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ELEMENTARY TOPICS ON WEAK POLYGROUPS

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Abstract. In this paper, we further develop the weak polygroup theory, we define quotient weak polygroup and then the fundamental homomorphism theorem of group theory is derived in the context of weak polygroups. Also, we consider the fundamental relation β^* defined on a weak polygroup and define a functor from the category of all weak polygroups into the category of all fundamental groups.

1. Introduction

In [7], [8], a new class of hyperstructures called H_v-structures, was introduced. This class is larger than the known ones originated from the hypergroup in the sense of Marty [6] and they satisfy the weak axioms where the non-empty intersection replaces the equality. The basic definitions on the subject can be found in [9].

In [4], [5], we studied the concept of weak polygroups which is a generalization of polygroups [1], [3]. We defined the fundamental relation β^* on the weak polygroups in a similar way in the known hyperstructures (see [2]), and proved some results in this respect. Moreover, we defined a semi-direct hyperproduct of two weak polygroups in order to obtain an extension of weak polygroups by weak polygroups.

In this paper, we further develop the weak polygroup theory, we define quotient weak polygroup and then the fundamental homomorphism theorem of group theory is derived in the context of weak polygroups. Also, we consider the fundamental relation β^* defined on a weak polygroup and define a functor from the category of all weak polygroups into the category of all fundamental groups.

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2. Quotient weak polygroups

We recall the following definition from [4], [5].

DEFINITION 2.1. A multivalued system $\langle P, \cdot, e, ^{-1}\rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P is called a weak polygroup if the following axioms hold for all x, y, z in P:

(i) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ (weak associative),
(ii) $e \cdot x = x \cdot e = x$,
(iii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

In the above definition if $A, B \subseteq P$ and $x \in P$, then

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad A \cdot x = A \cdot \{x\} \text{ and } x \cdot B = \{x\} \cdot B.$$

The following elementary facts about weak polygroups follow easily from the axioms:

$$e \in x \cdot x^{-1} \cap x^{-1} \cdot x, \; e^{-1} = e, \; (x^{-1})^{-1} = x, \text{ and } (x \cdot y)^{-1} = y^{-1} \cdot x^{-1},$$

where $A^{-1} = \{a^{-1} | \; a \in A\}$.

EXAMPLE 2.2. [5]. Let G be a group and θ an equivalence relation on G such that (i) $x\theta e$ implies $x = e$, (ii) $x\theta y$ implies $x^{-1}\theta y^{-1}$. Let $\theta[x]$ be the equivalence class of the element $x \in G$. Suppose that $G/\theta = \{\theta[x] \mid x \in G\}$. Then $\langle G/\theta, \odot, \theta[e], ^{-I}\rangle$ is a weak polygroup, when the hyperoperation $\odot$ is defined as follows:

$$\odot : G/\theta \times G/\theta \longrightarrow \mathcal{P}^*(G/\theta),$$
$$\theta[x] \odot \theta[y] = \{\theta[z] \mid z \in \theta[x] \cdot \theta[y]\},$$

and $\theta[x]^{-I} = \theta[x^{-1}]$.

A non-empty subset A of a weak polygroup P is said to be a subpolygroup of P if, under the hyperoperation in P, A itself forms a weak polygroup.

DEFINITION 2.3. If A is a subpolygroup of a weak polygroup P, then we define the relation $a \equiv b$ (mod$A$) if and only if there exists a set $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$, where $c_0 = a$, $c_{k+1} = b$ such that

$$a \cdot c_1^{-1} \cap A \neq \emptyset, \quad c_1 \cdot c_2^{-1} \cap A \neq \emptyset, \quad \ldots, \quad c_k \cdot b^{-1} \cap A \neq \emptyset.$$

This relation is denoted by aA_P^*b.

LEMMA 2.4. *The relation A_P^* is an equivalence relation.*

Proof. i) Since $e \in a \cdot a^{-1} \cap A$ for all $a \in P$; then aA_P^*a, i.e., A_P^* is reflexive.

ii) Suppose that aA_P^*b then there exists $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$, where $c_0 = a,\ c_{k+1} = b$ such that

$$a \cdot c_1^{-1} \cap A \neq \emptyset, \quad c_1 \cdot c_2^{-1} \cap A \neq \emptyset, \quad \ldots, \quad c_k \cdot b^{-1} \cap A \neq \emptyset.$$

Therefore there exists $x_i \in c_i \cdot c_{i+1}^{-1} \cap A \quad (i = 1, \ldots, k)$ which implies $x_i^{-1} \in c_{i+1} \cdot c_i^{-1}$ and $x_i^{-1} \in A$, this means that bA_P^*a, and so A_P^* is symmetric.

iii) Let aA_P^*b and bA_P^*c, where $a, b, c \in P$. Then there exist $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$ and $\{d_0, d_1, \ldots, d_{r+1}\} \subseteq P$, where $c_0 = a, c_{k+1} = b = d_0, d_{r+1} = c$ such that

$$a \cdot c_1^{-1} \cap A \neq \emptyset, \quad c_1 \cdot c_2^{-1} \cap A \neq \emptyset, \quad \ldots, \quad c_k \cdot b^{-1} \cap A \neq \emptyset,$$

$$b \cdot d_1^{-1} \cap A \neq \emptyset, \quad d_1 \cdot d_2^{-1} \cap A \neq \emptyset, \quad \ldots, \quad d_r \cdot c^{-1} \cap A \neq \emptyset.$$

We take $\{c_0, c_1, \ldots, c_{k+1}, d_1, d_2, \ldots, d_{r+1}\} \subseteq P$ which satisfies the condition for aA_P^*c, and so A_P^* is transitive.

Therefore A_P^* is an equivalence relation. □

We denote $A_P^*[x]$ the equivalence class with representative x.

THEOREM 2.5. *Let P be a weak polygroup. If A is a subpolygroup of P, then on the set $[P : A] = \{A_P^*[a] \mid a \in P\}$ we define the hyperoperation $\odot$ as follows:*

$$A_P^*[a] \odot A_P^*[b] = \{A_P^*[c] \mid c \in A_P^*[a] \cdot A_P^*[b]\},$$

what gives the weak polygroup $\langle [P : A], \odot, A_P^*[e], ^{-I} \rangle$, where $A_P^*[a]^{-I} = A_P^*[a^{-1}]$.

Proof. For $a, b, c \in P$, we have

$$(a \cdot b) \cdot c \subseteq (A_P^*[a] \cdot A_P^*[b]) \cdot A_P^*[c],$$
$$a \cdot (b \cdot c) \subseteq A_P^*[a] \cdot (A_P^*[b] \cdot A_P^*[c]),$$

therefore $\odot$ is weak associative. Now, we show that A is the unit element in $[P : A]$. Obviously, we have $A \subseteq A_P^*[e]$. On the other hand, suppose $a \in A_P^*[e]$ then there exists a set $\{c_0, c_1, \ldots, c_{n+1}\} \subseteq P$ where $c_0 = a, c_{n+1} = e$ such that

$$a \cdot c_1^{-1} \cap A \neq \emptyset, \quad c_1 \cdot c_2^{-1} \cap A \neq \emptyset, \quad \ldots, \quad c_n \cdot e^{-1} \cap A \neq \emptyset.$$

So $c_n \in A$. Since $c_{n-1} \cdot c_n^{-1} \cap A \neq \emptyset$, there exists $x \in c_{n-1} \cdot c_n^{-1} \cap A$ which implies $c_{n-1} \in x \cdot c_n$, and so $c_{n-1} \in A$. By induction, we obtain $a \in A$. Therefore $A_P^*[e] = A$. Now, we show that $A_P^*[a] \odot A_P^*[e] = A_P^*[a]$. Suppose $A_P^*[z] \in A_P^*[a] \odot A_P^*[e]$, we claim that $A_P^*[z] = A_P^*[a]$. We have $z \in A_P^*[a] \cdot A_P^*[e]$. Hence there exist $x \in A_P^*[a]$ and $y \in A$ such that $z \in x \cdot y$ which implies $y \in x^{-1} \cdot z$ then $x^{-1} \cdot z \cap A \neq \emptyset$, and so $A_P^*[x] = A_P^*[z]$. Therefore $A_P^*[z] = A_P^*[a]$. It is easy to see that $A_P^*[a^{-1}]$ is the inverse of $A_P^*[a]$ in $[P : A]$. Now, we show that $A_P^*[c] \in A_P^*[a] \odot A_P^*[b]$ implies $A_P^*[a] \in A_P^*[c] \odot A_P^*[b^{-1}]$ and $A_P^*[b] \in A_P^*[a^{-1}] \odot A_P^*[c]$. Since $A_P^*[c] \in A_P^*[a] \odot A_P^*[b]$, we have $A_P^*[c] = A_P^*[x]$ for some $x \in A_P^*[a] \cdot A_P^*[b]$. Therefore there exist $y \in A_P^*[a]$ and $z \in A_P^*[b]$ such that $x \in y \cdot z$, so $y \in x \cdot z^{-1}$. This implies that $A_P^*[y] \in A_P^*[x] \odot A_P^*[z^{-1}]$, and so $A_P^*[a] \in A_P^*[c] \odot A_P^*[b^{-1}]$. Similarly, we get $A_P^*[b] \in A_P^*[a^{-1}] \odot A_P^*[c]$. Therefore $\langle [P : A], \odot, A, ^{-I} \rangle$ is a weak polygroup. □

If A is a subpolygroup of a weak polygroup P, then the weak polygroup $[P : A]$, as in Theorem 2.5, is called the quotient weak polygroup of P by A.

DEFINITION 2.6. Let $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $\langle P_2, *, e_2, ^{-I} \rangle$ be two weak polygroups. A mapping ρ from P_1 into P_2 is said to be a strong homomorphism if for all $a, b \in P_1$,

i) $\varphi(e_1) = e_2$,
ii) $\rho(a \cdot b) = \rho(a) * \rho(b)$.

Clearly, a strong homomorphism ρ is an isomorphism if ρ is one to one and onto. We write $P_1 \cong P_2$ if P_1 is isomorphic to P_2.

Let ρ be a strong homomorphism from P_1 into P_2. The subset $K = \{x \mid x \in P_1,\ \rho(x) \text{ is the identity of } P_2\}$ is called the kernel of ρ.

COROLLARY 2.7. *Let ρ be a strong homomorphism from a weak polygroup P_1 into a weak polygroup P_2. The following propositions hold:*

i) *For all $a \in P_1,\ \rho(a^{-1}) = \rho(a)^{-1}$;*
ii) *The kernel of ρ is a subpolygroup of P_1;*
iii) *Let A be a subpolygroup of P_1. The image $\rho(A) = \{\rho(x) \mid x \in A\}$ is a subpolygroup of P_2. For a subpolygroup B of P_2, the inverse image $\rho^{-1}(B) = \{x \mid x \in P_1,\ \rho(x) \in B\}$ is a subpolygroup of P_1.*

Let P_1, P_2 be two weak polygroups and ρ a strong homomorphism of P_1onto P_2. If K is the kernel of ρ then we can form $[P_1 : K]$. It is fairly natural to expect that there should be a very close relationship between

P_2 and $[P_1 : K]$. The fundamental homomorphism theorem, which we are about to prove, spells out this relationship in exact detail.

THEOREM 2.8. (Fundamental Homomorphism Theorem). *Let ρ be a strong homomorphism from P_1 onto P_2 with kernel K. Then*

$$[P_1 : K] \cong P_2.$$

Proof. We define $\varphi : [P_1 : K] \longrightarrow P_2$ as follows:

$$\varphi(K^*_{P_1}[x]) = \rho(x) \quad \text{for all } x \in P_1.$$

This mapping is well defined, because if $K^*_{P_1}[x] = K^*_{P_1}[y]$, then there exists $\{z_0, z_1, \ldots, z_{k+1}\} \subseteq P_1$ where $z_0 = x$, $z_{k+1} = y$ such that

$$x \cdot z_1^{-1} \cap K \neq \emptyset, \quad z_1 \cdot z_2^{-1} \cap K \neq \emptyset, \quad \ldots, \quad z_k \cdot y^{-1} \cap K \neq \emptyset.$$

Thus $e_2 \in \rho(x \cdot z_1^{-1})$, $e_2 \in \rho(z_1 \cdot z_2^{-1})$, $\ldots$ $, e_2 \in \rho(z_k \cdot y^{-1})$ or $e_2 \in \rho(x) * \rho(z_1)^{-1}$, $e_2 \in \rho(z_1) * \rho(z_2)^{-1}$, $\ldots$ $, e_2 \in \rho(z_k) * \rho(y)^{-1}$ and so $\rho(x) = \rho(y)$.

Now, for every $K^*_{P_1}[x]$, $K^*_{P_1}[y] \in [P_1 : K]$, we have

$$\begin{aligned}
\varphi(K^*_{P_1}[x] \odot K^*_{P_1}[y]) &= \varphi(\{K^*_{P_1}[z] \mid z \in K^*_{P_1}[x] \cdot K^*_{P_1}[y]\} \\
&= \{\rho(z) \mid z \in K^*_{P_1}[x] \cdot K^*_{P_1}[y]\} \\
&= \rho(K^*_{P_1}[x] \cdot K^*_{P_1}[y]) \\
&= \rho(K^*_{P_1}[x]) * \rho(K^*_{P_1}[y]) \\
&= \rho(x) * \rho(y) \\
&= \varphi(K^*_{P_1}[x]) * \varphi(K^*_{P_1}[y]),
\end{aligned}$$

and $\varphi(K) = \varphi(K^*_{P_1}[e_1]) = \rho(e_1) = e_2$. Therefore φ is a strong homomorphism.

Furthermore if $\varphi(K^*_{P_1}[x]) = \varphi(K^*_{P_1}[y])$ then $\rho(x) = \rho(y)$ which implies $x \cdot y^{-1} \cap K \neq \emptyset$, and so $K^*_{P_1}[x] = K^*_{P_1}[y]$. Thus φ is a one to one mapping. □

3. On the fundamental relation β^*

Let $\langle P, \cdot, e, ^{-1} \rangle$ be a weak polygroup. We define the relation β^* as the smallest equivalence relation on P such that the quotient P/β^*, the set of all equivalence classes, is a group. In this case β^* is called the fundamental equivalence relation on P and P/β^* is called the fundamental group. The product $\otimes$ in P/β^* is defined as follows:

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z) \quad \text{for all} \quad z \in \beta^*(x) \cdot \beta^*(y).$$

This relation is studied by Corsini [2] concerning hypergroups, see also [5], [9].

Let $\mathcal{U}_P$ be the set of all finite products of elements of P. We define the relation β as follows:

$$x\beta y \text{ if and only if } \{x,y\} \subseteq u \text{ for some } u \in \mathcal{U}_P.$$

Let us denote $\widehat{\beta}$ the transitive closure of β. Then we can rewrite the definition of $\widehat{\beta}$ on P as follows:

$a\widehat{\beta}b$ if and only if there exist $z_1, z_2, \ldots, z_{n+1} \in P$ with $z_1 = a, z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathcal{U}_P$ such that $\{z_i, z_{i+1}\} \subseteq u_i \quad (i = 1, \ldots, n)$.

THEOREM 3.1. *The fundamental relation β^* is the transitive closure of the relation β.*

Proof. The proof is similar to the proof of Theorem 1.2.2 [9], also see [5]. □

The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the core of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*. We have the following statements (see [5]),

i) $\omega_P = \beta^*(e)$,
ii) $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$.

LEMMA 3.2. *Let $f : P_1 \longrightarrow P_2$ be a strong homomorphism of weak polygroups and β_1^*, β_2^* fundamental equivalence relations on P_1, P_2 respectively. Then the map $F : P_1/\beta_1^* \longrightarrow P_2/\beta_2^*$ defined by $F(\beta_1^*(x)) = \beta_2^*(f(x))$ is a homomorphism of fundamental groups.*

Proof. First, we show that F is well-defined. Suppose that $\beta_1^*(x) = \beta_1^*(y)$. Then there exist $x_1, x_2, \ldots, x_{n+1} \in P_1$ with $x_1 = x, x_{n+1} = y$ and $u_1, \ldots, u_n \in \mathcal{U}_{P_1}$ such that $\{x_i, x_{i+1}\} \subseteq u_i \quad (i = 1, \ldots, n)$. Since f is a strong homomorphism and $u_i \in \mathcal{U}_{P_1}$, we get $f(u_i) \in \mathcal{U}_{P_2}$. Therefore $f(x)\beta_2^* f(y)$ which implies $\beta_2^*(f(x)) = \beta_2^*(f(y))$, and so $F(\beta_1^*(x)) = F(\beta_1^*(y))$. Thus F is well-defined. Now, we have

$$\begin{aligned} F(\beta_1^*(x) \otimes \beta_1^*(y)) &= F(\beta_1^*(x \cdot y)) = \beta_2^*(f(x \cdot y)) \\ &= \beta_2^*(f(x) \cdot f(y)) = \beta_2^*(f(x)) \otimes \beta_2^*(f(y)) \\ &= F(\beta_1^*(x)) \otimes F(\beta_1^*(y)). \end{aligned}$$

□

DEFINITION 3.3. Let f be a strong homomorphism from P_1 into P_2 and let β_1^*, β_2^* be the fundamental relations on P_1, P_2 respectively, then we define

$$\overline{\ker f} = \{\beta_1^*(x) \mid x \in P_1,\ \beta_2^*(f(x)) = \omega_{P_2}\}.$$

The next corollary summarize the results of Lemma 3.2.

COROLLARY 3.4. *i)* $\overline{\ker f}$ *is a normal subgroup of the fundamental group* P_1/β_1^*.

ii) We have

$$(P_1/\beta_1^*)/\overline{\ker f} \cong P_2/\beta_2^*.$$

Let $\mathcal{P}$ be the set of all weak polygroups and all strong homomorphisms. One can show that $\mathcal{P}$ is a category. We set $\mathcal{P}_{\beta^*}$ the category of fundamental groups and homomorphisms of groups, then we have the following theorem:

THEOREM 3.5. *Let* $\mathcal{F}$ *be the function from* $\mathcal{P}$ *into* $\mathcal{P}_{\beta^*}$ *defined by* $\mathcal{F}(P) = P/\beta^*$ *and when* $f : P_1 \longrightarrow P_2$ *is a strong homomorphism*

$$\begin{array}{c} \mathcal{F}(f) : P_1/\beta_1^* \longrightarrow P_2/\beta_2^*, \\ \beta_1^*(x) \longrightarrow \beta_2^*(f(x)), \end{array}$$

where β_1^*, β_2^* *are the fundamental relations on* P_1, P_2 *respectively. Then* $\mathcal{F}$ *is a functor.*

Proof. Clearly, $\mathcal{F}$ is well-defined. Now, we have the following:

i) If $P_1 \in obj\mathcal{P}$ then $P_1/\beta_1^* \in obj\mathcal{P}_{\beta^*}$.

ii) If $f : P_1 \longrightarrow P_2$ is a strong homomorphism, by Lemma 3.2, $\mathcal{F}(f)$ is a homomorphism of groups.

iii) Suppose β_3^* is the fundamental relation on P_3. If $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3$ is a sequence of strong homomorphisms in $\mathcal{P}$, then

$$\begin{aligned} \mathcal{F}(gf)(\beta_1^*(x)) &= \beta_3^*(gf(x)) = \beta_3^*(g(f(x))) = \mathcal{F}(g)\beta_2^*(f(x)) \\ &= \mathcal{F}(g)(\mathcal{F}(f)(\beta_1^*(x))), \end{aligned}$$

so $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$.

iv) For every $P_1 \in obj\mathcal{P}$, we have $\mathcal{F}(1_{P_1})(\beta_1^*(x)) = \beta_1^*(1_{P_1}(x)) = \beta_1^*(x)$, therefore $\mathcal{F}(1_{P_1}) = 1_{\mathcal{F}(P_1)}$. Thus $\mathcal{F}$ is a functor. □

References

[1] S. D. Comer, *Polygroups derived from cogroups*, J. Algebra **89** (1984), 397–405.

[2] P. Corsini, *Prolegomena of Hypergroup Theory*, Supplemento alla "Rivista di matematica pura ed applicata", Aviani Editor, 1993.

[3] B. Davvaz and N. S. Poursalavati, *On polygroup hyperrings and representations of polygroups*, J. Korean Math. Soc. **36** (1999), no. 6, 1021–1031.

[4] B. Davvaz, *Weak polygroups*, Proc. 28th Iranian Math. Conference University of Tabrize, (1997), 28–31.

[5] ———, *On polygroups and permutation polygroups*, Math. Balkanica (N.S.) **14** (2000), no. 1-2, 41–58.

[6] F. Marty, *Sur une generalization de la notion de group*, 8^{th} Congress Math. Scandenaves, Stockholm, (1934), 45–49.

[7] T. Vougiouklis, *The fundamental relations in hyperrings. The general hyperfield*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), World scientific, (1991), 203–211.

[8] ———, *A new class of hyperstructures*, J. Combin. Inf. System Sci. **20** (1995), 229–235.

[9] ———, *Hyperstructures and their representations*, Hadronic Press, Inc., Palm Harbor, FL, 1994.

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