes \dots \otimes \Delta} \underbrace{\wedge^2\varphi \otimes \dots \otimes \wedge^2\varphi}_{k\text{-times}} \xrightarrow{\frac{1}{k!} \cdot m_{S(\wedge^2\varphi)}} S_k(\wedge^2\varphi).$$

Moreover, for a partition $\lambda \in \Omega_r^-$, the following composite map is denoted by θ_λ :

$$\begin{aligned} \wedge^{2\lambda}\varphi &= \wedge^{2\lambda_1}\varphi \otimes \dots \otimes \wedge^{2\lambda_q}\varphi \\ &\quad \downarrow \theta_{\lambda_1} \otimes \dots \otimes \theta_{\lambda_q} \\ S_{\lambda_1}(\wedge^2\varphi) &\otimes \dots \otimes S_{\lambda_q}(\wedge^2\varphi) \\ &\quad \downarrow m_{S(\wedge^2\varphi)} \\ S_{\lambda_1 + \dots + \lambda_q}(\wedge^2\varphi) &= S_r(\wedge^2\varphi). \end{aligned}$$

NOTE. We can extend the map θ_λ to one for any sequence μ , so that $\text{Im } \theta_\mu = \text{Im } \theta_{\tilde{\mu}}$.

DEFINITION 3.2. For each $k \in \mathbb{N}$, we define a map ϖ_k as the compositions

$$\begin{aligned} \varpi_k : \wedge^k\varphi \otimes \wedge^k\varphi &\xrightarrow{\Delta^{\otimes(k)} \otimes \Delta^{\otimes(k)}} \underbrace{(\wedge_{l_1}^1\varphi \otimes \cdots \otimes \wedge_{l_k}^1\varphi)}_{k\text{-times}} \otimes \underbrace{(\wedge_{r_1}^1\varphi \otimes \cdots \otimes \wedge_{r_k}^1\varphi)}_{k\text{-times}} \\ &\xrightarrow{T} (\wedge_{l_1}^1\varphi \otimes \wedge_{r_1}^1\varphi) \otimes \cdots \otimes (\wedge_{l_k}^1\varphi \otimes \wedge_{r_k}^1\varphi) \xrightarrow{m_{\wedge\varphi} \otimes \cdots \otimes m_{\wedge\varphi}} \underbrace{\wedge^2\varphi \otimes \cdots \otimes \wedge^2\varphi}_{k\text{-times}} \\ &\xrightarrow{(\pm)^{\frac{1}{k!}} \cdot m_{S(\wedge^2\varphi)}} S_k(\wedge^2\varphi), \end{aligned}$$

where (\pm) means that we can take the appropriate sign as the situation i.e., for each $0 \leq i, j \leq k$ and $(\wedge^k\varphi)_i \otimes (\wedge^k\varphi)_j \subseteq \wedge^k\varphi \otimes \wedge^k\varphi$, we give

$$\begin{cases} + & \text{when } k-j \equiv 0 \text{ or } k+j \equiv 1 \pmod{4} \\ - & \text{otherwise} \end{cases}$$

and $\wedge_{l_*}^1\varphi$ is obtained from the left $\wedge^k\varphi$, $\wedge_{r_*}^1\varphi$ is obtained from the right $\wedge^k\varphi$, but these are just used in order to distinguish the origin, i.e., $\wedge_{l_*}^1\varphi = \wedge_{r_*}^1\varphi = \wedge^1\varphi$.

NOTE. We can easily see that the diagram

$$\begin{array}{ccc} \wedge^{2k}\varphi & \xrightarrow{\Delta} & \wedge^k\varphi \otimes \wedge^k\varphi \\ \pm 2^k \cdot \theta_k \downarrow & & \varpi_k \downarrow \\ S_k(\wedge^2\varphi) & \xrightarrow{id} & S_k(\wedge^2\varphi) \end{array}$$

is commutative.

DEFINITION 3.3. For a partition $\lambda \in \Omega_r^-$, $GL(\mathbb{F}) \times GL(\mathbb{G})$ -subcomplexes \mathcal{M}_λ and $\dot{\mathcal{M}}_\lambda$ of $S_r(\wedge^2\varphi)$ are defined as

$$\mathcal{M}_\lambda = \sum_{\mu \in \Omega_r^-, \mu \geq \lambda} \text{Im } \theta_\mu \quad \text{and} \quad \dot{\mathcal{M}}_\lambda = \sum_{\mu \in \Omega_r^-, \mu > \lambda} \text{Im } \theta_\mu.$$

NOTE. When $\lambda_0 = (1^r)$, $\mathcal{M}_{\lambda_0} = S_r(\wedge^2\varphi)$.

THEOREM 3.4 (Plethysm Formula on $S_r(\wedge^2\varphi)$). *For any arbitrary commutative ring R and a non-negative integer r , $\{\mathcal{M}_\lambda\}_{\lambda \in \Omega_r^-}$ is a natural filtration of $S_r(\wedge^2\varphi)$ such that its associated graded complex is $\sum_{\lambda \in \Omega_r^-} L_{2\lambda}\varphi$.*

Proof. We have only to show that for any partition $\lambda \in \Omega_r^-$, $\mathcal{M}_\lambda/\dot{\mathcal{M}}_\lambda$ is isomorphic to $L_{2\lambda}\varphi$ as a $\text{GL}(\mathbb{F}) \times \text{GL}(\mathbb{G})$ -module. Now consider the diagram

$$\begin{array}{ccc} \wedge^{2\lambda}\varphi & \xrightarrow{\theta_\lambda} & \mathcal{M}_\lambda \subseteq S_r(\wedge^2\varphi) \\ d_{2\lambda}(\varphi) \downarrow & & \rho_\lambda \downarrow \\ L_{2\lambda}\varphi & \xrightarrow{\beta_\lambda} & \mathcal{M}_\lambda/\dot{\mathcal{M}}_\lambda, \end{array}$$

where ρ_λ is the projection. Let ψ_λ be the composite map $\rho_\lambda \circ \theta_\lambda$. In order to construct the isomorphism $\beta_\lambda : L_{2\lambda}\varphi \cong \mathcal{M}_\lambda/\dot{\mathcal{M}}_\lambda$, it is sufficient to show that $\ker d_{2\lambda}(\varphi) = \ker \psi_\lambda$.

1) We will prove $\ker d_{2\lambda}(\varphi) \subseteq \ker \psi_\lambda$. Since $L_{2\lambda}\varphi = \wedge^{2\lambda}\varphi/\text{Im}(\square)$, this is equivalent to show that $\text{Im}(\square) \subseteq \ker \psi_\lambda$, i.e., $\psi_\lambda \circ \square = 0$ for each partition $\lambda \in \Omega_r^-$.

Let $\lambda \in \Omega_r^-$ be of the form $(\lambda_1, \lambda_2, \dots, \lambda_q)$. Then 2λ is $(2\lambda_1, 2\lambda_2, \dots, 2\lambda_q)$ and the map \square is

$$\begin{aligned} & \sum_{t=1}^{q-1} \sum_{\nu=1}^{2\lambda_{t+1}} \wedge^{2\lambda_1}\varphi \otimes \dots \otimes \wedge^{2\lambda_{t-1}}\varphi \otimes \wedge^{2\lambda_t+\nu}\varphi \otimes \wedge^{2\lambda_{t+1}-\nu}\varphi \otimes \dots \otimes \wedge^{2\lambda_q}\varphi \\ & \square = \sum_{t=1}^{q-1} \sum_{\nu=1}^{2\lambda_{t+1}} 1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(q-t-1)} \downarrow \\ & \wedge^{2\lambda}\varphi = \wedge^{2\lambda_1}\varphi \otimes \dots \otimes \wedge^{2\lambda_{t-1}}\varphi \otimes \wedge^{2\lambda_t}\varphi \otimes \wedge^{2\lambda_{t+1}}\varphi \otimes \dots \otimes \wedge^{2\lambda_q}\varphi. \end{aligned}$$

In order to prove $\psi_\lambda \circ \square = 0$, we have only to show that $\psi_\lambda \circ (1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(q-t-1)}) = 0$ for each t and ν . And it is sufficient to show that

$$\theta_\lambda \circ \square_\nu (\wedge^{2\lambda_1+\nu}\varphi \otimes \wedge^{2\lambda_2-\nu}\varphi) \in \sum_{\mu \in \Omega_{\lambda_1+\lambda_2}^-, \mu > \lambda} \text{Im } \theta_\mu = \dot{\mathcal{M}}_\lambda$$

for any partition $\lambda = (\lambda_1, \lambda_2)$ of length 2 and $1 \leq \nu \leq 2\lambda_2$.

Fortunately we have a nice claim : Let $l = \min\{\nu, 2\lambda_2 - \nu\}$.

The diagram

$$\begin{array}{ccc} \wedge^{2\lambda_1+\nu}\varphi \otimes \wedge^{2\lambda_2-\nu}\varphi & \xrightarrow{\square_\nu} & \wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2}\varphi \\ \alpha \downarrow & & \theta_{(\lambda_1, \lambda_2)} \downarrow \\ \bigoplus_k (\wedge^{2\lambda_1+\nu+k}\varphi \otimes \wedge^{2\lambda_2-\nu-k}\varphi) & \xrightarrow{\beta} & S_{\lambda_1+\lambda_2}(\wedge^2\varphi) \end{array}$$

is commutative, where

$$\alpha = \sum_{\nu \equiv k \pmod{2}} \tilde{\square}_k \text{ and } \beta = \bigoplus_k C_{(\lambda_1, \lambda_2)}^{[\nu, k]} \cdot \theta_{(\lambda_1 + \frac{\nu+k}{2}, \lambda_2 - \frac{\nu+k}{2})}$$

for some $C_{(\lambda_1, \lambda_2)}^{[\nu, k]} \in \mathbb{Z}$.

If we proved the above claim, we obtain the result $\theta_\lambda \cdot \square_\nu \in \dot{\mathcal{M}}_\lambda$.

We will use the induction on λ_1 and l .

i) For $\lambda_1 = 1$, then $\lambda = (1, 1)$, $2\lambda = (2, 2)$, $\nu = 1, 2$ and $l = 0, 1$.

a) When $l = 0$, i.e., $\nu = 2$: Since $\theta_1 = id_{(\wedge^2\varphi)}$, $\theta_{(2)} = \frac{1}{2!}(m_{S(\wedge^2\varphi)} \circ \Delta)$ and

$$\begin{array}{ccc} \wedge^4\varphi & \xrightarrow{\square_2} & \wedge^2\varphi \otimes \wedge^2\varphi \\ \Delta \downarrow & & \theta_{(1,1)} \downarrow \\ \wedge^2\varphi \otimes \wedge^2\varphi & \xrightarrow{m_{S(\wedge^2\varphi)}} & S_2(\wedge^2\varphi) \end{array}$$

is commutative, so $\theta_{(1,1)} \circ \square_2 = 2 \cdot \theta_{(2)} \circ \tilde{\square}_0$ and $C_{(1,1)}^{[2,0]} = 2$.

b) When $l = 1$, i.e., $\nu = 1$: Let $x \otimes y \in \wedge^3\varphi \otimes \wedge^1\varphi$, since $\theta_1 = id_{\wedge^2\varphi}$,

$$\begin{aligned} \theta_{(1,1)} \circ \square_1(x \otimes y) &= \theta_{(1,1)} \left(\sum_{x'} (\text{sgn}(x', x'')) x' \otimes x'' \cdot y \right) \\ &= \sum_{x'} (\text{sgn}(x', x'')) m_{S(\wedge^2\varphi)}(x' \otimes x'' \cdot y), \end{aligned}$$

where $\Delta(x) = \sum_{x'} (\text{sgn}(x', x'')) x' \otimes x'' \in \wedge^2 \varphi \otimes \wedge^1 \varphi$. And

$$\begin{aligned}
 \theta_{(2)} \circ \tilde{\square}_1(x \otimes y) &= \theta_{(2)}(x \cdot y) = \frac{1}{2!} \cdot [m_{S(\wedge^2 \varphi)} \circ \Delta](x \cdot y) \\
 &= \frac{1}{2!} \cdot m_{S(\wedge^2 \varphi)} \\
 &\quad \cdot \left[\sum_{x'} (\text{sgn}(x', x'')) x' \otimes x'' \cdot y + \sum_{x'} (\text{sgn}(x', x'')) x'' \cdot y \otimes x' \right] \\
 &= \frac{1}{2!} \cdot m_{S(\wedge^2 \varphi)} (2 \sum_{x'} (\text{sgn}(x', x'')) x' \otimes x'' \cdot y) \\
 &= \sum_{x'} (\text{sgn}(x', x'')) m_{S(\wedge^2 \varphi)}(x' \otimes x'' \cdot y).
 \end{aligned}$$

Hence we proved this case and $C_{(1,1)}^{[1,1]} = 1$.

ii) If $\lambda_1 > 1$, i.e., $\lambda = (\lambda_1, \lambda_2)$, $2\lambda = (2\lambda_1, 2\lambda_2)$, $\nu = 1, \dots, 2\lambda_2$ and $l = 0, 1, \dots, \lambda_2$.

a) When $l = 0$ i.e., $\nu = 2\lambda_2$: We have the commutative diagram

$$\begin{array}{ccc}
 \wedge^{2\lambda_1+2\lambda_2} \varphi & & \\
 \square_{2\lambda_2} \downarrow & & \\
 \wedge^{2\lambda_1} \varphi \otimes \wedge^{2\lambda_2} \varphi & \xrightarrow{\theta_{(\lambda_1, \lambda_2)}} & S_{\lambda_1+\lambda_2}(\wedge^2 \varphi) \\
 \Delta^{\otimes(\lambda_1)} \otimes \Delta^{\otimes(\lambda_2)} \downarrow & & m_{S(\wedge^2 \varphi)} \uparrow \\
 \underbrace{(\wedge^2 \varphi \otimes \dots \otimes \wedge^2 \varphi)}_{\lambda_1\text{-times}} \otimes \underbrace{(\wedge^2 \varphi \otimes \dots \otimes \wedge^2 \varphi)}_{\lambda_2\text{-times}} & \xrightarrow{\eta} & S_{\lambda_1}(\wedge^2 \varphi) \otimes S_{\lambda_2}(\wedge^2 \varphi),
 \end{array}$$

where $\eta = \frac{1}{\lambda_1!} \cdot m_{S(\wedge^2 \varphi)} \otimes \frac{1}{\lambda_2!} \cdot m_{S(\wedge^2 \varphi)}$. And $\theta_{(\lambda_1+\lambda_2)}$ is the compositions

$$\wedge^{2\lambda_1+2\lambda_2} \varphi \xrightarrow{\Delta^{\otimes \dots \otimes \Delta}} \underbrace{\wedge^2 \varphi \otimes \dots \otimes \wedge^2 \varphi}_{\lambda_1+\lambda_2\text{-times}} \xrightarrow{\frac{1}{(\lambda_1+\lambda_2)!} \cdot m_{S(\wedge^2 \varphi)}} S_{\lambda_1+\lambda_2}(\wedge^2 \varphi).$$

So, $\theta_{(\lambda_1, \lambda_2)} \circ \square_{2\lambda_2} = \binom{\lambda_1+\lambda_2}{\lambda_1} \cdot \theta_{(\lambda_1+\lambda_2)} \circ \tilde{\square}_0$ and $C_{(\lambda_1, \lambda_2)}^{[2\lambda_2, 0]} = \binom{\lambda_1+\lambda_2}{\lambda_1}$.

When $l = 1$, then $\nu = 1$ or $2\lambda_2 - 1$. We consider first ($\nu = 1$)- case.

b) When $l = 1$ and $\nu = 1$: We have two commutative diagrams

$$\begin{array}{ccc}
 \wedge^{2\lambda_1+1}\varphi \otimes \wedge^{2\lambda_2-1}\varphi & \xrightarrow{\square_1} & \wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2}\varphi \xrightarrow{\theta_{(\lambda_1,\lambda_2)}} S_{\lambda_1+\lambda_2}(\wedge^2\varphi) \\
 \Delta \otimes \Delta \downarrow & & \uparrow \theta_{(\lambda_1,1,\lambda_2-1)} \\
 \wedge^{2\lambda_1}\varphi \otimes [\wedge^1\varphi \otimes \wedge^1\varphi] \otimes \wedge^{2\lambda_2-2}\varphi & \xrightarrow{1 \otimes m \otimes 1} & \wedge^{2\lambda_1}\varphi \otimes \wedge^2\varphi \otimes \wedge^{2\lambda_2-2}\varphi \\
 \Delta \otimes \Delta \uparrow & & \downarrow \theta_{(\lambda_1,1,\lambda_2-1)} \\
 \wedge^{2\lambda_1+1}\varphi \otimes \wedge^{2\lambda_2-1}\varphi & \xrightarrow{\tilde{\square}_1} & \wedge^{2\lambda_1+2}\varphi \otimes \wedge^{2\lambda_2-2}\varphi \xrightarrow{\theta_{(\lambda_1+1,\lambda_2-1)}} S_{\lambda_1+\lambda_2}(\wedge^2\varphi).
 \end{array}$$

Hence $\theta_{(\lambda_1,\lambda_2)} \circ \square_1 = \theta_{(\lambda_1+1,\lambda_2-1)} \circ \tilde{\square}_1$ and $C_{(\lambda_1,\lambda_2)}^{[1,1]} = 1$.

c) When $l = 1$, and $\nu = 2\lambda_2 - 1$: We have two commutative diagrams

$$\begin{array}{ccc}
 \wedge^{2\lambda_1+2\lambda_2-1}\varphi \otimes \wedge^1\varphi & \xrightarrow{\square_{2\lambda_2-1}} & \wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2}\varphi \xrightarrow{\theta_{(\lambda_1,\lambda_2)}} S_{\lambda_1+\lambda_2}(\wedge^2\varphi) \\
 \Delta \otimes id \downarrow & & \uparrow \theta_{(\lambda_1,\lambda_2-1,1)} \\
 \wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2-1}\varphi \otimes \wedge^1\varphi & \xrightarrow{id \otimes \square_1} & [\wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2-2}\varphi] \otimes \wedge^2\varphi
 \end{array}$$

and

$$\begin{array}{ccc}
 \wedge^{2\lambda_1+2\lambda_2-1}\varphi \otimes \wedge^1\varphi & \xrightarrow{\tilde{\square}_1} & \wedge^{2\lambda_1+2\lambda_2}\varphi \\
 \square_1 \downarrow & & \downarrow \theta_{(\lambda_1+\lambda_2)} \\
 [\wedge^{2\lambda_1+2\lambda_2-2}\varphi] \otimes \wedge^2\varphi & \xrightarrow{\theta_{(\lambda_1+\lambda_2-1,1)}} & S_{\lambda_1+\lambda_2}(\wedge^2\varphi).
 \end{array}$$

By the way, for the above two []'s, by ii)'s a), the diagram

$$\begin{array}{ccc}
 \wedge^{2\lambda_1+2\lambda_2-2}\varphi & \xrightarrow{\binom{\lambda_1+\lambda_2-1}{\lambda_1} \cdot \theta_{(\lambda_1+\lambda_2-1)}} & S_{\lambda_1+\lambda_2-1}(\wedge^2\varphi) \\
 \square_{2\lambda_2-2} \downarrow & & id \uparrow \\
 \wedge^{2\lambda_1}\varphi \otimes \wedge^{2\lambda_2-2}\varphi & \xrightarrow{\theta_{(\lambda_1,\lambda_2-1)}} & S_{\lambda_1+\lambda_2-1}(\wedge^2\varphi)
 \end{array}$$

is commutative. So, $\theta_{(\lambda_1,\lambda_2)} \circ \square_{2\lambda_2-1} = \binom{\lambda_1+\lambda_2-1}{\lambda_1} \cdot \theta_{(\lambda_1+\lambda_2)} \circ \tilde{\square}_1$ and $C_{(\lambda_1,\lambda_2)}^{[2\lambda_2-1,1]} = \binom{\lambda_1+\lambda_2-1}{\lambda_1}$.

d) When $l > 1$ and $\nu = 2, 3, \dots, 2\lambda_2 - 2$: Let $x \otimes y \in \wedge^{2\lambda_1 + \nu} \varphi \otimes \wedge^{2\lambda_2 - \nu} \varphi$ and for each $i \equiv l \pmod{2}$, let

$$\begin{aligned} \mathcal{U}_i &= \sum_{x'} (\text{sgn}(x', x'')) \cdot \sum_{y'} (\text{sgn}(y', y'')) \\ &\quad \cdot \theta_{\lambda_1 + \frac{\nu-i}{2}}(x') \cdot \varpi_i(x'' \otimes y'') \cdot \theta_{\lambda_2 - \frac{\nu+i}{2}}(y''), \end{aligned}$$

where $\Delta(x) = \sum_{x'} (\text{sgn}(x', x'')) x' \otimes x'' \in \wedge^{2\lambda_1 + \nu - i} \varphi \otimes \wedge^i \varphi$ and

$$\Delta(y) = \sum_{y'} (\text{sgn}(y', y'')) y' \otimes y'' \in \wedge^{2\lambda_2 - \nu - i} \varphi \otimes \wedge^i \varphi.$$

Now consider

$$\theta_{(\lambda_1, \lambda_2)} \circ \square_{\nu}(x \otimes y) = \theta_{(\lambda_1, \lambda_2)} \left(\sum_{x'_{(1)}} (\text{sgn}(x'_{(1)}, x''_{(1)})) (x'_{(1)} \otimes x''_{(1)} \cdot y) \right),$$

where $\Delta(x) = \sum_{x'_{(1)}} (\text{sgn}(x'_{(1)}, x''_{(1)})) x'_{(1)} \otimes x''_{(1)} \in \wedge^{2\lambda_1} \varphi \otimes \wedge^{\nu} \varphi$.

Then it is

$$\begin{aligned} & \sum_{x'_{(1)}} (\text{sgn}(x'_{(1)}, x''_{(1)})) \theta_{\lambda_1}(x'_{(1)}) \\ & \cdot \left[\sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \sum_{x'_{(2)}} \sum_{y'_{(1)}} (\text{sgn}(x'_{(2)}, x''_{(2)})) \right. \\ & \quad \left. (\text{sgn}(y'_{(1)}, y''_{(1)})) \theta_{\frac{\nu-k}{2}}(x'_{(2)}) \cdot \varpi_k(x''_{(2)} \otimes y'_{(1)}) \cdot \theta_{\lambda_2 - \frac{\nu+k}{2}}(y''_{(1)}) \right] \\ &= \sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \left[\sum_{x'_{(1)}} \sum_{x'_{(2)}} \sum_{y'_{(1)}} (\text{sgn}(x'_{(1)}, x''_{(1)})) (\text{sgn}(x'_{(2)}, x''_{(2)})) \right. \\ & \quad (\text{sgn}(y'_{(1)}, y''_{(1)})) [\theta_{\lambda_1}(x'_{(1)}) \cdot \theta_{\frac{\nu-k}{2}}(x'_{(2)})] \\ & \quad \left. \cdot \varpi_k(x''_{(2)} \otimes y'_{(1)}) \cdot \theta_{\lambda_2 - \frac{\nu+k}{2}}(y''_{(1)}) \right], \end{aligned}$$

where $\Delta(x'_{(1)}) = \sum_{x'_{(2)}} (\text{sgn}(x'_{(2)}, x''_{(2)})) x'_{(2)} \otimes x''_{(2)} \in \wedge^{\nu-k} \varphi \otimes \wedge^k \varphi$, and $\Delta(y) = \sum_{y'_{(1)}} (\text{sgn}(y'_{(1)}, y''_{(1)})) y'_{(1)} \otimes y''_{(1)} \in \wedge^k \varphi \otimes \wedge^{2\lambda_2 - \nu - k} \varphi$.

Since by ii)'s a), the diagram

$$\begin{array}{ccc} \Lambda^{2\lambda_1+\nu-k}\varphi & \xrightarrow{\binom{\lambda_1+\frac{\nu-k}{2}}{\lambda_1}\cdot\theta_{(\lambda_1+\frac{\nu-k}{2})}} & S_{\lambda_1+\frac{\nu-k}{2}}(\Lambda^2\varphi) \\ \Delta \downarrow & & \uparrow id \\ \Lambda^{2\lambda_1}\varphi \otimes \Lambda^{\nu-k}\varphi & \xrightarrow{\theta_{(\lambda_1, \frac{\nu-k}{2})}} & S_{\lambda_1+\frac{\nu-k}{2}}(\Lambda^2\varphi) \end{array}$$

is commutative, so the above equation is

$$\begin{aligned} (2) \quad & \sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \binom{\lambda_1 + \frac{\nu-k}{2}}{\lambda_1} \left[\sum_{x'_{(3)}} (\text{sgn}(x'_{(3)}, x''_{(3)})) \right. \\ & \cdot \sum_{y'_{(1)}} (\text{sgn}(y'_{(1)}, y''_{(1)})) \theta_{\lambda_1+\frac{\nu-k}{2}}(x'_{(3)}) \cdot \varpi_k(x''_{(3)}) \otimes y'_{(1)} \\ & \left. \cdot \theta_{\lambda_2-\frac{\nu+k}{2}}(y''_{(1)}) \right] \\ & = \sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \binom{\lambda_1 + \frac{\nu-k}{2}}{\lambda_1} \cdot \mathcal{U}_k, \end{aligned}$$

where $\Delta(x) = \sum_{x'_{(3)}} (\text{sgn}(x'_{(3)}, x''_{(3)})) x'_{(3)} \otimes x''_{(3)} \in \Lambda^{2\lambda_1+\nu-k}\varphi \otimes \Lambda^k\varphi$.

By the way, for $1 \leq k \leq l$ and $\nu \equiv k \pmod{2}$,

$$\begin{aligned} & \theta_{(\lambda_1+\frac{\nu+k}{2}, \lambda_2-\frac{\nu+k}{2})} \circ \tilde{\square}_k(x \otimes y) \\ & = \theta_{(\lambda_1+\frac{\nu+k}{2}, \lambda_2-\frac{\nu+k}{2})} \left(\sum_{y'_{(1)}} (\text{sgn}(y'_{(1)}, y''_{(1)})) (x \cdot y'_{(1)} \otimes y''_{(1)}) \right) \\ & = \sum_{y'_{(1)}} (\text{sgn}(y'_{(1)}, y''_{(1)})) \left[\sum_{1 \leq m \leq k, \nu \equiv k \equiv m \pmod{2}} \sum_{x'_{(4)}} \sum_{y'_{(2)}} \right. \\ & \quad (\text{sgn}(x'_{(4)}, x''_{(4)})) (\text{sgn}(y'_{(2)}, y''_{(2)})) \theta_{\lambda_1+\frac{\nu-m}{2}}(x'_{(4)}) \cdot \varpi_m(x''_{(4)}) \otimes y'_{(2)} \\ & \quad \left. \cdot \theta_{\frac{k-m}{2}}(y''_{(2)}) \cdot \theta_{\lambda_2-\frac{\nu+k}{2}}(y''_{(1)}) \right], \end{aligned}$$

where $\Delta(x) = \sum_{x'_{(4)}} (\text{sgn}(x'_{(4)}, x''_{(4)})) x'_{(4)} \otimes x''_{(4)} \in \Lambda^{2\lambda_1+\nu-m}\varphi \otimes \Lambda^m\varphi$

$$\begin{aligned}
 \text{and } \Delta(y'_{(1)}) &= \sum_{y'_{(2)}} (\text{sgn}(y'_{(2)}, y''_{(2)})) y'_{(2)} \otimes y''_{(2)} \in \wedge^m \varphi \otimes \wedge^{k-m} \varphi, \\
 &= \sum_{1 \leq m \leq k, \nu \equiv k \equiv m \pmod{2}} \sum_{y'_{(1)}} \sum_{x'_{(4)}} \sum_{y'_{(2)}} (\text{sgn}(y'_{(1)}, y''_{(1)})) \\
 &\quad (\text{sgn}(x'_{(4)}, x''_{(4)})) (\text{sgn}(y'_{(2)}, y''_{(2)})) \theta_{\lambda_1 + \frac{\nu-m}{2}}(x'_{(4)}) \cdot \varpi_m(x''_{(4)} \otimes y'_{(2)}) \\
 &\quad \cdot \left[\theta_{\frac{k-m}{2}}(y''_{(2)}) \cdot \theta_{\lambda_2 - \frac{\nu+k}{2}}(y''_{(1)}) \right].
 \end{aligned}$$

Since by ii)'s a), we have a commutative diagram

$$\begin{array}{ccc}
 \wedge^{2\lambda_2 - \nu - m} \varphi & \xrightarrow{\binom{\lambda_2 - \frac{\nu+m}{2}}{\frac{k-m}{2}} \cdot \theta_{(\lambda_2 - \frac{\nu+m}{2})}} & S_{\lambda_2 - \frac{\nu+m}{2}}(\wedge^2 \varphi) \\
 \Delta \downarrow & & \uparrow id \\
 \wedge^{k-m} \varphi \otimes \wedge^{2\lambda_2 - \nu - k} \varphi & \xrightarrow{\theta_{(\frac{k-m}{2}, \lambda_2 - \frac{\nu+k}{2})}} & S_{\lambda_2 - \frac{\nu+m}{2}}(\wedge^2 \varphi),
 \end{array}$$

the above equation is

$$\begin{aligned}
 &\sum_{1 \leq m \leq k, \nu \equiv k \equiv m \pmod{2}} \binom{\lambda_2 - \frac{\nu+m}{2}}{\frac{k-m}{2}} \sum_{x'_{(4)}} \sum_{y'_{(3)}} (\text{sgn}(x'_{(4)}, x''_{(4)})) \\
 &(\text{sgn}(y'_{(3)}, y''_{(3)})) \theta_{\lambda_1 + \frac{\nu-m}{2}}(x'_{(4)}) \cdot \varpi_m(x''_{(4)} \otimes y'_{(3)}) \cdot \theta_{\lambda_2 - \frac{\nu+m}{2}}(y''_{(3)}) \\
 (3) \quad &= \sum_{1 \leq m \leq k, \nu \equiv k \equiv m \pmod{2}} \binom{\lambda_2 - \frac{\nu+m}{2}}{\frac{k-m}{2}} \cdot \mathcal{U}_m,
 \end{aligned}$$

where $\Delta(y) = \sum_{y'_{(3)}} (\text{sgn}(y'_{(3)}, y''_{(3)})) y'_{(3)} \otimes y''_{(3)} \in \wedge^m \varphi \otimes \wedge^{2\lambda_2 - \nu - m} \varphi$

In (3), when $m = k$, $\binom{\lambda_2 - \frac{\nu+m}{2}}{\frac{k-m}{2}} = 1$. So

$$\begin{aligned}
 \mathcal{U}_k &= \theta_{(\lambda_1 + \frac{\nu+k}{2}, \lambda_2 - \frac{\nu+k}{2})} \circ \tilde{\square}_k(x \otimes y) \\
 &\quad - \sum_{1 < m < k, \nu \equiv k \equiv m \pmod{2}} \binom{\lambda_1 - \frac{\nu+m}{2}}{\frac{k-m}{2}} \cdot \mathcal{U}_m.
 \end{aligned}$$

Thus in (2),

$$\begin{aligned} \theta_{(\lambda_1, \lambda_2)} \circ \square_\nu(x \otimes y) &= \sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \binom{\lambda_1 + \frac{\nu-k}{2}}{\lambda_1} \cdot \mathcal{U}_k \\ &= \sum_{1 \leq k \leq l, \nu \equiv k \pmod{2}} \binom{\lambda_1 + \frac{\nu-k}{2}}{\lambda_1} \cdot \left[\theta_{(\lambda_1 + \frac{\nu+k}{2}, \lambda_2 - \frac{\nu+k}{2})} \circ \tilde{\square}_k(x \otimes y) \right. \\ &\quad \left. - \sum_{1 \leq m < k, \nu \equiv k \equiv m \pmod{2}} \binom{\lambda_1 - \frac{\nu+m}{2}}{\frac{k-m}{2}} \cdot \mathcal{U}_m \right]. \end{aligned}$$

By induction hypothesis, for $1 \leq m < k$, $\nu \equiv k \equiv m \pmod{2}$,

$$\mathcal{U}_m \in \text{Im}(\oplus_{1 \leq k' \leq m, \nu \equiv k' \equiv m \pmod{2}} \theta_{(\lambda_1 + \frac{\nu+k'}{2}, \lambda_2 - \frac{\nu+k'}{2})} \circ \tilde{\square}_{k'}).$$

Hence $\theta_{(\lambda_1, \lambda_2)} \circ \square_\nu = \oplus_{1 \leq k \leq l, \nu \equiv k \pmod{2}} (C_{(\lambda_1, \lambda_2)}^{[\nu, k]} \cdot \theta_{(\lambda_1 + \frac{\nu+k}{2}, \lambda_2 - \frac{\nu+k}{2})} \circ \tilde{\square}_k)$ and $C_{(\lambda_1, \lambda_2)}^{[\nu, k]} \in \mathbb{Z}$. Therefore, we completed the claim.

2) To prove $L_{2\lambda}\varphi \cong \mathcal{M}_\lambda/\dot{\mathcal{M}}_\lambda$, it remains only to show that

$$\sum_{\lambda \in \Omega_r^-} \text{rank}(L_{2\lambda}\varphi) = \text{rank}(S_r(\wedge^2\varphi)).$$

Let $\text{rank}(F) = m$ and $\text{rank}(G) = n$.

$$\begin{aligned} &\text{rank}(S_r(\wedge^2\varphi)) \\ &= \sum_{a+b+c=r} \text{rank}[S_a(\wedge^2 F) \otimes \wedge^b(F \otimes G) \otimes D_c(D_2G)] \\ &= \sum_{a+b+c=r} \text{rank}(S_a(\wedge^2 F)) \cdot \text{rank}(\wedge^b(F \otimes G)) \cdot \text{rank}(D_c(D_2G)). \end{aligned}$$

Since

$$\begin{aligned} \text{rank}(S_a(\wedge^2 F)) &= \sum_{\omega \in \Omega_a^-} \text{rank}(L_{2\lambda}F) = \sum_{\omega \in \Omega_a^-} s_{2\tilde{\omega}}(1_m), \\ \text{rank}(\wedge^b(F \otimes G)) &= \sum_{\mu \in \Omega_b^-} \text{rank}(L_\mu F \otimes K_\mu G) = \sum_{\mu \in \Omega_b^-} s_{\tilde{\mu}}(1_m) \cdot s_\mu(1_n), \\ \text{rank}(D_c(D_2G)) &= \sum_{\tau \in \Omega_c^-} \text{rank}(K_{2\tau}G) = \sum_{\tau \in \Omega_c^-} s_{2\tau}(1_n), \end{aligned}$$

so

$$\begin{aligned} & \text{rank}(S_r(\wedge^2 \varphi)) \\ &= \sum_{a+b+c=r} \left[\sum_{\omega \in \Omega_a^-} s_{2\omega}^{\sim}(1_m) \cdot \sum_{\mu \in \Omega_b^-} (s_{\bar{\mu}}(1_m) \cdot s_{\mu}(1_n)) \cdot \sum_{\tau \in \Omega_c^-} s_{2\tau}(1_n) \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \text{rank}(L_{2\lambda}(\varphi)) \\ &= \sum_{0 \subseteq \gamma \subseteq 2\lambda} \text{rank}(L_{\gamma}F \otimes K_{2\lambda/\gamma}G) = \sum_{0 \subseteq \gamma \subseteq 2\lambda} s_{2\lambda/\gamma}(1_n) \cdot s_{\bar{\gamma}}(1_m). \end{aligned}$$

Hence, we have only to show that

$$\begin{aligned} & \sum_{\lambda \in \Omega_r^-} \sum_{0 \subseteq \gamma \subseteq 2\lambda} s_{2\lambda/\gamma}(1_n) \cdot s_{\bar{\gamma}}(1_m) \\ &= \sum_{a+b+c=r} \left[\sum_{\omega \in \Omega_a^-} s_{2\omega}^{\sim}(1_m) \cdot \sum_{\mu \in \Omega_b^-} (s_{\bar{\mu}}(1_m) \cdot s_{\mu}(1_n)) \cdot \sum_{\tau \in \Omega_c^-} s_{2\tau}(1_n) \right]. \end{aligned}$$

Now consider

$$\sum_{\lambda \in \Omega_r^-} \sum_{0 \subseteq \gamma \subseteq 2\lambda} s_{2\lambda/\gamma}(1_n) \cdot s_{\bar{\gamma}}(1_m) = \sum_{j=0}^{2r} \sum_{\gamma \in \Omega_j^-} \sum_{\lambda \in \Omega_r^-} s_{2\lambda/\gamma}(1_n) \cdot s_{\bar{\gamma}}(1_m)$$

For each $\gamma \in \Omega_j^-$, $j = 0, 1, \dots, 2r$, by Lemma 2.19,

$$\sum_{\lambda \in \Omega_r^-} s_{2\lambda/\gamma} = \prod_{i \leq j} (1 - x_i x_j)^{-1} \cdot \sum_{2\omega} s_{\gamma/2\omega}.$$

And by Lemma 2.18,

$$\sum_{\eta: \text{even}} s_{\eta} = \prod_{i \leq j} (1 - x_i x_j)^{-1}.$$

So we have

$$\sum_{\lambda \in \Omega_r^-} s_{2\lambda/\gamma} = \sum_{2\tau} s_{2\tau} \cdot \sum_{2\omega} s_{\gamma/2\omega}.$$

Since

$$s_{\gamma/2\omega} = \sum_{\mu} c_{2\omega, \mu}^{\gamma} \cdot s_{\mu} \quad \text{where } |\gamma| = |2\omega| + |\mu|,$$

$$\sum_{\lambda \in \Omega_r^-} s_{2\lambda/\gamma} = \sum_{2\tau} s_{2\tau} \cdot \sum_{2\omega} \left[\sum_{\mu} c_{2\omega, \mu}^{\gamma} \cdot s_{\mu} \right].$$

And we have

$$\begin{aligned} & \sum_{j=0}^{2r} \sum_{\gamma \in \Omega_j^-} \left[\sum_{\lambda \in \Omega_r^-} s_{2\lambda/\gamma}(1_n) \right] \cdot s_{\tilde{\gamma}}(1_m) \\ &= \sum_{j=0}^{2r} \sum_{\gamma \in \Omega_j^-} \left[\sum_{2\tau} s_{2\tau}(1_n) \cdot \sum_{2\omega} \sum_{\mu} c_{2\omega, \mu}^{\gamma} \cdot s_{\mu}(1_n) \right] \cdot s_{\tilde{\gamma}}(1_m) \\ &= \sum_{j=0}^{2r} \sum_{2\tau} s_{2\tau}(1_n) \cdot \sum_{2\omega} \sum_{\mu} s_{\mu}(1_n) \cdot \left[\sum_{\gamma \in \Omega_j^-} c_{2\omega, \mu}^{\gamma} \cdot s_{\tilde{\gamma}}(1_m) \right]. \end{aligned}$$

By Lemma 2.16 and Lemma 2.17,

$$= \sum_{j=0}^{2r} \sum_{2\tau} s_{2\tau}(1_n) \cdot \sum_{2\omega} \sum_{\mu} s_{\mu}(1_n) \cdot \left[s_{\tilde{2\omega}}(1_m) \cdot s_{\tilde{\mu}}(1_m) \right],$$

where $|2\omega| + |\mu| = |\gamma| = j$. By the way $|2\lambda| = 2r$, $|2\lambda| = |\gamma| + |2\tau| + |\mu|$, and $|2\omega| + |\mu| = |\gamma|$, so

$$2r = |\gamma| + |2\tau| + |\mu| = (|2\omega| + |\mu|) + |2\tau| + |\mu| = 2|\omega| + 2|\mu| + 2|\tau|.$$

And so $r = |\omega| + |\mu| + |\tau|$. Hence the above equation is

$$\begin{aligned} & \sum_{|\omega|+|\mu|+|\tau|=r} \sum_{2\tau} s_{2\tau}(1_n) \cdot \sum_{\mu} (s_{\mu}(1_n) \cdot s_{\tilde{\mu}}(1_m)) \cdot \sum_{2\omega} s_{\tilde{2\omega}}(1_m) \\ &= \sum_{a+b+c=r} \left[\sum_{\omega \in \Omega_a^-} s_{\tilde{2\omega}}(1_m) \cdot \sum_{\mu \in \Omega_b^-} (s_{\tilde{\mu}}(1_m) \cdot s_{\mu}(1_n)) \cdot \sum_{\tau \in \Omega_c^-} s_{2\tau}(1_n) \right]. \end{aligned}$$

Therefore we completed the proof of Theorem 3.4. \square

COROLLARY 3.5. *For an arbitrary commutative ring R and an arbitrary non-negative integer r , $\{\mathcal{M}_\lambda\}_{\lambda \in \Omega_r^-}$ is a natural filtration of $S_r(\wedge^2 F)$ such that its associated graded module is $\sum_{\lambda \in \Omega_r^-} L_{2\lambda} F$.*

Proof. We have a plethysm formula for $S_r(\wedge^2 F)$, when we take $\varphi: 0 \rightarrow F$ in the Theorem 3.4. \square

COROLLARY 3.6. *For an arbitrary commutative ring R and an arbitrary non-negative integer r , $\{\mathcal{M}_\lambda\}_{\lambda \in \Omega_r^-}$ is a natural filtration of $D_r(D_2 G)$ such that its associated graded module is $\sum_{\lambda \in \Omega_r^-} K_{2\lambda} G$.*

Proof. We have a plethysm formula for $S_r(\wedge^2 F)$, when we take $\varphi: G \rightarrow 0$ in the Theorem 3.4. \square

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