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MAPPINGS OF CONSERVATIVE DISTANCES

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Abstract. In this paper, we will deal with the Aleksandrov-Rassias problem. More precisely, we prove some theorems concerning the mappings preserving one or two distances.

1. Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry (or a congruence) if f satisfies

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x-y\| = \rho$ always implies $\|f(x)-f(y)\| \leq \rho$. Similarly, a distance ρ is said to be extensive (or non-shrinking) by f if the inequality $\|f(x)-f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x-y\| = \rho$. We say that ρ is conservative (or preserved) by f if ρ is contractive and extensive by f simultaneously.

If f is an isometry, then every distance $\rho > 0$ is conservative by f, and conversely. At this point, we can raise a question:

Is a mapping that preserves certain distances an isometry?

In 1970, A. D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $\rho > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $\rho = 1$ when X is a normed space (see [15]).

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Indeed, earlier than Aleksandrov, F. S. Beckman and D. A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = E^n$:

If a mapping $f : E^n \to E^n$ $(1 < n < \infty)$ preserves distance 1, then f is a linear isometry up to translation.

For $n = 1$, they suggested the mapping $f : E^1 \to E^1$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for } x \in \mathbb{Z}, \\ x & \text{otherwise} \end{cases}$$

as an example for a non-isometric mapping that preserves distance 1. For $X = E^\infty$, Beckman and Quarles also presented an example for a unit distance preserving mapping that is not an isometry (cf. [12]).

We may find a number of valuable papers on a variety of topics in the Aleksandrov problem (see [6]-[23] and also the references cited therein).

In 1985, W. Benz [3] introduced a sufficient condition under which a mapping, with a contractive distance ρ and an extensive one $N\rho$, is an isometry (see also [5]):

THEOREM 1. *Let X and Y be real normed spaces such that $\dim X \geq 2$ and Y is strictly convex. Suppose $f : X \to Y$ is a mapping and $N \geq 2$ is a fixed integer. If a distance $\rho > 0$ is contractive and $N\rho$ is extensive by f, then f is a linear isometry up to translation.*

In this connection, Th. M. Rassias [14] raised a question whether a mapping $f : X \to Y$ preserving two distances with a non-integral ratio is an isometry. Such kind of problems are called the Aleksandrov-Rassias problems.

In this paper, by using theorems of W. Benz, we obtain some results concerning the Aleksandrov-Rassias problem.

2. Mappings with one conservative distance

The Aleksandrov problem still remains open even for the mappings $f : E^m \to E^n$ with $1 < m < n < \infty$.

We now generalize the theorem of Beckman and Quarles by considering the mappings between real Hilbert spaces but with an additional condition.

THEOREM 2. *Let X and Y be real Hilbert spaces with $\dim X \geq 3$ and $\dim Y \geq 3$. If a mapping $f : X \to Y$ preserves a distance $\rho > 0$ and if f maps the vertices of each regular quadrilateral of side length ρ (and*

$\sqrt{2}\rho$) *in* X *onto the vertices of a rhombus of side length* ρ *(resp.* $\sqrt{2}\rho$) *in* Y, *then* f *is a linear isometry up to translation.*

Proof. Without loss of generality, we assume that $\rho = 1$ throughout the proof. Let p_0, p_1, p_2, p_3, p_4 comprise the vertices of a regular quadrangular pyramid of unit side, i.e.,

$$(1) \qquad \|p_3 - p_1\| = \|p_4 - p_2\| = \sqrt{2}\,,\ \|p_{i+1} - p_i\| = 1,\ \|p_i - p_0\| = 1$$

for $i = 1, 2, 3, 4$, where we set $p_5 = p_1$. Put $s_i = f(p_i)$ for any $i = 0, 1, 2, 3, 4$. Then, the hypothesis and (1) imply

$$(2) \qquad \|s_{i+1} - s_i\| = 1,\ \|s_i - s_0\| = 1$$

for all $i = 1, 2, 3, 4$ with $s_5 = s_1$. If we set $x = s_2 - s_1$, $y = s_4 - s_1$ and $z = s_0 - s_1$, it follows from (2) that

$$(3) \qquad \|x\| = \|y\| = \|z\| = 1,\ \|x - z\| = \|y - z\| = \|x + y - z\| = 1.$$

The last equality follows from the fact that s_1, s_2, s_3, s_4 comprise the vertices of a unit rhombus. From (3) we get

$$1 = \|x - z\|^2 = \|x\|^2 - 2\langle x, z\rangle + \|z\|^2,\ 1 = \|y - z\|^2 = \|y\|^2 - 2\langle y, z\rangle + \|z\|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on Y, and hence

$$(4) \qquad \langle x, z\rangle = \langle y, z\rangle = \frac{1}{2}.$$

From (3) again, it follows that

$$1 = \|x + y - z\|^2 = \|x\|^2 + \|y\|^2 + \|z\|^2 + 2\langle x, y\rangle - 2\langle y, z\rangle - 2\langle x, z\rangle$$

and by (4) we get

$$\langle x, y\rangle = 0.$$

Therefore, we use (3) to obtain

$$\|s_2 - s_4\|^2 = \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = 2,$$
$$\|s_3 - s_1\|^2 = \|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = 2,$$

which means that f preserves the distance $\sqrt{2}$.

Taking $\rho = \sqrt{2}$ instead of 1 for next time, we can apply the same argument as before to proving that f also preserves the distance 2. According to Theorem 1 (see also [5]), f is a linear isometry up to translation. □

3. Mappings with two conservative distances

In 1985, W. Benz introduced a sufficient condition under which a mapping, with a contractive distance ρ and an extensive one $N\rho$, is an isometry (see Theorem 1 or [3]).

In the following theorem, we introduce a sufficient condition under which a mapping preserving two distinct distances, where the ratio is not integral, is an isometry.

THEOREM 3. *Let X and Y be real Hilbert spaces with $\dim X \geq 2$ and $\dim Y \geq 2$. Suppose that the distance $\rho > 0$ is contractive by a mapping $f : X \to Y$ and that there exists an integer $n > 1$ such that $\sqrt{n^2+1}\,\rho$ is extensive by f. If f maps the midpoint of every segment joining v and w of length $2n\rho$ into the segment between $f(v)$ and $f(w)$, then f is a linear isometry up to translation.*

Proof. Assume that x, y are points in X separated from each other by distance $n\rho$. Choose a point $z \in X$ such that x is the midpoint of y and z. Moreover, let $u \in X$ be a point at distance ρ from x such that the segment between u and x is perpendicular to the line through y and z. Applying Pythagorean theorem, we see that

$$\|y-u\| = \|z-u\| = \sqrt{n^2+1}\,\rho.$$

By the hypotheses, we have

$$\|f(x)-f(u)\| \leq \rho,\ \|f(y)-f(x)\| \leq n\rho,\ \|f(z)-f(x)\| \leq n\rho, \tag{5}$$
$$\|f(y)-f(u)\| \geq \sqrt{n^2+1}\,\rho \text{ and } \|f(z)-f(u)\| \geq \sqrt{n^2+1}\,\rho.$$

The last two inequalities in (5) are due to the triangle inequality, more precisely, due to the fact that the distance $n\rho$ is contractive by f because the distance ρ is so. Furthermore, we know from the hypothesis that $f(x)$ is on the segment joining $f(y)$ and $f(z)$.

Let m be a point on the segment joining $f(y)$ and $f(z)$ such that the segment between $f(u)$ and m is perpendicular to the line through $f(y)$ and $f(z)$. Denote by a, b, c the distances from m to $f(y)$, $f(z)$, $f(u)$, respectively. Since $\rho \geq \|f(x)-f(u)\| \geq c$, by applying Pythagorean theorem, we have

$$a^2+\rho^2 \geq a^2+c^2 \geq (n^2+1)\,\rho^2 \text{ and } b^2+\rho^2 \geq b^2+c^2 \geq (n^2+1)\,\rho^2$$

which imply that $a \geq n\rho$ and $b \geq n\rho$. Since Y is strictly convex, it follows from (5) that

$$2n\rho \leq a+b = \|f(y)-f(z)\| \leq \|f(y)-f(x)\| + \|f(z)-f(x)\| \leq 2n\rho.$$

Hence, it has to be $a = b = n\rho$ and $c = \rho$. Thus, m should coincide with $f(x)$. Consequently, we conclude that

$$\|f(x) - f(y)\| = a = n\rho.$$

This fact means that distance $n\rho$ is conservative by f. According to Theorem 1 (see also [5]), f is a linear isometry up to translation. □

Following W. Benz [4], a set of n distinct points of an n-dimensional real Euclidean space E^n is called a β-set if the points are pairwise of distance $\beta > 0$.

Benz proved that for any $\alpha, \beta > 0$ with $\gamma(\alpha, \beta) = 4\alpha^2 - 2\beta^2(1-1/n) > 0$ and for any β-set P of E^n, there exist exactly two distinct points in E^n which have distance α from all $p \in P$ and he proved also that those two points are separated from each other by a distance $\sqrt{\gamma(\alpha, \beta)}$. Conversely, he also proved that for any x and y with $\|x-y\| = \sqrt{\gamma(\alpha, \beta)}$ there exists a β-set P such that x and y have distance α from all $p \in P$ (see [4]).

The proof of the following theorem may be interesting, even though this theorem is a special case of a theorem of Beckman and Quarles [2].

THEOREM 4. *Given integers $k, n \geq 2$, let E^n be an n-dimensional real Euclidean space and let $f : E^n \to E^n$ be a mapping that preserves the distances $\rho > 0$ and $\beta_n(k)\rho$, where we define*

$$\beta_n(k) = \sqrt{\frac{(4k^2-1)n}{2k^2(n-1)}}.$$

Then, f is a linear isometry up to translation.

Proof. Let x and y be points of E^n with $\|x - y\| = \sqrt{\gamma(\rho, \beta_n(k)\rho)} = \rho/k$. According to [4] (or see above), there exists a $\beta_n(k)\rho$-set, $\{p_1, ..., p_n\}$, such that x and y have distance ρ from all p_i.

Let us define $q_i = f(p_i)$ for $i = 1, ..., n$. Then, by the hypothesis, the set $\{q_1, ..., q_n\}$ is also a $\beta_n(k)\rho$-set and, according to [4], there exist exactly two distinct points s and t in E^n which have distance ρ from all q_i and further $\|s - t\| = \sqrt{\gamma(\rho, \beta_n(k)\rho)} = \rho/k$. It hence holds that $\|f(x) - f(y)\| \in \{0, \rho/k\}$.

Put $u_i = x + i(y-x)$ for $i \in \{0, ..., k\}$. Then, $\|u_k - x\| = \|u_k - u_0\| = \rho$ and $\|u_i - u_{i-1}\| = \|y - x\| = \rho/k$ for $i \in \{1, ..., k\}$. Thus, by a slight modification of the last paragraph, we may see that

$$\rho = \|f(u_k) - f(x)\| \leq \|f(y) - f(x)\| + \sum_{i=2}^{k} \|f(u_i) - f(u_{i-1})\| \leq k(\rho/k) = \rho.$$

Hence, we conclude that $\{f(x), f(y)\} = \{s, t\}$ and $\|f(x) - f(y)\| = \rho/k$, i.e., f preserves the third distance ρ/k. By Theorem 1, our assertion is true. □

For more detailed information on the subjects of Aleksandrov-Rassias problem, we can refer to [19, 20, 23].

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