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$\mathbb{Z}_p$-EQUIVARIANT SPIN^c-STRUCTURES

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ABSTRACT. Let X be a closed, oriented, Riemannian 4-manifold with $b_2^+(X) > 1$ and of simple type. Suppose that $\sigma : X \to X$ is an involution preserving orientation with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set with $\Sigma \cdot \Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$. We show that if $\chi(\Sigma) + \Sigma \cdot \Sigma \neq 0$ then the Spin^c bundle $\tilde{P}$ is not $\mathbb{Z}_2$-equivariant, where $\det \tilde{P} = L$ is a basic class with $c_1(L)[\Sigma] = 0$.

1. Introduction

Let X be a closed, oriented Riemannian 4-manifold. Let L be a complex line bundle over X satisfying $c_1(L) = w_2(TX) \bmod 2$. Then there are a principal Spin$^c(4)$-bundle $\tilde{P} \to X$ with $\det \tilde{P} = L$ and the twisted $(\pm\frac{1}{2})$-spinor bundles $W^{\pm}$ associated with L. In this paper, we say that $\tilde{P}$ is a Spin^c-structure on X.

Let $\mathcal{A}(L)$ be the set of Riemannian connections on L and $\Gamma(W^+)$ be the space of sections of W^+. The gauge group $\mathcal{G}(\tilde{P})$ of bundle automorphisms on L acts on $\mathcal{A}(L) \times \Gamma(W^+)$ by $g(A,\psi) = (A + g^{-1}dg, g^{\frac{1}{2}}\psi)$ for all $g \in \mathcal{G}(\tilde{P})$ and $(A,\psi) \in \mathcal{A}(L) \times \Gamma(W^+)$.

For $(A,\psi) \in \mathcal{A}(L) \times \Gamma(W^+)$ and a real-valued self-dual 2-form $\delta \in \Omega_X^+(\mathbb{R})$ on X, the perturbed Seiberg-Witten equations are defined by

$$(*)\begin{cases} F_A^+ + i\delta = q(\psi) \\ D_A\psi = 0, \end{cases}$$

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where $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator associated to the connection A. $q : C^\infty(W^+) \to \Omega^+_X(i\mathbb{R})$ is a quadratic map defined by $q(\psi) = \psi \otimes \psi^* - \frac{||\psi||^2}{2}\mathrm{Id}$.

Let $\mathcal{M}(\tilde{P})$ be the moduli space, the gauge equivalence classes of all solutions of the perturbed Seiberg-Witten equations $(*)$. For a generic self-dual 2-form δ on X, $\mathcal{M}(\tilde{P})$ is a smooth, compact manifold with its dimension $\dim \mathcal{M}(\tilde{P}) = \frac{1}{4}\{c_1(L)^2[X] - (2\chi(X) + 3\mathrm{sign}(X))\}$, where $\chi(X)$ and $\mathrm{sign}(X)$ are the Euler characteristic and the signature of X respectively. The orientations of the cohomology spaces determines an orientation of the moduli space.

If $\dim \mathcal{M}(\tilde{P})$ is even, say equal to $2d \geq 0$ then the Seiberg-Witten invariant is defined by $\mathcal{SW}(\tilde{P}) = \int_{\mathcal{M}(\tilde{P})} \mu^d$, the integral of the maximal power of the Chern class $\mu = c_1(\mathcal{M}(\tilde{P})_0)$ of the circle bundle $\mathcal{M}(\tilde{P})_0 \to \mathcal{M}(\tilde{P})$ where $\mathcal{M}(\tilde{P})_0$ is the framed moduli space.

If $\dim \mathcal{M}(\tilde{P})$ is odd or negative then the Seiberg-Witten invariant $\mathcal{SW}(\tilde{P})$ is defined to be zero. For details, see [16].

In general, there are infinitely many elements $c_1(L) \in H^2(X;\mathbb{Z})$ satisfying $c_1(L) = w_2(TX)$ mod 2. Each such element induces a Spin^c-structure on X. However there are only finitely many elements in $H^2(X;\mathbb{Z})$ such that their Seiberg-Witten invariants are non-zero. Such an element in $H^2(X;\mathbb{Z})$ is called a basic class. So the set of basic classes is finite. Furthermore X is said to be of simple type if all basic classes satisfy $c_1(L)^2[X] = 2\chi(X) + 3\mathrm{sign}(X)$.

It has been a conjecture in Kähler geometry that complex curves in compact Kähler surfaces X should minimize the genus in their respective homology classes. This conjecture is attributed to Thom [10] for the case $X = \mathbb{C}P^2$. Using the Seiberg-Witten invariants, Kronheimer and Mrowka [10] and Morgan-Szabó-Taubes [11] proved the Thom conjecture when the complex curves have non-negative self-intersection numbers. If X has a basic class, it gives a minimal genus bounds for the embedded surface so called the adjunction inequality.

THEOREM 1.1 (ADJUNCTION INEQUALITY [10]). *Let X be a smooth 4-manifold with $b_2^+(X) > 1$ and a basic class L and let Σ be an embedded connected, oriented surface with $\Sigma \cdot \Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$. Then we have an inequality*

$$-\chi(\Sigma) \geq \Sigma \cdot \Sigma + |c_1(L)[\Sigma]|.$$

Ozsváth and Szabó [13] also had the adjunction inequality for a 4-manifold X of simple type with $b_2^+(X) > 1$ and $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma < 0$

and proved the Thom conjecture for an embedded symplectic surface in a closed, symplectic 4-manifold.

Suppose that a cyclic group $\mathbb{Z}_p$ (p is prime) acts on X by orientation preserving isometry. The induced action of $\mathbb{Z}_p$ on the orthonormal frame bundle $P_{SO(4)}$ commutes with the right action of $SO(4)$ on $P_{SO(4)}$. Choose an action of $\mathbb{Z}_p$ over the principal $U(1)$-bundle P_L associated to L which is compatible with the $\mathbb{Z}_p$-action on X, and commutes with the canonical right action of $U(1)$ on P_L.

If the $\mathbb{Z}_p$ action on the product $P_{SO(4)} \times P_L$ lifts to a $\mathbb{Z}_p$ action on the Spin^c-structure $\tilde{P}$ then we say that $\tilde{P}$ is $\mathbb{Z}_p$-equivariant.

Note that the lifting group on $\tilde{P}$ induced from the $\mathbb{Z}_p$ action on $P_L \times P_{SO(4)}$ might have form a larger group. In general, this group acting on $\tilde{P}$ is not necessary to be $\mathbb{Z}_p$. In particular, when $p = 2$, we say that the lifting group on $\tilde{P}$ is of even type if $\tilde{P}$ is $\mathbb{Z}_2$-equivariant and is of odd type if otherwise.

About the $\mathbb{Z}_2$-equivariant Spin^c-structure, Ruan and Wang [14] showed that if the virtual dimensions of the moduli space $\mathcal{M}(\tilde{P})$ and the fixed moduli space $\mathcal{M}(\tilde{P})^\tau$ are zero and $\mathcal{M}(\tilde{P})^\tau$ is a smooth manifold then $\mathcal{SW}(\tilde{P}) = \mathcal{SW}(\tilde{P})^\tau \bmod 2$ where $\mathcal{M}(\tilde{P})^\tau$ is the fixed point set of a $\mathbb{Z}_2$ action $\tau : \mathcal{M}(\tilde{P})^* \to \mathcal{M}(\tilde{P})^*$ and $\mathcal{SW}(\tilde{P})^\tau$ is a Seiberg-Witten invariants defined from $\mathcal{M}(\tilde{P})^\tau$. We call $\mathcal{SW}(\tilde{P})^\tau$ as a fixed Seiberg-Witten invariant.

Feng [8] showed that if $\tilde{P}$ is $\mathbb{Z}_2$-equivariant and $b_1(X) = 0$ and $b_2^+(X) > 2$ and $b_2^+(X/\mathbb{Z}_p) = b_2^+(X)$ then $\mathcal{SW}(\tilde{P}) = 0 \bmod p$ if $k_j \leq \frac{1}{2}(b_2^+(X) - 1)$ for $j = 0, 1, \dots, p-1$ where $\Sigma_{j=0}^{p-1} k_j = \mathrm{ind} D_A$ is the index of the Dirac operator D_A.

In this paper, we consider a $\mathbb{Z}_p$ action on a closed, connected, oriented Riemannian 4-manifold X with an oriented, connected, compact, 2-dimensional submanifold Σ as a fixed point set. We consider the $\mathbb{Z}_p$-equivariant Spin^c-structure $\tilde{P}$ over X for all prime p and the condition of the fixed point set Σ.

In Section 2, we prove that if $\tilde{P}$ is a $\mathbb{Z}_p$-equivariant Spin^c-structure then we compute the virtual dimension of the τ'-invariant moduli space $\mathcal{M}(\tilde{P})_{\tau'}$.

In Section 3, let X be a closed, oriented, Riemannian 4-manifold with $b_2^+(X) > 1$. Suppose that $\sigma : X \to X$ is an involution preserving orientation with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set and $\Sigma \cdot \Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X; \mathbb{Z})$. Suppose that $\tilde{P}$ is a $\mathbb{Z}_2$-equivariant Spin^c-structure over X and $\det \tilde{P} = L$ is a basic class with $c_1(L)[\Sigma] = 0$. Then we show that the fixed moduli space $\mathcal{M}(\tilde{P})^\tau$ is not empty.

Furthermore, if X is of simple type, then Σ satisfies

$$\chi(\Sigma) + \Sigma \cdot \Sigma = 0.$$

This means that if X is a closed, oriented, Riemannian 4-manifold with $b_2^+(X) > 1$ and is of simple type and if $\sigma : X \to X$ is an involution preserving orientation with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set with $\Sigma \cdot \Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$, then we show that if $\chi(\Sigma) + \Sigma \cdot \Sigma \neq 0$ then the Spin^c-structure $\tilde{P}$ is not $\mathbb{Z}_2$-equivariant where $\det \tilde{P} = L$ is a basic class over X with $c_1(L)[\Sigma] = 0$.

2. $\mathbb{Z}_p$ action on the Spin^c-bundle and the fixed point set

Let X be a closed, oriented Riemannian 4-manifold. Let L be a complex line bundle over X satisfying $c_1(L) = w_2(TX)$ mod 2. Then there are a principal Spin$^c(4)$-bundle $\tilde{P} \to X$ with $\det \tilde{P} = L$ and the twisted $(\pm\frac{1}{2})$-spinor bundles $W^\pm$ associated with L.

Suppose that there is a $\mathbb{Z}_p$ action $\sigma : X \to X$ preserving orientation with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set.

Since Spin$^c(4)$ is a 2-fold covering space of $SO(4) \times U(1)$, by Bredon [3] the $\mathbb{Z}_p$ action on $P_{SO(4)} \times P_L$ can be lifted to an action of some group $\triangle$ on $\tilde{P}$ which is an extension as follows

$$0 \to \mathbb{Z}_2 \to \triangle \to \mathbb{Z}_p \to 0. \tag{1}$$

If p is odd, prime then $\triangle \cong \mathbb{Z}_{2p} \cong \mathbb{Z}_2 \times \mathbb{Z}_p$ and so there is a subgroup of $\triangle$ which is isomorphic to $\mathbb{Z}_p$. Then we can find a $\mathbb{Z}_p$-equivariant Spin^c-structure $\tilde{P}$ on X.

While, when $p = 2$, we can not always get a $\mathbb{Z}_2$-equivariant Spin^c-structure $\tilde{P}$ because exactly the exact sequence (1) is non-trivial and $\triangle \cong \mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Assume that there is a $\mathbb{Z}_p$ action $\tau : \tilde{P} \to \tilde{P}$ induced from the $\mathbb{Z}_p$ action $(\sigma_*, \det \tau)$ on $P_{SO(4)} \times P_L$. For simplicity we choose a metric on X which is $\mathbb{Z}_p$-invariant.

As in [14], for $(A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+)$, we define a $\mathbb{Z}_p$-action by $\tau^*(A,\psi) = ((\det \tau)^* A, (\tau^{-1})^*\psi)$ where $\nabla_{(\det\tau)^*A}(s) = \det \tau(\nabla_A(\det \tau^{-1} \circ s \circ \sigma))$ for all $s \in \Gamma(L)$ and $(\tau^{-1})^*\psi = \tau^{-1} \circ \psi \circ \sigma$. Then τ^* acts on the solution set of the perturbed Seiberg-Witten equations $(*)$. Note that the self-dual 2-form $\delta \in \Omega^+_X(\mathbb{R})$ in $(*)$ assumed to be invariant under the $\mathbb{Z}_p$ action σ on X.

For all gauge transformation $g \in \mathcal{G}(\tilde{P})$, define a $\mathbb{Z}_p$-action $\tau^* g = \tau \circ g \circ \tau^{-1}$.

Since any two liftings $\tau : \tilde{P} \to \tilde{P}$ differ by a gauge transformation, the induced $\mathbb{Z}_p$ action of τ^* on $\mathcal{A}(L) \times \Gamma(W^+)/\mathcal{G}(\tilde{P})$ is independent of the choice of τ. Let $\mathcal{M}(\tilde{P})^\tau$ be the fixed point set of the $\mathbb{Z}_2$-action $\tau^* : \mathcal{M}(\tilde{P})^* \to \mathcal{M}(\tilde{P})^*$ where $\mathcal{M}(\tilde{P})^* = \{[A, \psi] \in \mathcal{M}(\tilde{P}) | \psi \neq 0\}$ is the irreducible moduli space. In this paper, we say $\mathcal{M}(\tilde{P})^\tau$ as the fixed moduli space.

While, using the $\mathbb{Z}_p$ action τ^* on $\mathcal{A}(L) \times \Gamma(W^+)$ and $\mathcal{G}(\tilde{P})$, we consider a τ-invariant moduli space

$$\mathcal{M}(\tilde{P})_\tau = \frac{\{\tau^* - \text{invariant irreducible solutions of the equations}(*)\}}{\{\tau^* - \text{invariant gauge transformations}\}}$$

depending on the choice of the $\mathbb{Z}_p$ action $\tau : \tilde{P} \to \tilde{P}$. For details see [14].

Let $[A, \psi] \in \mathcal{M}(\tilde{P})^\tau$. Then there is a gauge g such that $\tau^*(A, \psi) = g^*(A, \psi)$ and so $[A, \psi] \in \mathcal{M}(\tilde{P})_{\tau'}$ where $\tau' = \tau \circ g^{-1}$.

Since $[A, \psi] \in \mathcal{M}(\tilde{P})_{\tau'}$ and $(\tau')^*(A, \psi) = ((\det \tau')^* A, (\tau')^{-1} \circ \psi \circ \sigma) = (A, \psi)$, we have

$$\begin{aligned}(\tau')^{p*}(A, \psi) &= ((\det \tau')^{p*} A, (\tau')^{-p} \circ \psi \circ \sigma^p) \\ &= ((\det \tau')^{p*} A, (\tau')^{-p} \circ \psi) = (A, \psi).\end{aligned}$$

Thus $(\tau')^p = \mathrm{Id}$ on $\tilde{P}$ and so $(\det \tau')^p = \mathrm{Id}$ on L. As in [14], denote $\tau' = s\tau$ for a unique map $s : X \to S^1$.

$(\tau')^p = \mathrm{Id}$ on $\tilde{P}$ implies that $s(\sigma^{p-1}(x))s(\sigma^{p-2}(x)) \cdots s(x) = 1 \in S^1$ for all $x \in X$.

If we restrict $x \in \Sigma$, then $s(\sigma^{p-1}(x))s(\sigma^{p-2}(x)) \cdots s(x) = s(x)^p = 1$. Thus $s = \exp \frac{2\pi h i}{p}$ on the fixed point set Σ, $h = 0, \cdots, p-1$.

Since the involution $\sigma : X \to X$ is an orientation preserving isometry with Σ as a fixed point set and $(\tau')^p = \mathrm{Id}$ on $\tilde{P}$, $\det \tau' : L \to L$ acts as an orientation preserving isometry.

Thus for each point $x \in \Sigma$,

$$\det \tau' = \exp \frac{2\pi m i}{p} : L|_x \to L|_x, \quad m = 0, \cdots, p-1.$$

By the Lefschetz theorem of Atiyah-Siegal in [15], the dimension of the τ'-invariant moduli space $\mathcal{M}(\tilde{P})_{\tau'}$ is

$$\dim \mathcal{M}(\tilde{P})_{\tau'} = \mathrm{ind}(\bar{D}_A) + 2\mathrm{ind}(D_A),$$

where $\bar{D}_A : \Gamma(S^+ \otimes S^-)^{\mathbb{Z}_p} \to \Gamma(S^+ \otimes S^+)^{\mathbb{Z}_p}$ and $D_A : \Gamma(W^+)^{\mathbb{Z}_p} \to \Gamma(W^-)^{\mathbb{Z}_p}$ are the Dirac operators. $S^\pm$ are $(\pm\frac{1}{2})$-spinor bundles of X with $W^\pm = S^\pm \otimes L^{\frac{1}{2}}$.

The indices $\mathrm{ind}(\bar{D}_A)$ and $\mathrm{ind}(D_A)$ are averages of the Lefschetz numbers $\mathrm{L}(\tau', \bar{D}_A)$ and $\mathrm{L}(\tau', D_A)$ for each $\tau' \in \mathbb{Z}_p$ such as

$$\mathrm{ind}(\bar{D}_A) = \frac{1}{p}\Sigma_{\tau' \in \mathbb{Z}_p} \mathrm{L}(\tau', \bar{D}_A) = \frac{1}{p}\Sigma_{k=1}^{p} \mathrm{L}((\tau')^k, \bar{D}_A),$$

$$\mathrm{ind}(D_A) = \frac{1}{p}\Sigma_{\tau' \in \mathbb{Z}_p} \mathrm{L}(\tau', D_A) = \frac{1}{p}\Sigma_{k=1}^{p} \mathrm{L}((\tau')^k, D_A).$$

By Atiyah and Singer [15], the Lefschetz numbers $\mathrm{L}(\tau', \bar{D}_A)$ and $\mathrm{L}(\tau', D_A)$ are defined by

$$\mathrm{L}(\tau', \bar{D}_A)|_\Sigma = (-1)^{\frac{\dim \Sigma}{2}} \frac{ch_{\tau'}(S^+)ch_{\tau'}(S^- - S^+)td(T\Sigma \otimes \mathbb{C})}{e(T\Sigma)ch_{\tau'}(\wedge_{-1}N_\Sigma^{\tau'} \otimes \mathbb{C})}[\Sigma],$$

$$\mathrm{L}(\tau', D_A)|_\Sigma = (-1)^{\frac{\dim \Sigma}{2}} \frac{ch_{\tau'}(W^+ - W^-)td_{\tau'}(T\Sigma \otimes \mathbb{C})}{e(T\Sigma)ch_{\tau'}(\wedge_{-1}N_\Sigma^{\tau'} \otimes \mathbb{C})}[\Sigma],$$

where $N_\Sigma^{\tau'}$ is the normal bundle of Σ in X. For details see [15].

The Lefschetz numbers can be calculated by

$$\mathrm{L}(\tau', \bar{D}_A)|_\Sigma = \frac{(e^{\frac{x_1}{2}} e^{\frac{\theta_N i + x_2}{2}} + e^{\frac{-x_1}{2}} e^{\frac{-\theta_N i - x_2}{2}})(e^{\frac{\theta_N i + x_2}{2}} - e^{\frac{-\theta_N i - x_2}{2}})}{(1 - e^{x_2 + \theta_N i})(1 - e^{-x_2 - \theta_N i})}[\Sigma],$$

$$\mathrm{L}(\tau', D_A)|_\Sigma = -\frac{e^{\frac{1}{2}c_1(L)[\Sigma]} e^{\frac{i\theta_L}{2}} (e^{\frac{\theta_N i + x_2}{2}} - e^{\frac{-\theta_N i - x_2}{2}})}{(1 - e^{x_2 + \theta_N i})(1 - e^{-x_2 - \theta_N i})}[\Sigma],$$

where x_i are the Chern's roots and θ_L is determined by the $\mathbb{Z}_p$-action $\det \tau' = \exp \theta_L i$ on $L|_\Sigma$. The angle θ_N is determined by the $\mathbb{Z}_p$-action $\sigma_* = \exp \theta_N i$ on the normal bundle N_Σ.

THEOREM 2.1. *In the above notations, the virtual dimension of the $\mathbb{Z}_p$-invariant moduli space* $\dim \mathcal{M}(\tilde{P})_{\tau'}$, $\tau' = s\tau \in \mathbb{Z}_p$, *is*

$$\dim \mathcal{M}(\tilde{P})_{\tau'} = \frac{1}{p}\Big[\dim \mathcal{M}(\tilde{P}) - \frac{p-1}{2}\chi(\Sigma) + \frac{1}{2}\Sigma_{k=1}^{p-1} \frac{-1 + \cos\frac{k\theta_L}{2}\cos\frac{k\theta_N}{2}}{\sin^2\frac{k\theta_N}{2}} \Sigma \cdot \Sigma + \frac{1}{2}\Sigma_{k=1}^{p-1}\frac{\sin\frac{k\theta_L}{2}}{\sin\frac{k\theta_N}{2}} c_1(L)[\Sigma]\Big],$$

and $\Sigma_{k=1}^{p-1} \frac{\cos\frac{k\theta_L}{2}}{\sin\frac{k\theta_N}{2}} c_1(L)[\Sigma] = \Sigma_{k=1}^{p-1} \frac{\sin\frac{k\theta_L}{2}\cos\frac{k\theta_N}{2}}{\sin^2\frac{k\theta_N}{2}} \Sigma \cdot \Sigma,$

where $k\theta_L$ and $k\theta_N$ are determined by the action $\det {\tau'}^k = \exp k\theta_L i$ on $L|_\Sigma$ and $\sigma_*^k = \exp k\theta_N i$ on N_Σ respectively, $k = 1, \ldots, p-1$.

Proof. Since the induced $\mathbb{Z}_p$-action σ_* on $TX|_\Sigma = T\Sigma \oplus N_\Sigma$ satisfies $\sigma_*|_{T\Sigma} = \mathrm{Id}$ and $\sigma_*|_{N_\Sigma} = \exp \frac{2\pi n i}{p}$, $n = 1, \ldots, p-1$, we have $\theta_N = \frac{2\pi n}{p}$, $n = 1, \ldots, p-1$.

Since ${\tau'}^*(A, \psi) = ((\det \tau')^* A, (\tau')^{-1} \circ \psi \circ \sigma)$, we have

$$({\tau'}^k)^*(A, \psi) = ((\det \tau')^{k*} A, (\tau')^{-k} \circ \psi \circ \sigma^k)$$

for all $\tau' \in \mathbb{Z}_p$.

Thus we conclude that if $\det \tau' = \exp 2\pi i(\frac{m}{p})$ on $L|_\Sigma$ and $\sigma_*|_{N_\Sigma} = \exp \frac{2\pi n i}{p}$, then $\det {\tau'}^k = \exp \frac{2\pi m k i}{p}$ on $L|_\Sigma$ and $\sigma_*^k = \exp \frac{2\pi n k i}{p}$ on N_Σ, $m = 0, \ldots, p-1$, $n = 1, \ldots, p-1$, and so $k\theta_L = \frac{2\pi k m}{p}$ and $k\theta_N = \frac{2\pi k n}{p}$.

The Lefschetz numbers can be calculated as follows

$$\begin{aligned}
&\mathrm{L}(\tau', \bar{D}_A)|_\Sigma \\
&= \frac{(e^{\frac{x_1}{2}} e^{\frac{\theta_N i + x_2}{2}} + e^{\frac{-x_1}{2}} e^{\frac{-\theta_N i - x_2}{2}})(e^{\frac{\theta_N i + x_2}{2}} - e^{\frac{-\theta_N i - x_2}{2}})}{(1 - e^{x_2 + \theta_N i})(1 - e^{-x_2 - \theta_N i})}[\Sigma] \\
&= \frac{\Sigma \cdot \Sigma}{(\cos\theta_N - 1)} - \frac{\chi(\Sigma)}{2}, \mathrm{L}(\tau', D_A)|_\Sigma \\
&= -\frac{e^{\frac{1}{2}c_1(L)[\Sigma]} e^{\frac{i\theta_L}{2}} (e^{\frac{\theta_N i + x_2}{2}} - e^{\frac{-\theta_N i - x_2}{2}})}{(1 - e^{x_2 + \theta_N i})(1 - e^{-x_2 - \theta_N i})}[\Sigma] \\
&= -e^{\frac{i\theta_L}{2}} \Big(\frac{i}{4 \sin \frac{\theta_N}{2}} c_1(L)[\Sigma] - \frac{\cos \frac{\theta_N}{2}}{4 \sin^2 \frac{\theta_N}{2}} \Sigma \cdot \Sigma \Big) \\
&= \frac{\cos \frac{\theta_L}{2} \cos \frac{\theta_N}{2}}{4 \sin^2 \frac{\theta_N}{2}} \Sigma \cdot \Sigma + \frac{\sin \frac{\theta_L}{2}}{4 \sin \frac{\theta_N}{2}} c_1(L)[\Sigma] \\
&\quad - i \Big(\frac{\cos \frac{\theta_L}{2}}{4 \sin \frac{\theta_N}{2}} c_1(L)[\Sigma] - \frac{\sin \frac{\theta_L}{2} \cos \frac{\theta_N}{2}}{4 \sin^2 \frac{\theta_N}{2}} \Sigma \cdot \Sigma \Big).
\end{aligned}$$

Then we show that

$$\begin{aligned}
\mathrm{L}({\tau'}^k, \bar{D}_A) &= \frac{\Sigma \cdot \Sigma}{(\cos k\theta_N - 1)} - \frac{\chi(\Sigma)}{2}, \\
\mathrm{L}({\tau'}^k, D_A)|_\Sigma &= -e^{\frac{ik\theta_L}{2}} \Big(\frac{i}{2 \sin \frac{k\theta_N}{2}} c_1(L)[\Sigma] - \frac{\cos \frac{k\theta_N}{2}}{4 \sin^2 \frac{k\theta_N}{2}} \Sigma \cdot \Sigma \Big).
\end{aligned}$$

When $k = p$, we have $(\tau')^p = \mathrm{Id}$ on $\tilde{P}$ and so

$$\begin{aligned} &\mathrm{L}(\tau'^p, \bar{D}_A) + 2\mathrm{L}(\tau'^p, D_A) = \mathrm{L}(\mathrm{Id}, \bar{D}_A) + 2\mathrm{L}(\mathrm{Id}, D_A) \\ &= \frac{1}{4}(c_1(L)^2[X] - (2\chi(X) + 3\mathrm{sign}(X))) = \dim \mathcal{M}(\tilde{P}). \end{aligned}$$

Thus we conclude that

$$\begin{aligned} \dim \mathcal{M}(\tilde{P})_{\tau'} &= \frac{1}{p}\Big[\dim \mathcal{M}(\tilde{P}) - \frac{p-1}{2}\chi(\Sigma) \\ &+ \frac{1}{2}\Sigma_{k=1}^{p-1}\frac{-1+\cos\frac{k\theta_L}{2}\cos\frac{k\theta_N}{2}}{\sin^2\frac{k\theta_N}{2}}\Sigma\cdot\Sigma + \frac{1}{2}\Sigma_{k=1}^{p-1}\frac{\sin\frac{k\theta_L}{2}}{\sin\frac{k\theta_N}{2}}c_1(L)[\Sigma] \\ &- i\Big(\frac{1}{2}\Sigma_{k=1}^{p-1}\frac{\cos\frac{k\theta_L}{2}}{\sin\frac{k\theta_N}{2}}c_1(L)[\Sigma] - \frac{1}{2}\Sigma_{k=1}^{p-1}\frac{\sin\frac{k\theta_L}{2}\cos\frac{k\theta_N}{2}}{\sin^2\frac{k\theta_N}{2}}\Sigma\cdot\Sigma\Big)\Big]. \end{aligned}$$

Since the virtual dimension of the $\mathbb{Z}_p$-invariant moduli space $\mathcal{M}(\tilde{P})_{\tau'}$ should be non-negative integer, we have

$$\Sigma_{k=1}^{p-1}\frac{\cos\frac{k\theta_L}{2}}{\sin\frac{k\theta_N}{2}}c_1(L)[\Sigma] = \Sigma_{k=1}^{p-1}\frac{\sin\frac{k\theta_L}{2}\cos\frac{k\theta_N}{2}}{\sin^2\frac{k\theta_N}{2}}\Sigma\cdot\Sigma.$$

□

REMARK 2.2. From Theorem 2.1 we calculate the dimension of the $\mathbb{Z}_p$-invariant moduli space $\mathcal{M}(\tilde{P})_{\tau'}$ for all prime p which is dependent on the $\mathbb{Z}_p$ actions on $L|_\Sigma$ by $\det\tau'$ and on the normal bundle N_Σ by σ_*, respectively.

3. Applications

If X is a spin 4-manifold with trivial canonical class and $\sigma : X \to X$ is an involution preserving orientation and $P_{\mathrm{Spin}(4)}$ is a principal Spin(4)-bundle then $P_{\mathrm{Spin}(4)}$ is $\mathbb{Z}_2$-equivariant if and only if $\mathbb{Z}_2$ acts on X with only isolated fixed points. Then the dimension of the fixed point set is zero. And the order of the lifting group on $P_{\mathrm{Spin}(4)}$ is 4 if and only if $\mathbb{Z}_2$ acts on X with a 2-dimensional compact submanifold as a fixed point set. See [1] for details.

PROPOSITION 3.1. *Let X be a closed, oriented, Riemannian 4-manifold with $b_2^+(X) > 1$. Suppose that $\sigma : X \to X$ is an involution preserving orientation with an oriented, connected, compact 2-dimensional* *submanifold Σ as a fixed point set and $\Sigma\cdot\Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$. Suppose that $\tilde{P}$ is a $\mathbb{Z}_2$-equivariant Spin^c-structure over X with non-zero Seiberg-Witten invariants and $c_1(L)[\Sigma] = 0$. Then the fixed moduli space $\mathcal{M}(\tilde{P})^\tau$ is not empty.*

Proof. Since $L = \det\tilde{P}$ is a basic class over X, $\mathcal{M}(\tilde{P})$ is not empty and $\dim\mathcal{M}(\tilde{P}) \geq 0$. Since $\mathcal{M}(\tilde{P})^\tau$ is the fixed point set of the $\mathbb{Z}_p$ action $\tau^* : \mathcal{M}(\tilde{P})^* \to \mathcal{M}(\tilde{P})^*$, we have $\dim\mathcal{M}(\tilde{P}) \geq \dim\mathcal{M}(\tilde{P})^\tau$.

By using Theorem 1.1 (the adjunction inequality) and the condition $c_1(L)[\Sigma] = 0$, we conclude that

$$-(\chi(\Sigma) + \Sigma\cdot\Sigma) \geq 0.$$

By Theorem 3.8 in [14] there is a disjoint decomposition

$$\mathcal{M}(\tilde{P})^\tau = \amalg_{[s]\in K}\mathcal{M}(\tilde{P})_{s\tau}$$

indexed by the set K of equivalence classes of maps $s : X \to S^1$ with $s(\sigma(x))s(x) = 1$ for all $x \in X$ where $s \sim s'$ if and only if $g(\sigma(x))s(x) = g(x)s'(x)$ for some map $g : X \to S^1$ where $(s\tau)^2 = \mathrm{Id}$ on $\tilde{P}$ and $s^2 \sim 1$.

As above, for simplicity we let $\tau' = s\tau$. Since $\sigma_* = -\mathrm{Id}$ on N_Σ and $\det\tau' = \mathrm{Id}$ or $-\mathrm{Id}$ on $L|_\Sigma$, we have $\theta_N = \pi$ and $\theta_L = 0$ or π.

By Theorem 2.1,

$$\dim\mathcal{M}(\tilde{P})_{\tau'} = \frac{1}{2}[\dim\mathcal{M}(\tilde{P}) - \frac{1}{2}\chi(\Sigma) - \frac{1}{2}\Sigma\cdot\Sigma + \frac{1}{2}(\sin\frac{\theta_L}{2})c_1(L)[\Sigma]],$$

$$(\cos\frac{\theta_L}{2})c_1(L)[\Sigma] = 0.$$

Thus we have

$$\dim\mathcal{M}(\tilde{P})_{\tau'} = \frac{1}{2}[\dim\mathcal{M}(\tilde{P}) - \frac{1}{2}(\chi(\Sigma) + \Sigma\cdot\Sigma)].$$

Since $\dim\mathcal{M}(\tilde{P}) \geq 0$ and $-(\chi(\Sigma) + \Sigma\cdot\Sigma) \geq 0$, we conclude that

$$\dim\mathcal{M}(\tilde{P})_{\tau'} \geq 0$$

for all $[s] \in K$.

If $\dim\mathcal{M}(\tilde{P}) = 0$ then $\dim\mathcal{M}(\tilde{P})^\tau = 0$. By [5] there is a generic self-dual 2-form $\delta \in \Omega^+_X(\mathbb{R})$ on X such that $\mathcal{M}(\tilde{P})^\tau$ is a smooth manifold. Then by Theorem 2.2 in [14], any point in $\mathcal{M}(\tilde{P})^\tau$ is a smooth point of $\mathcal{M}(\tilde{P})$ and $\mathcal{SW}(\tilde{P}) = \mathcal{SW}(\tilde{P})^\tau \bmod 2$. Since the Seiberg-Witten invariant $\mathcal{SW}(\tilde{P})$ is non trivial for the basic class L, $\mathcal{M}(\tilde{P})^\tau$ is not empty.

If $\dim\mathcal{M}(\tilde{P}) > 0$, then $\dim\mathcal{M}(\tilde{P})_{\tau'} > 0$ and so $\dim\mathcal{M}(\tilde{P})^\tau > 0$. Thus for a generic self-dual 2-form δ, $\mathcal{M}(\tilde{P})^\tau$ is not empty. □

COROLLARY 3.2. *Let X be a smooth, closed, connected, oriented 4-manifold with $b_2^+(X) > 1$ and $b_1(X) = 0$. Suppose that there is an involution $\sigma : X \to X$ with a fixed point set, an oriented, connected, compact 2-dimensional submanifold Σ with $g(\Sigma) > 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$ and $\Sigma \cdot \Sigma \geq 0$. If $\tilde{P}$ is a $\mathbb{Z}_2$-equivariant Spin^c-structure with non-zero Seiberg-Witten invariants and $c_1(L)[\Sigma] = 0$ for $\det \tilde{P} = L$ then $\mathcal{M}(\tilde{P})^\tau$ is not empty and $\dim \mathcal{M}(\tilde{P}) = \dim \mathcal{M}(\tilde{P})^\tau$.*

Proof. By Proposition 3.1, $\mathcal{M}(\tilde{P})^\tau$ is not empty. By Theorem 1.1 (the higher type adjunction inequality) in [12], we have

$$-(\chi(\Sigma) + \Sigma \cdot \Sigma) \geq |c_1(L)[\Sigma]| + 2\dim \mathcal{M}(\tilde{P}).$$

Thus for all $s\tau$-invariant moduli spaces $\mathcal{M}(\tilde{P})_{s\tau}$, $[s] \in K$, we have

$$\text{(a)} \qquad \begin{aligned} \dim \mathcal{M}(\tilde{P})_{s\tau} &= \frac{1}{2}[\dim \mathcal{M}(\tilde{P}) - \frac{1}{2}(\chi(\Sigma) + \Sigma \cdot \Sigma)] \\ &\geq \frac{1}{2}(\dim \mathcal{M}(\tilde{P}) + \frac{1}{2}|c_1(L)[\Sigma]| + \dim \mathcal{M}(\tilde{P})) = \dim \mathcal{M}(\tilde{P}). \end{aligned}$$

Since $\dim \mathcal{M}(\tilde{P})^\tau \leq \dim \mathcal{M}(\tilde{P})$ and a disjoint decomposition $\mathcal{M}(\tilde{P})^\tau = \amalg_{[s]\in K} \mathcal{M}(\tilde{P})_{s\tau}$, we conclude that

$$\text{(b)} \qquad \dim \mathcal{M}(\tilde{P})_{s\tau} \leq \dim \mathcal{M}(\tilde{P}) \quad \text{for all} \quad [s] \in K.$$

Equations (a) and (b) imply that $\dim \mathcal{M}(\tilde{P})_{s\tau} = \dim \mathcal{M}(\tilde{P})$ for all $[s] \in K$ and so $\dim \mathcal{M}(\tilde{P})^\tau = \dim \mathcal{M}(\tilde{P})$. □

COROLLARY 3.3. *Let X be a closed, oriented, Riemannian 4-manifold with $b_2^+(X) > 1$ and is of simple type. Suppose that $\sigma : X \to X$ is an involution preserving orientation with an oriented, connected, compact 2-dimensional submanifold Σ as a fixed point set and $\Sigma \cdot \Sigma \geq 0$ and $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$. Suppose that $\tilde{P}$ is a $\mathbb{Z}_2$-equivariant Spin^c-structure over X with non-zero Seiberg-Witten invariants and $c_1(L)[\Sigma] = 0$. Then the fixed point set Σ satisfies $\chi(\Sigma) + \Sigma \cdot \Sigma = 0$.*

Proof. By Proposition 3.1, the fixed moduli space $\mathcal{M}(\tilde{P})^\tau$ is not empty. Since X is of simple type, for the basic class $L = \det \tilde{P}$, $\dim \mathcal{M}(\tilde{P}) = 0$. Thus we conclude that

$$\chi(\Sigma) + \Sigma \cdot \Sigma = c_1(L)[\Sigma] = 0.$$

□

REMARK 3.4. (1) In the same condition with Corollary 3.3, if Σ does not satisfy the condition $\chi(\Sigma)+\Sigma\cdot\Sigma\neq 0$, then $\tilde{P}$ is not $\mathbb{Z}_2$-equivariant. (2) We can find many examples satisfying the condition $c_1(L)[\Sigma]=0$ in Proposition 3.1, Corollaries 3.2 and 3.3. For example, if we consider a Lagrangian surface Σ then $\chi(\Sigma)+\Sigma\cdot\Sigma=0$. Under the condition of Theorem 1.1 (the adjunction inequality), we have

$$0\leq c_1(L)[\Sigma]\leq -(\chi(\Sigma)+\Sigma\cdot\Sigma)=0,$$

and so $c_1(L)[\Sigma]=0$.

EXAMPLE 3.5. In Section 1 in [7] they constructed a double cover R_p of $\mathbb{C}P^2$ branched along B where B is a smooth complex curve of degree $2p$ in $\mathbb{C}P^2$. When $p=3$, R_3 is a $K3$ surface and $g(B)=10$ and $B\cdot B=36$.

Denote $X=R_3$ and let $\pi: R_3\to\mathbb{C}P^2$ be the projection map. Then there is a canonical involution $\sigma: X\to X$ which is just the covering map with the fixed point set $\pi^{-1}(B)=\bar{B}$. The basic class on X is only trivial line bundle $L=X\times\mathbb{C}\to X$. Also the basic class L and the associated positive spinor bundles W^+ can be written by

$$\det\tilde{P}=L=K_X^*,\quad W^+=K_X^*\oplus II,$$

where $II\to X$ is a trivial line bundle and K_X is the canonical class of X.

Thus there is a $\mathbb{Z}_2$ action on $L=X\times\mathbb{C}$ and $\tilde{P}$ is a $\mathbb{Z}_2$-equivariant Spin^c-structure over X.

Since $\chi(\bar{B})+\bar{B}\cdot\bar{B}=-18+18=c_1(L)[\bar{B}]=0$ and $\dim\mathcal{M}(\tilde{P})=\dim\mathcal{M}(\tilde{P})^\tau=0$, R_3 and $\bar{B}$ satisfy Corollary 3.3.

EXAMPLE 3.6. Let (X,ω) be a closed, symplectic 4-manifold with a symplectic structure ω and $b_2^+(X)>1$. Suppose that $\sigma: X\to X$ is an anti-symplectic involution (that is, $\sigma^*\omega=-\omega$) with a fixed point set Σ, an embedded, oriented, connected, compact 2-dimensional submanifold with $[\Sigma]\neq 0\in H_2(X;\mathbb{Z})$ and $\Sigma\cdot\Sigma\geq 0$. Then Σ is a Lagrangian surface and $\chi(\Sigma)+\Sigma\cdot\Sigma=0$.

Suppose that $\tilde{P}$ is a Spin^c-structure with non trivial Seiberg-Witten invariants. Then by Remark 3.4, we have $c_1(L)[\Sigma]=0$.

Since X is of simple type and $\det\tilde{P}=L$ is a basic class, we get $\dim\mathcal{M}(\tilde{P})=\dim\mathcal{M}(\tilde{P})^\tau=0$ and by [14] $\mathcal{SW}(\tilde{P})^\tau=\mathcal{SW}(\tilde{P})$ mod 2.

If $\tilde{P}=K_X$ or K_X^* then $\mathcal{SW}(\tilde{P})\neq 0$ mod 2 and so $\mathcal{SW}(\tilde{P})^\tau\neq 0$ mod 2 where K_X is the canonical class of X.

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