J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 10, Number 1 (February 2003), Pages 25–35

REFLECTIONS OF COMPLETELY REGULAR AND ZERO-DIMENSIONAL QUASI-ORDERED SPACES

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ABSTRACT. We study equivalent definitions and some categorical properties of completely regular quasi-ordered spaces and zero-dimensional quasi-ordered spaces. Using the o-completely regular (*resp.* o-zero-dimensional) filters on a completely regular (*resp.* zero-dimensional) quasi-ordered space, we show that the category **COMPOS** (*resp.* **ZCOMPOS**) of compact (*resp.* compact zero-dimensional) partially ordered spaces is reflective in the category **CRQOS** (*resp.* **ZQOS**) of completely regular (*resp.* zero-dimensional) quasi-ordered spaces and continuous isotones.

0. INTRODUCTION

There have been many attempts to study reflections of topological partially ordered spaces by many authors (*cf.* Choe & Garcia [6], Choe & Y. Hong [7], S. Hong [10], Y. Hong [11], Park [14], etc.).

The concept of completely regular quasi-ordered spaces have been introduced by Nachbin [12], and it is known that every compact partially ordered space is a completely regular partially ordered space. Y. Hong [11] has shown that the category **COMPOS** of compact partially ordered spaces is extensive in the category **CRPOS** of completely regular partially ordered spaces, using the concept of o-completely regular filters (see also Choe & Y. Hong [7]).

The concept of zero-dimensional quasi-ordered spaces have been introduced categorically by Nailana [13]. S. Hong [10] and Y. Hong [11] have shown the concept of zero-dimensional partially ordered spaces as a continuous partially ordered space satisfying two conditions (Z1) and (Z2).

Received by the editors August 7, 2002.
2000 *Mathematics Subject Classification*. 06A06, 18A40, 54F05.
Key words and phrases. completely regular quasi-ordered spaces, zero-dimensional quasi-ordered spaces, o-completely regular filters, o-zero-dimensional filters, reflection, an universal initial completion, MacNeille completion, compact partially ordered spaces, compact zero-dimensional partially ordered spaces.

In this paper, we extend the above result in S. Hong [10] and Y. Hong [11] to completely regular quasi-ordered spaces and zero-dimensional quasi-ordered spaces. Firstly, we study equivalent definitions of completely regular quasi-ordered spaces, and we show that for a continuous quasi-ordered space $(X, \tau, \leq)$, the definition of zero-dimensional quasi-ordered space by Nailana [13] is equivalent to the two conditions (Z1) and (Z2) by S. Hong [10] and Y. Hong [11].

And we show that the category **CRPOS** of completely regular partially ordered spaces (*resp.* **ZPOS** of zero-dimensional partially ordered spaces) is epireflective in the category **CRQOS** (*resp.* **ZQOS**) of completely regular (*resp.* zero-dimensional) quasi-ordered spaces and continuous isotones. Moreover, we show that the topological category **CRQOS** is both the MacNeille and the universal initial completion of the category **CRPOS**, and that the topological category **ZQOS** is an initial completion of the mono-topological category **ZPOS**.

And finally, using o-completely regular filters on a completely regular quasi-ordered space, we construct a dense continuous isotone $\beta_1 : (X, \tau, \leq) \longrightarrow (\beta_1 X, \tau^*, \leq^*)$ from a completely regular quasi-ordered space $(X, \tau, \leq)$ to a compact partially ordered space $(\beta_1 X, \tau^*, \leq^*)$, and show that the category **COMPOS** is epireflective in the category **CRQOS**. Moreover, substituting the three point chain $\{-1, 0, 1\}$ with the discrete topology (which will be denoted by **3**) for the closed interval $[-1, 1]$, we show that the category **ZCOMPOS** of compact zero-dimensional partially ordered spaces is reflective in the category **ZQOS**.

For the terminology not introduced in the paper, we refer to Adámek, Herrlich & Strecker [1] for the category theory and Bourbaki [3, 4] for topology and Davey & Priestley [8] for the order theory. Also we assume throughout this paper that a subcategory of a category is full and isomorphism closed.

1. COMPLETELY REGULAR QUASI-ORDERED SPACES AND ZERO-DIMENSIONAL QUASI-ORDERED SPACES

A *continuous quasi-ordered space* $(X, \tau, \leq)$ is a topological quasi-ordered space with a continuous order $\leq$, *i. e.*, for any $x \not\leq y$ in X, there are neighborhoods U, V of x, y, respectively such that $u \not\leq v$ for all $u \in U$ and $v \in V$ (*cf.* Choe [5], Nachbin [12] and Ward [18]).

The following definition is due to Nachbin [12].

Definition 1.1. A continuous quasi-ordered space $(X, \tau, \leq)$ is called a *completely regular quasi-ordered space* if

(1) for any $x \not\leq y$ in X, there is a continuous isotone $f : X \longrightarrow \mathbb{R}$ such that $f(y) \lneq f(x)$, and

(2) for any $x \in X$ and any neighborhood U of x, there is a continuous isotone $f : X \longrightarrow \mathbb{R}$ and a continuous anti-isotone $g : X \longrightarrow \mathbb{R}$ such that $0 \leq f \leq 1$, $0 \leq g \leq 1$, $f(x) = g(x) = 1$ and $f(y) \wedge g(y) = 0$ for all $y \in \mathbf{C}U$, where $\mathbf{C}U$ is the complement of U.

Remark 1.2. In Definition 1.1, the condition (2) is equivalent to the following condition:

(2′) For any $x \in X$ and any neighborhood V of x, there exist finitely many continuous isotones $f_1, f_2, \ldots, f_n : X \longrightarrow [-1, 1]$ such that $f_i(x) = 0$ for each $i = 1, 2, \ldots, n$ and $\mathbf{C}V \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1, 1\})$, where $[-1, 1]$ is endowed with the usual topology and usual order.

Proof. For any $x \in X$ and any neighborhood V of x, by the condition (2), there is a continuous isotone $f : X \longrightarrow \mathbb{R}$ and a continuous anti-isotone $g : X \longrightarrow \mathbb{R}$ such that $0 \leq f, g \leq 1$, $f(x) = g(x) = 1$ and $f(y) \wedge g(y) = 0$ for all $y \in \mathbf{C}V$; hence $\mathbf{C}V \subseteq f^{-1}(0) \cup g^{-1}(0)$. Let $f_1 = f - 1$ and $f_2 = 1 - g$. Then $f_1, f_2 : X \longrightarrow [-1, 1]$ are continuous isotones and satisfy the condition (2′).

Conversely, for any $x \in X$ and any neighborhood U of x, there are finitely many continuous isotones $f_1, f_2, \ldots, f_n : X \longrightarrow [-1, 1]$ such that $f_i(x) = 0$ for all i and $\mathbf{C}U \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1, 1\})$. Put $f = 1 - (f_1 \vee f_2 \vee \cdots \vee f_n)^-$ and $g = 1 - (f_1 \vee f_2 \vee \cdots \vee f_n)^+$. Then clearly f and g have the required conditions. □

The class of continuous quasi-ordered spaces and continuous isotones forms a category which will be denoted by **WQOS** (*cf.* Shin [16]).

Remark 1.3. In 1984, Salbany [15] has shown that $(X, \tau, \leq)$ is a completely regular quasi-ordered space if and only if the source $\mathbf{WQOS}((X, \tau, \leq), \mathbf{I}_0)$ is initial, where $\mathbf{I}_0$ denotes the unit interval with the usual topology and usual order.

The following definition is due to Nailana [13].

Definition 1.4. A continuous quasi-ordered space $(X, \tau, \leq)$ is called a *zero-dimensional quasi-ordered space* if the source $\mathbf{WQOS}((X, \tau, \leq), D_0)$ is initial, where D_0 is the two-point chain $\{0, 1\}$ with the discrete topology.

Zero-dimensional partially ordered spaces have been studied by S. Hong [10] and Y. Hong [11].

We generalize the same concept to topological quasi-ordered spaces.

Remark 1.5. In S. Hong [10] and Y. Hong [11], a zero-dimensional partially ordered space is a continuous partially ordered space $(X, \tau, \leq)$ satisfying the following conditions:

(Z1) For each $x \in X$ and each open neighborhood V of x, there exist finitely many continuous isotones $f_1, f_2, \ldots, f_n : X \longrightarrow \mathbf{3}$ such that $f_i(x) = 0$ for all i and $\mathbf{C}V \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1, 1\})$, where $\mathbf{3}$ denotes the three point chain $\{-1, 0, 1\}$ with the discrete topology.

(Z2) For each $x \not\leq y$ in X, there exists a continuous isotone $f : X \longrightarrow \mathbf{3}$ such that $f(x) \gtrsim f(y)$.

Proposition 1.6. *For* $(X, \tau, \leq) \in \mathbf{WQOS}$, *the following are equivalent:*

(1) $(X, \tau, \leq)$ *is a zero-dimensional quasi-ordered space.*

(2) (Z1) *and* (Z2) *of Remark 1.5 hold.*

Proof. (1) $\Rightarrow$ (2). For each $x \in X$ and each open neighborhood V of x, there exists finitely many continuous isotones $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m : (X, \tau, \leq) \longrightarrow D_0$ such that

$$x \in (\bigcap_{i=1}^{n} f_i^{-1}(\{1\})) \cap (\bigcap_{j=1}^{m} g_j^{-1}(\{0\})) \subseteq V.$$

We consider continuous isotones $u, v : D_0 \longrightarrow \mathbf{3}$ defined by $u(1) = 1$ and $u(0) = 0$, $v(1) = 0$ and $v(0) = -1$. Let $k_i = v \circ f_i$ $(i = 1, 2, \ldots, n)$ and $l_j = u \circ g_j$ $(j = 1, 2, \ldots, m)$. Then clearly $k_i, l_j : (X, \tau, \leq) \longrightarrow \mathbf{3}$ are continuous isotones for all i, j.

And $k_i(x) = v(f_i(x)) = v(1) = 0$, $l_j(x) = u(g_j(x)) = u(0) = 0$. Take any $a \in \mathbf{C}V$, then either $f_i(a) = 0$ for some i or $g_j(a) = 1$ for some j. So $k_i(a) = v(f_i(a)) = v(0) = -1$ or $l_j(a) = u(g_j(a)) = u(1) = 1$. Hence

$$a \in (\bigcup_{i=1}^{n} k_i^{-1}(\{-1, 1\})) \cup (\bigcup_{j=1}^{m} l_j^{-1}(\{-1, 1\})).$$

Thus (Z1) holds.

For the second statement, suppose $x \not\leq y$ in X. Then there is continuous isotone $f : (X, \tau, \leq) \longrightarrow D_0$ such that $f(x) \not\leq f(y)$, *i. e.*, $f(y) = 0$ and $f(x) = 1$. Hence

$e \circ f : (X, \tau, \leq) \longrightarrow \mathbf{3}$ is a continuous isotone such that $e(f(x)) = 1 \gneq 0 = e(f(y))$, where $e : D_0 \hookrightarrow \mathbf{3}$ is the inclusion map. Thus (Z2) holds.

(2) $\Rightarrow$ (1). We first claim that $\mathbf{WQOS}((X, \tau, \leq), \mathbf{3})$ is initial. Since every $f \in \mathbf{WQOS}((X, \tau, \leq), \mathbf{3})$ is isotone, $x \leq y$ implies $f(x) \leq' f(y)$, where $\leq'$ is the usual order on $\mathbf{3}$. Conversely, suppose $x \not\leq y$, then by (Z2), there is a continuous isotone $g : (X, \tau, \leq) \longrightarrow \mathbf{3}$ such that $g(x) \gneq' g(y)$. Hence $g(x) \not\leq' g(y)$. Thus $\leq$ is an initial quasi-order.

So it remains to show that $\mathbf{WQOS}((X, \tau, \leq), \mathbf{3})$ is initial in $\mathbf{Top}$. Clearly each $f \in \mathbf{WQOS}((X, \tau, \leq), \mathbf{3})$ is continuous. Take a map $h : (Y, \tau') \longrightarrow (X, \tau)$ such that for any $f \in \mathbf{WQOS}((X, \tau, \leq), \mathbf{3})$ $f \circ h$ is continuous. For each $y \in Y$ and each neighborhood V of $h(y)$, by (Z1), there are finitely many continuous isotones $f_1, f_2, \ldots, f_n : (X, \tau, \leq) \longrightarrow \mathbf{3}$ such that $f_i(h(y)) = 0$ for all i and $\mathbf{C}V \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1, 1\})$. Since $f_i \circ h$ is continuous, $(f_i \circ h)^{-1}(\{0\}) = h^{-1}(f_i^{-1}(\{0\}))$ is a neighborhood of y. Hence $\bigcap_{i=1}^{n} h^{-1}(f_i^{-1}(\{0\}))$ is a neighborhood of y and

$$\begin{aligned} h\Big(\bigcap_{i=1}^{n} h^{-1}(f_i^{-1}(\{0\}))\Big) &\subseteq \bigcap_{i=1}^{n} h\Big(h^{-1}(f_i^{-1}(\{0\}))\Big) \\ &\subseteq \bigcap_{i=1}^{n} f_i^{-1}(\{0\}) = \mathbf{C}\Big(\bigcup_{i=1}^{n} f_i^{-1}(\{-1, 1\})\Big) \subseteq V. \end{aligned}$$

So h is continuous at y. Hence h is continuous on Y. We note that $\mathbf{WQOS}(\mathbf{3}, D_0)$ is initial. Since

$$\{u \circ f \mid f \in \mathbf{WQOS}((X, \tau, \leq), \mathbf{3}), u \in \mathbf{WQOS}(\mathbf{3}, D_0)\} \subseteq \mathbf{WQOS}((X, \tau, \leq), D_0),$$

$\mathbf{WQOS}((X, \tau, \leq), D_0)$ is initial. This completes the proof. □

Let **CRQOS** (*resp.* **ZQOS**) be the full subcategory of **WQOS** determined by completely regular (*resp.* zero-dimensional) quasi-ordered spaces and **CRPOS** (*resp.* **ZPOS**) its full subcategory determined by partial order relation instead of quasi-order relation. Then clearly **CRPOS** $\subsetneq$ **CRQOS** $\subsetneq$ **WQOS** and **ZPOS** $\subsetneq$ **ZQOS** $\subsetneq$ **WQOS**. Moreover, **ZQOS** $\subsetneq$ **CRQOS**.

It is well known that the category **Pord** of partially ordered sets is epireflective, initially dense and finally dense in the category **Qord** of quasi-ordered sets and isotones (*cf.* Alderton [2]).

Theorm 1.7. *The category* **CRPOS** (*resp.* **ZPOS**) *is epireflective in the category* **CRQOS** (*resp.* **ZQOS**).

Proof. For any $(X,\tau,\leq) \in$ **CRQOS**, let $q : (X,\leq) \longrightarrow (X/\mathcal{R},\leq_{\mathcal{R}})$ be the **Pord**-reflection of $(X,\leq)$, *i. e.*, $\mathcal{R} = \{(x,y) \in X \times X \mid x \leq y \text{ and } y \leq x\}$, $[x] \leq_{\mathcal{R}} [y]$ if and only if $x \leq y$ and q is the quotient map. For any $f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)$, there is a unique isotone $\overline{f} : X/\mathcal{R} \longrightarrow \mathbf{I}_0$ with $\overline{f} \circ q = f$, for $\mathbf{I}_0 \in$ **Pord**.

Let τ_p be the initial topology on $X/\mathcal{R}$ with respect to

$$\{\overline{f} \mid f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)\}.$$

Then the source $(\overline{f} : (X/\mathcal{R},\tau_p,\leq_{\mathcal{R}}) \longrightarrow \mathbf{I}_0)_{f\in\mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)}$ is initial in the category **TQOS** of topological quasi-ordered spaces. Indeed $[x] \leq_{\mathcal{R}} [y]$ if and only if $x \leq y$ if and only if for all $f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)$, $f(x) \leq f(y)$ if and only if $\overline{f}(q(x)) \leq \overline{f}(q(y))$ for all $f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)$ if and only if for all $f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)$, $\overline{f}([x]) \leq \overline{f}([y])$.

Since $\mathbf{I}_0 \in$ **WQOS**, $(X/\mathcal{R},\tau_p,\leq_{\mathcal{R}}) \in$ **WQOS** by the fact that the category **WQOS** is closed under initial sources in the category **TQOS** (*cf.* Shin [16]); hence the source $(\overline{f})_{f\in\mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)}$ is initial in the category **WQOS**.

Thus $(X/\mathcal{R},\ \tau_p,\ \leq_{\mathcal{R}})$ is a completely regular partially ordered space. Clearly $q : X \longrightarrow X/\mathcal{R}$ is a continuous isotone, for $\overline{f} \circ q = f$ $(f \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0))$.

Now take any continuous isotone $g : X \longrightarrow Y$, where $Y \in$ **CRPOS**, then there is a unique isotone $\overline{g} : X/\mathcal{R} \longrightarrow Y$ with $\overline{g} \circ q = g$.

It remains to show that $\overline{g}$ is continuous. Take any $h \in \mathbf{WQOS}(Y,\mathbf{I}_0)$, then $h \circ g \in \mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)$. Let $\overline{h} : X/\mathcal{R} \longrightarrow \mathbf{I}_0$ be the unique isotone with $\overline{h} \circ q = h \circ g$, then $\overline{h}$ is also a continuous isotone, because $\overline{h}$ is a member of the initial source $(\overline{f} : X/\mathcal{R} \longrightarrow \mathbf{I}_0)_{f\in\mathbf{WQOS}((X,\tau,\leq),\mathbf{I}_0)}$. Since $h \circ \overline{g} \circ q = h \circ g = \overline{h} \circ q$ and q is onto, $h \circ \overline{g} = \overline{h}$ $(h \in \mathbf{WQOS}(Y,\mathbf{I}_0))$. Since the source $\mathbf{WQOS}(Y,\mathbf{I}_0)$ is initial in the category **Top** of topological spaces, $\overline{g}$ is also continuous.

For the case of the category **ZQOS**, substituting D_0 for $\mathbf{I}_0$ in the above proof, we have the same results for **ZPOS** in the place of **ZQOS**. □

It is immediate from the definition that the category **CRPOS** (*resp.* **ZPOS**) is initially dense in the category **CRQOS** (*resp.* **ZQOS**).

Proposition 1.8. *The category* **CRPOS** *is finally dense in the category* **CRQOS**.

Proof. Let $(X,\tau,\leq) \in$ **CRQOS** and

$$S = \{f \in \mathbf{CRQOS} \mid f : (\tilde{X},\tilde{\tau},\tilde{\leq}) \longrightarrow (X,\tau,\leq),\ (\tilde{X},\tilde{\tau},\tilde{\leq}) \in \mathbf{CRPOS}\}.$$

We note that every topological space is a quotient of a paracompact Hausdorff space (*cf.* Herrlich [9]). So there is a paracompact Hausdorff space (X', τ') and a quotient map $q : (X', \tau') \longrightarrow (X, \tau)$. Then $(X', \tau', =) \in$ **CRPOS** and

$$q : (X', \tau', =) \longrightarrow (X, \tau, \leq)$$

is continuous isotone. Hence $q \in S$, and so $S \neq \varnothing$. We claim that S is a final sink. Let

$$G_0 = \bigcup\{(f \times f)(G_{\tilde{\leq}}) \mid f : (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \longrightarrow (X, \tau, \leq) \text{ in } S\}.$$

Then $G_0 = G_{\leq}$. Indeed, if $x \leq y$ in X, and let $(Y, D, \leq') \in$ **CRPOS**, where $(Y, \leq')$ is the two-point chain $\{0, 1\}$ and D is the discrete topology. Let the map $g : (Y, D, \leq') \longrightarrow (X, \tau, \leq)$ be defined by $g(0) = x$ and $g(1) = y$. Then g is continuous isotone. Hence $g \in S$, *i. e.*, $(x, y) \in G_0$. Hence $G_{\leq} \subseteq G_0$.

Conversely, suppose that $(x, y) \in G_0$. Since each $f \in S$ is isotone, $G_0 \subseteq G_{\leq}$. Since q is final in **Top** and $q \in S$, the sink $(f : (\tilde{X}, \tilde{\tau}) \longrightarrow (X, \tau))_{f \in S}$ is final in **Top**. Thus S is a final sink in **TQOS** (*cf.* Shin [17]) and hence in **CRQOS**. This completes the proof. □

Proposition 1.9.

(1) **CRQOS** (*resp.* **ZQOS**) *is a topological category.*

(2) **CRPOS** (*resp.* **ZPOS**) *is a mono-topological category.*

Proof. (1) Since the category **CRQOS** is the initial hull of $\{\mathbf{I}_0\}$ in the category **WQOS**, the category **CRQOS** is bireflective in the category **WQOS**. Note that any bireflective subcategory of a topological category is topological. Since the category **WQOS** is topological (*cf.* Shin [16]), the category **CRQOS** is topological.

For the second half, substituting D_0 for $\mathbf{I}_0$, we have the results.

(2) It follows from the fact that the categories **CRPOS** and **ZPOS** are closed under initial mono-sources in the category **WPOS**. □

Corollary 1.10.

(1) *The category* **CRQOS** *is both the MacNeille and the universal initial completion of the category* **CRPOS**.

(2) *The category* **ZQOS** *is an initial completion of the category* **ZPOS**.

2. COMPOS (*resp.* ZCOMPOS)-REFLECTIONS OF COMPLETELY REGULAR (*resp.* ZERO-DIMENSIONAL) QUASI-ORDERED SPACES

It is known (*cf.* Choe & Y. Hong [7] and Y. Hong [11]) that a completely regular partially ordered space is compact if and only if every maximal o-completely regular filter on the space is convergent.

The following definition is due to Choe & Y. Hong [7] and Y. Hong [11].

Definition 2.1. Let $(X, \tau, \leq)$ be a completely regular quasi-ordered space.

(1) A filter $\mathcal{F}$ on $(X, \tau, \leq)$ is said to be *o-completely regular* if $\mathcal{F}$ has an open base $\mathcal{B}$ satisfying that for all $U \in \mathcal{B}$, there exist $V \in \mathcal{B}$ with $V \subseteq U$ and finitely many continuous isotones $f_1, f_2, \ldots, f_n : X \longrightarrow [-1,1]$ with $f_i(V) = 0$ for all i and $CU \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1,1\})$.

(2) An o-completely regular filter $\mathcal{F}$ on $(X, \tau, \leq)$ is said to be *maximal* if it is a maximal element in the set of o-completely regular filters endowed with the inclusion.

Remark 2.2.

(1) For every o-completely regular filter, by Zorn's Lemma, there exists a maximal o-completely regular filter containing it.

(2) For $(X, \tau, \leq) \in$ **CRQOS** and $\mathcal{F}$ an o-completely regular filter on X, the following property is obtained by the same way in Choe & Y. Hong [7] and Y. Hong [11]: $\mathcal{F}$ is a maximal o-completely regular filter if and only if for each open sets U, V with $V \subseteq U$ and finitely many continuous isotones $f_1, f_2, \ldots, f_n : X \longrightarrow [-1,1]$ such that $f_i(V) = 0$ for $i = 1, 2, \ldots, n$ and $CU \subseteq \bigcup_{i=1}^{n} f_i^{-1}(\{-1,1\})$, either $U \in \mathcal{F}$ or $U \notin \mathcal{F}$ and there is $F \in \mathcal{F}$ with $F \bigcap V = \varnothing$.

Lemma 2.3. A filter $\mathcal{U}$ on a completely regular quasi-ordered space $(X, \tau, \leq)$ contains a maximal o-completely regular filter if and only if $f(\mathcal{U})$ is convergent for each continuous isotone $f : X \longrightarrow [-1,1]$.

Proof. Since a filter containing a convergent filter is again convergent, it is enough to show that for every maximal o-completely regular filter $\mathcal{F}$ and any continuous isotone $f : X \longrightarrow [-1,1]$, $f(\mathcal{F})$ is convergent. Detail of the proof is the same for **CRPOS** (see Y. Hong [11]). □

Corollary 2.4. *Every neighborhood filter of a completely regular quasi-ordered space* $(X, \tau, \leq)$ *is a maximal o-completely regular filter.*

A continuous quasi-ordered space $(X, \tau, \leq)$ is called a *compact quasi-ordered space* if the topological space (X, τ) is compact.

Using Lemma 2.3 and Corollary 2.4, we have the following remark.

Remark 2.5. A completely regular quasi-ordered space $(X, \tau, \leq)$ is compact if and only if every maximal o-completely regular filter on $(X, \tau, \leq)$ is convergent.

We will characterize the reflection of a completely regular quasi-ordered space $(X, \tau, \leq)$ by o-completely regular filters on a completely regular quasi-ordered space $(X, \tau, \leq)$.

For a completely regular quasi-ordered space $(X, \tau, \leq)$, let

$$\beta_1 X = \{\mathcal{M} \mid \mathcal{M} : \text{maximal } o\text{-completely regular filter on } X\}$$

endowed with the topology τ^* generated by

$$\{U^* \mid U^* = \{\mathcal{M} \in \beta_1 X, U \in \mathcal{M}\}, U \in \tau\}$$

and a relation $\leq^*$ defined as follows:

$$\mathcal{M} \leq^* \mathcal{N} \text{ in } \beta_1 X \text{ if and only if } \lim f(\mathcal{M}) \leq \lim f(\mathcal{N}) \text{ for all } f \in \hom(X, [-1, 1]).$$

It is obvious that $(\beta_1 X, \leq^*)$ is a partially ordered set and $\{U^* \mid U \in \tau\}$ forms a base for τ^*. Let $\beta_1 : X \longrightarrow \beta_1 X$ be a map defined by $\beta_1(x) = \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the neighborhood filter of x in X.

Remark 2.6. For $(X, \tau, \leq) \in$ **CRQOS**, $\beta_1 X = (\beta_0 \circ q)X$, where $q : (X, \leq, \tau) \longrightarrow (X/\mathcal{R}, \tau_p, \leq_{\mathcal{R}})$ is the **Pord**-reflection (See in the proof of Theorem 1.7) and β_0 is the reflection of **CRPOS** in Y. Hong [11].

Proposition 2.7. *$\beta_1 : (X, \tau, \leq) \longrightarrow (\beta_1 X, \tau^*, \leq^*)$ is a dense continuous isotone. Furthermore, for any continuous isotone $f : X \longrightarrow [-1, 1]$, there is a unique continuous isotone $\overline{f} : \beta_1 X \longrightarrow [-1, 1]$ with $\overline{f} \circ \beta_1 = f$.*

Proof. For any basic open set U^* in $(\beta_1 X, \tau^*)$, we have

$$(\beta_1)^{-1}(U^*) = \{x \in X \mid \beta_1(x) = \mathcal{N}(x) \in U^*\} = \{x \in X \mid U \in \mathcal{N}(x)\} = U,$$

so that β_1 is continuous. Suppose that $x \leq y$. For any $f \in \hom(X, [-1, 1])$, we have

$$\lim f(\beta_1(x)) = \lim f(\mathcal{N}(x)) = f(x) \leq f(y) = \lim f(\mathcal{N}(y)) = \lim f(\beta_1(y)),$$

which imply $\beta_1(x) \leq^* \beta_1(y)$. Thus β_1 is an isotone. Now take any non-empty basic open set U^* in $(\beta_1 X, \tau^*)$, then $U \neq \varnothing$. Pick $x \in U$, then $U \in \mathcal{N}(x) = \beta_1(x)$, *i. e.*, $\beta_1(x) \in U^*$; hence $\beta_1(X)$ is dense in $(\beta_1 X, \tau^*)$.

For the second half, we define $\overline{f} : \beta_1 X \longrightarrow [-1,1]$ by $\overline{f}(\mathcal{M}) = \lim f(\mathcal{M})$ for all $\mathcal{M} \in \beta_1 X$. Then by Lemma 2.3, $\overline{f}$ is a map and clearly

$$\overline{f}(\beta_1(x)) = \overline{f}(\mathcal{N}(x)) = \lim f(\mathcal{N}(x)) = f(x) \text{ for all } x \in X.$$

By the definition of $\leq^*$, $\overline{f}$ is an isotone. Take any closed neighborhood V of $\overline{f}(\mathcal{M}) = \lim f(\mathcal{M})$, there is an open set $U \in \mathcal{M}$ with $f(U) \subseteq V$. Then U^* is an open neighborhood of $\mathcal{M}$. Moreover, take any $\mathcal{N} \in U^*$, then $U \in \mathcal{N}$. Since $\overline{f}(\mathcal{N}) = \lim f(\mathcal{N})$, $\overline{f}(\mathcal{N}) \in \overline{f(U)} \subseteq \overline{V} = V$. Thus $\overline{f}(U^*) \subseteq V$; therefore $\overline{f}$ is continuous at $\mathcal{M}$. Since $[-1,1]$ is a Hausdorff space and β_1 is dense, such an $\overline{f}$ with $\overline{f} \circ \beta_1 = f$ is unique. This completes the proof. □

Corollary 2.8. *For $(X, \tau, \leq) \in$ **CRQOS**, $(\beta_1 X, \tau^*, \leq^*)$ is a compact partially ordered space.*

Using Proposition 2.7 and Corollary 2.8, we have the following theorem immediately:

Theorm 2.9. *The category* **COMPOS** *of compact partially ordered spaces is reflective in the category* **CRQOS** *via $\beta_1 : (X, \tau, \leq) \longmapsto (\beta_1 X, \tau^*, \leq^*)$ where $(X, \tau, \leq\ ) \in$* **CRQOS**.

Substituting **3** for $[-1,1]$ in the above argument in this section, we can define o-zero-dimensional filters and using this, we can conclude that the category **ZCOMPOS** of compact zero-dimensional partially ordered spaces is reflective in the category **ZQOS**.

Remark 2.10. $\beta_1 : (X, \tau, \leq) \longrightarrow (\beta_1 X, \tau^*, \leq^*)$ where $(X, \tau, \leq) \in$ **CRQOS** (*resp.* $(X, \tau, \leq) \in$ **ZQOS**) is an embedding if and only if $(X, \tau, \leq\ ) \in$ **CRPOS** (*resp.* $(X, \tau, \leq) \in$ **ZPOS**).

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